Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New series Vol.8, 1994, Fasc.1

Convolution Theorem for a Generalized Hermite Transform

Beatriz Gonzáles M. †, *Shyam Kalla* †, *I. H. Dimovski* ‡

Presented by P. Kenderov

In this work we define a new integral transform and deduce its inversion formula. The nucleus of this transform involves a generalization of the Hermite polynomials. The transform is considered for locally integrable functions $f(x)$ according to Lebesgue and of order $O(e^{ax^2})$, with $a < -1$ for $|x| \to \infty$, that is a vector space which is denoted by L^{exp}. Besides we present an explicit convolution for such transform, for even and odd functions. The results are the extended for arbitrary functions of the space considered.

1. Introduction

We define a generalization of the Hermite polynomials $H_n(x)$ in terms of the confluent hypergeometric function ϕ, and deduce its orthogonality on $(-\infty,\infty)$. Other properties and relations referred to these polynomials are given in a previous paper [4].

(i) Generalized Hermite polynomials.

We define the generalized Hermite polynomials, which are denoted by the symbol $H_{2n+r,a}(x)$, as

$$H_{2n+r,a} = \frac{(-1)^n(2n+r)!}{n!}(2x)^r\phi(-n; a+1 : x^2) \tag{1}$$

where r is a nonnegative fixed integer ($r \in \mathbb{N}$); $n \in \mathbb{N}$; x is real and $a > -1$. If $r = 0$ and $a = -1/2$, $H_{2n+r,a}(x)$ coincides with the even Hermite polynomial, $H_{2n}x$ and if $r = 1$, $a = 1/2$, (1) reduces to the odd Hermite polynomials.

If we substitute the result [5, p. 273 (9.13.10)] in (1), with $z = x^2$ we obtain a relation with the generalized Laguerre polynomials $L_n^a(x)$:

$$H_{2n+r,a} = \frac{(-1)^n(2n+r)!(2x)^r}{(a+1)_n} L_n^a(x^2) \tag{2}$$

x is real; $r \in \mathbb{N}$ (fixed); $a > -1$ and $n \in \mathbb{N}$.

(i) Orthogonality of the polynomials $H_{2n+r,a}(x)$.

Let's consider the function $w(x, a, r) = e^{-x^2}(x^2)^{a-r+1/2}$ and the integral.

$$I_{a,r} = \int_{-\infty}^{\infty} w(x, a, r) H_{2n+r,a}(x) H_{2m+r,a}(x) \mathrm{d}x.$$

x is real; $r \in \mathbb{N}$ (fixed); $a > -1; n \in \mathbb{N}$ and $m \in \mathbb{N}$.
If we substitute $H_{2n+r,a}(x)$ and $H_{2m+r,a}(x)$ according to expression (2) we obtain

$$I_{a,r} = \frac{2^{2r}(-1)^{m+n}(2n+r)!(2m+r)!}{(a+1)_n(a+1)_m} \int_0^{\infty} e^{-t} t^a L_n^a(t) L_m^a(t) \mathrm{d}t$$

where, using the orthogonality property of the Laguerre polynomials [5, p. 84], results

$$I_{a,r} = \begin{cases} 0 & , n \neq m \\ \dfrac{2^{2r}\Gamma(a+n+1)}{n!} \left[\dfrac{(2n+1)!}{(a+1)_n}\right]^2 & , n = m \end{cases} \tag{3}$$

Hence, the polynomials $H_{2n+r,a}(x)$ are orthogonal on $(-\infty, \infty)$ with respect to the weight function $w(x, a, r)$.

