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Sharp Upper Bounds for and Some Properties of Generalized Entropies

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0. Introduction

In order to fix ideas we first consider a stochastic variable ξ with the probability density function $f_\xi(x) = f(x)$, $-\infty < x < \infty$. We recall that the (Shannon-Wiener) *differential-entropy*, denoted by $H[f]$, is defined by

$$H[f] = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx.$$

The differential-entropy is of great importance in Information Theory as a measure of the expected amount of information in nats (see e.g. [2] and the references given there). In this paper we generalize this differential-entropy by considering the more general exponential entropies $E[\alpha, \beta; f]$ *of order* (α, β). In the light of this the aim of my talk is to review, motivate, complement, illustrate and unify some fairly new results concerning inequalities, generalized entropies and generalized Gini means, which can be of importance in Information Theory and the theory of data compression. The paper is organized in the following way: In section 1 we introduce, motivate and begin to analyze the scale of exponential entropies of order (α, β). In particular the close connections to the generalized Gini means (see [13]) and Renyi's scale of entropies (see [2]) are pointed out. In section 2 we discuss the fact that $E[\alpha, \beta; f]$ can represent the *extent* of a probability function, i.e. its degree of concentration with regard to a reference measure. We also present a precise description of $E[\alpha, 1; f]$ in terms of *intrinsic extent* of a probability function thereby generalizing some previous results of L.L Campell [3]. In section 3 we investigate $E[\alpha, \beta; f]$ for the exponential family of distributions. In particular we calculate $E[\alpha, \beta; f]$ for the Miller-Thomas scale of distributions (which in particular contain the univariate Gaussian and the Laplace distributions). We also include some illustrations, which, in particular, indicate the possibility to obtain *sharp upper bounds for the exponential entropies.* In section 4 we present such upper bounds and point out the corresponding extremal functions thereby generalizing the classical *differential-entropy inequality* in an essential way. Section 5 is reserved for some concluding remarks. In particular we include a new result we recently obtained by using interpolation theory.

In this paper $\Gamma(x)$ and $B(p, r)$ denote the usual Gamma and Beta functions, respectively (see e.g. [17]).

1. Scales of Generalized Entropies and Generalized Gini means

Let X, Σ, P denote a probability space and let μ be a σ-finite reference measure on X such that P is absolutely continuous with respect to μ. We denote the corresponding density by $f = dP/d\mu$. Then the ***exponential entropy*** $E[\alpha, \beta; f]$ ***of order*** (α, β), $\alpha, \beta \geq 0$, is defined as

$$E[\alpha, \beta; f] = \begin{cases} \left(\frac{\int_X f^\alpha d\mu}{\int_X f^\beta d\mu} \right)^{\frac{1}{\beta-\alpha}} & , \alpha \neq \beta, \\ \exp\left(\frac{-\int_X f^\alpha \ln f d\mu}{\int_X f^\alpha d\mu} \right) & , \alpha = \beta, \end{cases}$$

whenever these quantities exist.

Example. *(the uniform distribution) Let* $\Omega = R$, *let* μ *be the Lebesgue measure and let* $f(x) = 1/\theta$ *on* $I = [0, \theta]$, *where* θ *is a positive number, and* $f(x) = 0$ *outside of* I. *Then* $E[\alpha, \beta; f] = \theta$.

First we mention the following remarks/motivations for studying this scale of entropies:

1. For the case $\beta = 1$, $0 \leq \alpha \leq 1$, $E[\alpha, \beta; f]$ coincide with the entropies introduced by L.L Campell [3].

2. The quantity $E[1/(1 + r), 1; f]$ coincides with the integral in the expression for the lower bound of the average distortion obtained by J.L. Bucklew-G.L. Wise in the theory for quantization of data (see [9]).

3. We observe that $E[\alpha,\beta;f] = 1/G_A[\alpha,\beta;f]$, where $G_A[\alpha,\beta;f]$ are the *generalized Gini means* studied by J. Peetre-L.E Persson in [13], with A equal to the integral operator over X.

4. We define $H[\alpha,\beta;f] = \ln E[\alpha,\beta;f]$ and note that $H[\alpha,f] = H[\alpha,1;f]$ is the usual scale of *Renyi's entropies*. In particular if μ is the Lebesgue measure on the real line, then $H[1,1;f]$ coincides with the usual differential-entropy $H[f]$.

5. $E[\alpha,\beta;f]$ represent the extent of f, i.e. the degree of its concentration with respect to a reference measure, and $E[\alpha,1;f]$ coincides with the intrinsic extent of order α (see section 2).

6. An essential generalization of the classical differential-entropy inequality can be done in terms of the exponential entropies $E[\alpha,1;f]$ and the corresponding optimal functions can be pointed out in all cases (see section 4).

7. The quantities $E[\alpha,\beta;f]$ are usually easy to calculate (see e.g. section 3) and to use in practice e.g. depending on the fact that the following elementary but useful properties hold:

Theorem 1. *(a) The following representation formula holds:*

$$E[\alpha,\beta;f] = \exp\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \ln E[a,a;f]da\right).$$

(b) $E[\alpha,\beta;f]$ *is a nonincreasing function of* α *(*β *fixed) and of* β *(*α *fixed).*

(c) $E[\alpha,\beta;f]$ *is a continuous function for every fixed* f.

(d) $E[\alpha,\beta;f] = E[\beta,\alpha;f]$.

(e) $E[\alpha,\beta;cf] = c^{-1}E[\alpha,\beta;f]$ *for every* $c > 0$.

(f) $E[0,1;f] = \mu(\Omega)$, $E[\alpha,\beta;f] \to 1/\operatorname{ess\,sup} f$ *as* $\alpha \to \infty$ *or* $\beta \to \infty$ *and* $E[0,0;f] = (G(f))^{-1}$, *where* $G(f)$ *is the generalized geometric mean of* f.

(g) The functional $(E[\alpha,\beta;\bullet])^{-1}$ *is convex if* $0 \le \alpha \le 1 \le \beta$ *(or* $0 \le \beta \le 1 \le \alpha$*).*

The representation formula in (a) means that $E[\alpha,\beta;f]$ is equal to the generalized geometric mean of the intermediate "diagonal" means $G[a,a;f]$. The proof of (a) follows easily by using the definition of the derivative and elementary calculus (see[13]). Moreover, (b) can be proved in the following way: We note that the derivative of the function $h(a) = \ln E[a,a;f]$ is a quotient with positive denumerator and with the numerator

$$\left(\int_X f^\alpha \ln f d\mu\right)^2 - \left(\int_X f^\alpha d\mu\right)\left(\int_X f^\alpha (\ln f)^2 d\mu\right).$$

Thus, according to Schwarz's inequality, we find that $h(a)$ is a non-increasing function of a and the proof follows by using (a). Also the other statements in Theorem 1 are obvious or follow easily from the results in [13] (see motivation 3 above) and some straightforward calculations (see [9]).

2. Exponential Entropies and the Extent of a Probability Function

First we note that, according to a result of L.L. Campell [3], the entropy $E[1,1;f]$ can be characterized in terms of the intrinsic extent $K_1(\mu)$. Here we introduce *the intrinsic extent of order* α, denoted $K_\alpha(\mu)$, as $K_\alpha(\mu) = \inf \nu(X)$ for $0 < \alpha \leq 1$ and $K_\alpha(\mu) = \sup \nu(X)$ for $\alpha > 1$, where inf and sup, respectively, are taken over all σ-infinite measures ν which are equivalent to μ, and satisfying

$$\int_X \left(\frac{d\nu}{d\mu}\right)^{\frac{\alpha-1}{\alpha}} dP = 1, \alpha \neq 1, \text{ and } \exp\left(\int_X \ln\left(\frac{d\nu}{d\mu}\right) dP\right) = 1 \text{ if } \alpha = 1.$$

Theorem 2. *Let* $\alpha > 0$. *Then* $E[\alpha, 1; f] = K_\alpha(\mu)$.

Remark. We note that Theorem 2 means that $E[\alpha, 1; f]$ can be characterized in terms connected with the probability distribution P.