2. Generalized Hermite transform and its inversion formula

The generalized Hermite transform of a function $f(x) \in L^{exp}$ is defined by

$$\mathcal{H}\{f(x);\ 2n+r\} = \int_{-\infty}^{\infty} f(x) e^{-x^2}(x^2)^{a-r+1/2} H_{2n+r,a}(x) \mathrm{d}x \tag{4}$$

x is real; $r \in \mathbb{N}$ (fixed); $n \in \mathbb{N}$; $a - r + 1/2 \geq 0$ and $H_{2n+r,a}(x)$ are the polynomials defined according to (1).
Let's consider

$$f(x) = \sum_{n=0}^{\infty} C_{2n+r} H_{2n+r,a}(x)\ ;\ r \in \mathbb{N},$$

If we multiply (5) by $e^{-x^2}(x^2)^{a-r+1/2}H_{2m+r,a}(x)$, integrate with respect to x over $(-\infty,\infty)$ and utilize the results (3), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)e^{-x^2}(x^2)^{a-r+1/2}H_{2m+r,a}\mathrm{d}x = \\ = C_{2m+r}\frac{2^{2r}\Gamma(a+m+1)}{m!}\left[\frac{(2m+1)!}{(a+1)_m}\right]^2\end{aligned} \tag{6}$$

The results (4), (5) and (6) lead to the desired inversion formula:

$$f(x)=\sum_{n=0}^{\infty}\frac{n[(a+1)_n]^2}{2^{2r}\Gamma(a+n+1)[(2n+r)!]^2}\mathcal{H}\{f(x);\ 2n+r\}\,H_{2n+r,a}(x) \tag{7}$$

x real, $r\in\mathbb{N}$ (fixed); $a\geq r-1/2$
If r is even, we have the generalized even Hermite integral transform of the function $f(x)$, which we denote by $\mathcal{H}_p\{f(x);2n+r\}$ and if r is odd, we obtain the odd Hermite integral transform of the function $f(x)$ which is denoted as $\mathcal{H}_i\{f(x);2n+r\}$

Particular cases

(i) For $r=0$ and $a=-1/2$, the transform (4) coincides with the even Hermite integral transform [3, p. 345]

(ii) If in (4) we take $r=1$ and $a=1/2$, we obtain the Hermite integral transform for n odd.

3. Convolution of the generalized Hermite integral transform for odd and even functions

The convolution of an integral transform is considered in the sense of Churchill [1] and Dimovski [3], which is defined as follows:

Definition 1. An operation $f*g$ in L^{exp}, is said to be a convolution for generalized Hermite transform (4), if and only if

$$\mathcal{H}\{f*g;2n+r\}=\mu_n\mathcal{H}\{f;2n+r\}\mathcal{H}\{g;2n+r\};n\in\mathbb{N}$$

$r\in\mathbb{N}$ (fixed), with $\mu_n\neq 0$, where the sequence $\{\mu_n\}$does not depend on functions f and g.

If we denote by L_I^{exp} and L_p^{exp} the odd and even function subspaces in L^{exp} respectively, obviously we obtain:
If $f\in L_I^{exp}$, then $\mathcal{H}_p\{f;2n+r\}=0$ and if $f\in L_p^{exp}$, then $\mathcal{H}_i\{f;2n+r\}=0$

Therefore, we just consider:

$$\mathcal{H}_i\{f; 2n+r\}; r \text{ odd, if } f \in L_I^{exp} \text{ and}$$
$$\mathcal{H}_p\{f; 2n+r\}; r \text{ even, if } f \in L_p^{exp}$$

3.1 Convolution of the generalized Hermite transform for odd functions

Let $f \in L_I^{exp}$, then

$$\mathcal{H}_i\{f(x); 2n+r\} = 2\int_0^\infty e^{-x^2}(x^2)^{a-r+1/2} f(x) H_{2n+r,a}(x)\mathrm{d}x \tag{8}$$

$r \in \mathbb{N}$ (a fixed odd number); $a \geq r - 1/2$ and $n \in \mathbb{N}$.
If we substitute the result (2) in (8) and set $x^2 = t$, results

$$\mathcal{H}_i\{f(x); 2n+r\} = \frac{2^r(-1)^r(2n+r)!}{(a+1)_n}\int_0^\infty e^{-t}t^a \frac{f(t^{1/2})}{t^{r/2}} L_n^a(t)\mathrm{d}t \tag{9}$$