Proof. Let $\alpha > 1$. Then, by Hölder's inequality,

$$\nu(X) = \int_X \frac{d\nu}{d\mu}\frac{d\mu}{dP} dP \leq \left(\int_X \left(\frac{d\nu}{d\mu}\right)^{\frac{\alpha-1}{\alpha}} dP\right)^{\frac{\alpha}{\alpha-1}} \left(\int_X \left(\frac{d\mu}{dP}\right)^{1-\alpha} dP\right)^{\frac{1}{1-\alpha}} =$$

$$= \left(\int_X \left(\frac{d\mu}{dP}\right)^{1-\alpha} dP\right)^{\frac{1}{1-\alpha}} = \left(\int_X f^\alpha d\mu\right)^{\frac{1}{1-\alpha}}.$$

Now we introduce the (optimal) measure ν_0 by the relation

$$d\nu_0 = (E[a,1;f])^\alpha f^{\alpha-1} dP$$

and note that

$$\int_X \left(\frac{d\nu_0}{d\mu}\right)^{\frac{\alpha-1}{\alpha}} dP = (E[\alpha,1;f])^{\alpha-1} \int_X f^{\alpha-1} dP = (E[\alpha,1;f])^{\alpha-1} \int_X f^\alpha d\mu = 1$$

and

$$\nu_0(X) = (E[\alpha,1;f])^\alpha \int_X f^{\alpha-1} dP = (E[\alpha,1;f])^\alpha \int_X f^\alpha d\mu = E[\alpha,1;f].$$

Thus by taking supremum the proof follows. The proof of the case $\alpha < 1$ is similar and the proof of the case $\alpha = 1$ (Campell's case) is only a limiting case but a direct proof can also be carried out (see [9] and [3]).

Now let ξ be a realvalued and continuous random variable with the probability density function $f(x)$ (with respect to the Lebesgue measure) and with standard deviation σ. Then the random variable $\eta = a\xi$ has the density $f_a(x) = |a|^{-1}f(x/a)$ and the standard deviation $\sigma_a = |a|\sigma$. It follows that the *scale invariance property*

$$\frac{E[\alpha,\beta;f_a]}{\sigma_a} = \frac{E[\alpha,\beta;f_b]}{\sigma_b}$$

holds for all $a, b > 0$. This observation together with Theorem 1 suggest that the exponential entropies $E[\alpha,\beta;f]$ represent a continuous and nonincreasing scale of quantities measuring, in certain ranges of α and β, the **extent** of a probability function, a fact further illuminated in the next theorem.

Theorem 3. *Let* $0 < \alpha \leq 1 \leq \beta$ *and* $E_c = \{x|f(x) \geq c\}$ $(c > 0)$. *If* μ *and* P *have the same null sets and if* $\int_X f^\beta(x)d\mu \leq 1$, *then* $\mu(E_c) \leq c^{-\alpha}(E[\alpha,\beta;f])^{\beta-\alpha}$ *and* $P(E_c) \geq 1 - c^{1-\alpha}(E[\alpha,\beta;f])^{\beta-\alpha}$.

For the case $\beta = 1$ Theorem 3 was proved L.L. Campell [3] and the proof of this more general case is similar (see [9]).

Remark. In particular Theorem 3 means that if $E[\alpha,\beta;f]$ is "small", then a large part of the P-probability mass is concentrated on a set with a small μ-measure. The relationship of this fact to the theory of data compression in terms of scalar quantizers is further investigated in [7].

3. $E[\alpha,\beta;f]$ for the Exponential Family of Distributions

In this section we consider *the exponential family of distributions* i.e. we consider the distributions on an Euclidean space X defined by the densities $f_\theta(x) = C(\theta)e^{t(x)\theta}$, where $t(x)$ is a k-dimensional statistic, θ is a parameter and $C(\theta) = (\int_X e^{t(x)\theta}d\mu)^{-1}$. The set of parameters θ such that this integral-converges is called *the natural parameter space*. A straightforward calculation together with Theorem 1(c) yields the following statement (see [9]):

Proposition 4. *If* α *and* β *are such that* $\alpha\theta$ *and* $\beta\theta$ *belong to the natural* *parameter space, then*

$$E[\alpha,\beta,f_\theta] = \frac{1}{C(\theta)}\left(\frac{C(\beta\theta)}{C(\alpha\theta)}\right)^{\frac{1}{\beta-\alpha}} \quad \text{if } 0 \le \alpha < \beta \text{ and}$$

$$E[\alpha,\alpha;f_\theta] = \frac{1}{C(\theta)}e^{-E_{\alpha\theta}(t\theta)} = \frac{1}{C(\theta)}e^{\phi'(\alpha\theta)}, \quad \text{where}$$

$$\phi'(\alpha\theta) = \frac{\partial}{\partial\alpha}\ln(C(\alpha\theta)) \text{ and } E_{\alpha\theta}(t\theta) = \int_X t(x)\theta f_{\alpha\theta}(x)d\mu.$$

As a special case we obtain the following result for *the Miller-Thomas scale of distributions* (see [11]):

Corollary. *Let $s > 0$ and let μ be the Lebesgue measure on R, $t(x) = -|x|^s$, $\theta = \zeta^s$ and $C(\theta) = s\theta^{1/s}/2\Gamma(1/s)$, where $\zeta = \frac{1}{\sigma}\sqrt{\Gamma(3/s)/\Gamma(1/s)}$. Then*

$$E[\alpha,\beta;f_\theta] = \frac{2\Gamma(1/s)}{s\theta^{1/s}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{s(\beta-\alpha)}}, \alpha \ne \beta, \text{ and } E[\alpha,\alpha;f_\theta] = \frac{2\Gamma(1/s)}{s\theta^{1/s}}e^{1/s\alpha}.$$

In particular for the case $s = 2$ we obtain

Example 1. (the univariate Gaussian distribution). For simplicity we assume that the mean is equal to 0. Here we have $t(x) = -x^2$, $\theta = 1/2\sigma^2$ and $C(\theta) = \sqrt{\frac{\theta}{\pi}}$, where σ is the standard deviation, and thus,

$$E[\alpha,\beta;f_\theta] = \sqrt{2\pi}\sigma\left(\sqrt{\frac{\beta}{\alpha}}\right)^{\frac{1}{(\beta-\alpha)}}, \alpha \ne \beta, \text{ and } E[\alpha,\alpha;f_\theta] = \sqrt{2\pi}\sigma e^{1/2\alpha}.$$

Remark. By taking logarithms we obtain for the case $\alpha = \beta = 1$ the well-known differential entropy of the univariate Gaussian distribution (see [2]). Furthermore, for the case $s = 1$ we have

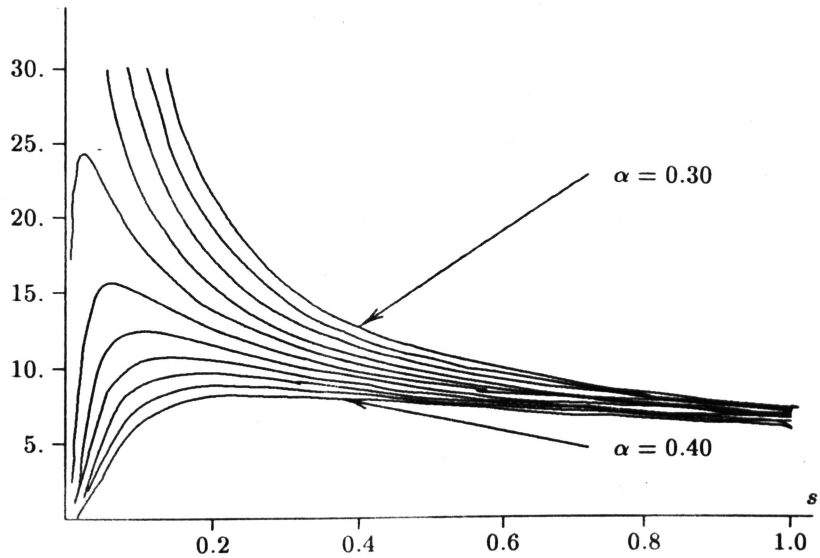
Example 2. (the Laplace distribution). Here we have $t(x) = -|x|$, $\theta = \frac{\sqrt{2}}{\sigma}$ and $C(\theta) = \theta/2$, where σ is the standard deviation, and thus,

$$E[\alpha,\beta;f_\theta] = \frac{2\sigma}{\sqrt{2}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{\beta-\alpha}}, \alpha \ne \beta, E[\alpha,\alpha;f_\theta] = \frac{2\sigma}{\sqrt{2}}e^{1/\alpha}.$$

Remark. By taking $(E[\alpha,1;f])^{1-\alpha}$ in the examples 1 and 2 we obtain the *information generating functions* for the univariate Gaussian and the Laplace distributions, respectively (compare with [6]).