Introducing the transformation

$$(Tf)(t) = \frac{f(t^{1/2})}{t^{r/2}}; \quad 0 \leq t < \infty \tag{10}$$

whose inverse transformation is

$$(T^{-1}\phi)(x) = x^r\phi(x^2) \tag{11}$$

we can write the relation (9), as

$$\mathcal{H}_i\{f(x); 2n+r\} = \frac{2^r(-1)^r(2n+r)!}{(a+1)_n}\int_0^\infty e^{-t}t^a (Tf)(t) L_n^a(t)\mathrm{d}t \text{ i.e.}$$

$$\mathcal{H}_i\{f(x); 2n+r\} = \frac{2^r(-1)^r(2n+r)!}{(a+1)_n} L^a\{Tf; n\} \tag{12}$$

where the symbol $L^a\{h(x); n\}$ denotes the Laguerre transform of function $h(x)$, defined in [2, p. 41].
Let's consider now the functions f and g in L_I^{exp} and the product of their generalized Hermite transforms. As indicated above

$$\mathcal{H}_i\{f; 2n+r\}\mathcal{H}_i\{g; 2n+r\} = \frac{2^{2r}[(2n+r)!]^2}{[(a+1)_n]^2} L^a\{Tf; n\} L^a\{Tg; n\} \tag{13}$$

If $a \geq r - 1/2$, for all $r \in \mathbb{N}$ (odd) (i.e. $a > -1/2$), then we can use the convolution theorem for the generalized Laguerre integral transform established in [2], according to which we can express

$$L^a\{f \overset{\wedge}{*} g; n\} = \mu_n L^a\{f; n\} L^a\{g; n\} \tag{14}$$

with

$$\mu_n = \frac{\sqrt{\pi} n!}{\Gamma(n + a + 1)} \tag{15}$$

and

$$(f \overset{\wedge}{*} g)(x) = \int_0^{\infty} e^{-t} t^a f(t) \int_0^{\pi} e^{-\sqrt{xt}\cos\phi} A \sin^{2a}\phi \mathrm{d}\phi \mathrm{d}t \tag{16}$$

with

$$A = g(x + t + 2\sqrt{xt}\cos\phi) \frac{J_{a-1/2}(\sqrt{xt}\sin\phi)}{(1/2\sqrt{xt}\sin\phi)^{a-1/2}}$$

and $J_\nu(x)$is the Bessel function of first kind and order ν. From (14), (15) and (13), we obtain

$$\begin{aligned} &\mathcal{H}_i\{f;2n + r\}\mathcal{H}_i\{g; 2n + r\} = \\ &= \frac{2^{2r}[(2n + r)!]^2\Gamma(n + a + 1)}{[(a + 1)_n]^2 \pi^{1/2} n!} L^a\{T_f \overset{\wedge}{*} T_g; n\} \end{aligned} \tag{17}$$

$r \in \mathbb{N}$ (a fixed odd number); $n \in \mathbb{N}$ and $a \geq r - 1/2$
where, according to (10) and (16)

$$(T_f \overset{\wedge}{*} T_g)(y) = \int_0^{\infty} e^{-t} t^a \frac{f(t^{1/2})}{t^{r/2}} \int_0^{\pi} e^{-\sqrt{yt}\cos\phi} B \mathrm{d}\phi \mathrm{d}t \tag{18}$$

with

$$B = \frac{\sin^{2a}\phi g[(y + t + 2\sqrt{yt}\cos\phi)^{1/2}] J_{a-1/2}(\sqrt{yt}\sin\phi)}{(y + t + 2\sqrt{yt}\cos\phi)^{1/2}(1/2\sqrt{yt}\sin\phi)^{a-1/2}}$$

Using (12) we obtain

$$L^a\{T_f \overset{\wedge}{*} T_g; n\} = \frac{(a + 1)_n(-1)^n}{2^r(2n + r)!} \mathcal{H}_i\{T^{-1}\{T_f \overset{\wedge}{*} T_g\}; 2n + r\} \tag{19}$$

From (12) and (19)

$$\mathcal{H}_i\{T^{-1}\{T_f \overset{\wedge}{*} T_g\}; 2n + r\} = \frac{(-1)^n(a + 1)_n \pi^{1/2} n!}{2^r(2n + r)!\Gamma(n + a + 1)} \mathcal{H}_i\{f; 2n + r\}\mathcal{H}_i\{g; 2n + r\}$$

Therefore $f \overset{i}{*} g = T^{-1}(T_f \overset{\wedge}{*} T_g)$ is a convolution of the generalized Hermite integral transform in L_I^{exp}. So,

$$\mathcal{H}_i\{f \overset{i}{*} g\}; 2n+r\} = \mu_n \mathcal{H}_i\{f; 2n+r\}\mathcal{H}_i\{g; 2n+r\}$$

with

$$\mu_n = \frac{(-1)^n (a+1)_n \pi^{1/2} n!}{2^r (2n+r)! \Gamma(n+a+1)} \tag{20}$$

Here $f \overset{i}{*} g$, from (18) and (11) is given by

$$\begin{aligned}(f \overset{i}{*} g)(y) &= T^{-1}(T_f \overset{\wedge}{*} T_g)(y) = \\ &= 2y^r \int_0^\infty e^{-x^2}(x^2)^{a-r/2} x f(x) \int_0^\pi e^{-yx\cos\phi} A d\phi dx\end{aligned} \tag{21}$$

with

$$A = \frac{\sin^{2a}\phi g[(y^2+x^2+2yx\cos\phi)]^{1/2} J_{a-1/2}(yx\sin\phi)}{(y^2+x^2+2yx\cos\phi)^{r/2}(\frac{yx}{2}\sin\phi)^{a-1/2}}$$

$r \in \mathbb{N}$ (a fixed odd number); $a \geq r - 1/2$; $n \in \mathbb{N}$ and $J_{a-1/2}(x)$ is the Bessel function of first kind and order ν

3.2 Convolution of the generalized Hermite transform for even functions

Let $f \in L_p^{exp}$, then

$$\mathcal{H}_p\{f; 2n+r\} = 2\int_0^\infty e^{-x^2}(x^2)^{a-r+1/2} f(x) H_{2n+r,a}(x) dx \tag{22}$$

The equations (9)-(12) are valid for functions $f \in L_p^{exp}$, when $a \geq r - 1/2$ and $r \in \mathbb{N}$ (a fixed even number); except that $r = 0$ and $a = -1/2$
Hence we have the following

Theorem 1. *–The operator $f * g$, given by (21) and denoted by $f \overset{i}{*} g$, if $r \in \mathbb{N}$ (odd) and $f \overset{p}{*} g$ if $r \in \mathbb{N}$ (even) is an internal operation in:*
(i) L_I^{exp} such that

$$\mathcal{H}_i\{f \overset{i}{*} g; 2n+r\} = \mu_n \mathcal{H}_i\{f; 2n+r\}\mathcal{H}_i\{g; 2n+r\}; \;\; a \geq r - 1/2$$

(ii) L_p^{exp} such that

$$\mathcal{H}_p\{f \overset{p}{*} g; 2n+r\} = \mu_n \mathcal{H}_p\{f; 2n+r\}\mathcal{H}_p\{g; 2n+r\} \tag{23}$$

$a \geq r - 1/2$ and $a \neq -1/2$
with $\mu_n \neq 0$, given by (20).

If $r = 1$ and $a = 1/2$, the former result coincides with Theorem 1; formulated and demonstrated by Dimovski and Kalla [3].
For $r \doteq 0$; $a = -1/2$, the expression (22) coincides with the Hermite integral transform of the function f, and the convolution $f \overset{p}{*} g$ (21) and the relation (23), correspond to theorem 2 given by Dimovski and Kalla [3].

4. Convolution of the generalized Hermite integral transform for arbitrary functions

Let f and g be arbitrary function in L^{exp}. Each function can be written as the sum of a function in L_I^{exp} and a function L_p^{exp}. So $f = f_I + f_p$ and $g = g_I + g_p$; where

$$f_I(x) = \frac{f(x) - f(-x)}{2} \in L_I^{exp} \tag{24}$$

$$f_p(x) = \frac{f(x) + f(-x)}{2} \in L_p^{exp} \tag{25}$$

Similarly for g.
Let's consider $\mathcal{H}\{f; 2n+r\} = \mathcal{H}\{f_I + f_p; 2n+r\}$. Then,

$$\mathcal{H}\{f; 2n+r\} = \mathcal{H}\{f_I; 2n+r\} + \mathcal{H}\{f_p; 2n+r\}$$

If $r \in \mathbb{N}$ (even), $\mathcal{H}\{f; 2n+r\} = \mathcal{H}_p\{f_p; 2n+r\}$ and if $r \in \mathbb{N}$ (odd), we have $\mathcal{H}\{f; 2n+r\} = \mathcal{H}_i\{f_I; 2n+r\}$.
Considering the operation $f * g = (f_I \overset{i}{*} g_I) + (f_p \overset{p}{*} g_p)$; where the operations $\overset{i}{*}$ and $\overset{p}{*}$ are given by (21) for $r \in \mathbb{N}$ (odd); $a \geq r - 1/2$ and $r \in \mathbb{N}$ (even); $a \geq r - 1/2$ ($a \neq -1/2$); respectively. Then we have:

$$\begin{aligned}\mathcal{H}\{f * g; 2n+r\} &= \mathcal{H}\{f_I \overset{i}{*} g_I; 2n+r\} + \mathcal{H}\{f_p \overset{p}{*} g_p; 2n+r\}\\ &= \mathcal{H}_i\{f_I \overset{i}{*} g_I; 2n+r\} \text{ if } r \in \mathbb{N} \text{ (odd)} ;\end{aligned}$$

according to Theorem 1

$$\mathcal{H}\{f * g; 2n+r\} = \mu_n \mathcal{H}_i\{f_I; 2n+r\} \mathcal{H}_i\{g_I; 2n+r\}$$

$r \in \mathbb{N}$ (a fixed odd number); $a \geq r - 1/2$ and with μ_n given by (20). Similarly, if $r \in \mathbb{N}$ (a fixed even number) we have:

$$\mathcal{H}\{f * g; 2n + r\} = \mu_n \mathcal{H}_p\{f_p; 2n + r\}\mathcal{H}_p\{g_p; 2n + r\}$$

$a \geq r - 1/2$ $(a \neq -1/2)$ and with μ_n given by (20).
The earlier results can be summed up in the following theorem

Theorem 2. *–The operation* $f * g = (f_I \overset{i}{*} g_I) + (f_p \overset{p}{*} g_p)$*; where the operators* $\overset{i}{*}$ *and* $\overset{p}{*}$ *are given by (21) for* $r \in \mathbb{N}$ *(odd) and* $r \in \mathbb{N}$ *(even), is a convolution of the generalized Hermite integral transform (4) in* L^{exp} *such that if* $f, g \in \mathbb{N}$ *and* $n = 0, 1, 2, \ldots$*, then*

(i) If $r \in \mathbb{N}$ *(odd)*

$$\mathcal{H}\{f * g; 2n + r\} = \mu_n \mathcal{H}_i\{f_I; 2n + r\}\mathcal{H}_i\{g_I; 2n + r\}; \quad a \geq r - 1/2$$

and $a \neq -1/2$

(ii) If $r \in \mathbb{N}$ *(even)*

$$\mathcal{H}\{f * g; 2n + r\} = \mu_n \mathcal{H}_p\{f_p; 2n + r\}\mathcal{H}_p\{g_p; 2n + r\}; \quad a \geq r - 1/2$$

and $a \neq -1/2$ *with* f_I*,* g_I*,* f_p*,* g_p *given by (24) and (25) and* μ_n *given by (20)*

If $r = 0$ and $a = -1/2$, the convolution of the generalized Hermite integral transform for arbitrary functions in L^{exp} correspond to the theorem 3 of Dimovski and Kalla [3].

The third of the authors was partially supported by Grant No. $MM65/91$ of the Ministry of Education and Science in Bulgaria.

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†*Department of Mathematics*
Centro de investigación de
Matemática Aplicada
Facultad de Ingenierial
Universidad de Zulia
Maracaibo,
VENEZUELA
‡*Institute of Mathematics*
Bulgarian Academy of Sciences
Sofia
BULGARIA

Received 06.07.1992