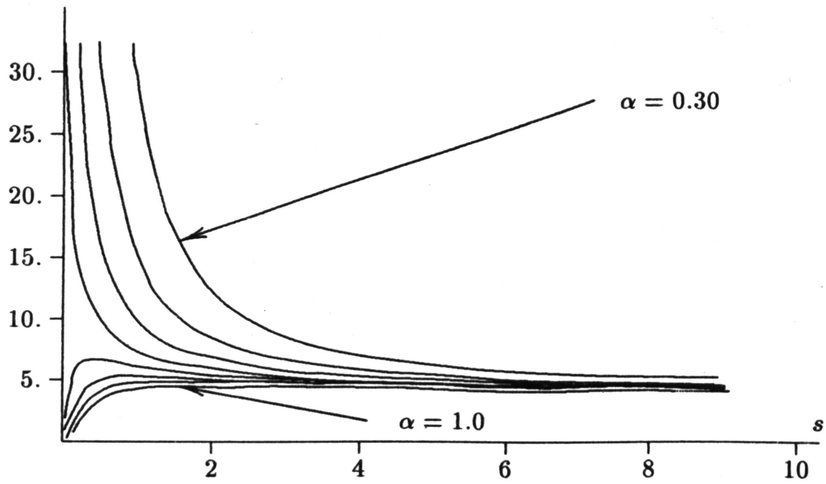
R e m a r k. The parameter s in the scale of Miller-Thomas distributions is called the *shape parameter*. Small values of s correspond to distributions with flat modes (see [9] and [11]).

We finish this section by presenting some illustrations of the exponential entropies for the scale of Miller-Thomas distributions for different s-values.



$E[\alpha, 1; f_\theta]$ for $\alpha = 0.30, 0.31, ..., 0.40$ ($\sigma = 1$)

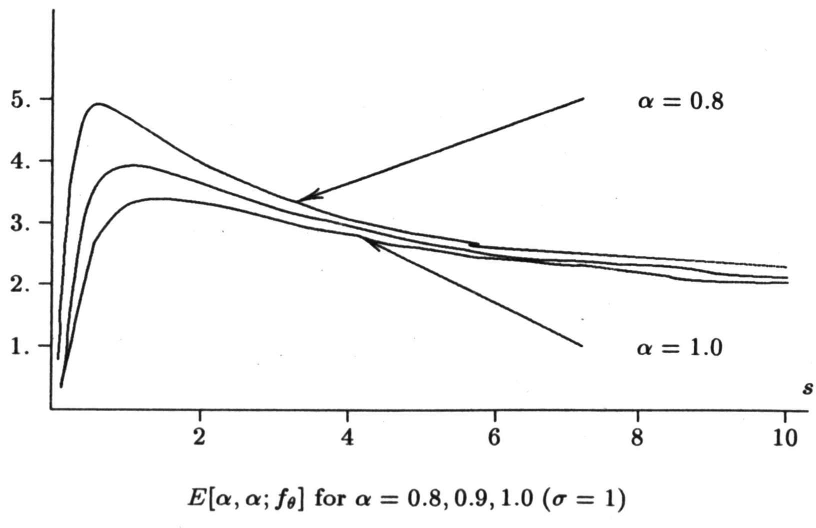
In particular we can make some elementary calculations to see that $E[\alpha, 1; f_\theta] \to 2\sigma 3^{1/2}$ as $s \to \infty$ and that $E[\alpha, 1; f_\theta] \to 0+$ iff $\alpha > 1/3$ (see [9]).



$E[\alpha, \alpha; f_\theta]$ for $\alpha = 0.30, 0.40, \ldots, 1.0$ ($\sigma = 1$)

In particular we can make some elementary calculations to see that $E[\alpha, \alpha; f_\theta] \to 2\sigma 3^{1/2}$ as $s \to \infty$ and that $E[\alpha, \alpha; f_\theta] \to 0+$ iff $\alpha > 2/3 \ln 3$ (see [9]).

According to the diffentialentropy-inequality (see section 4) we know that $E[1, 1; f_\theta]$ has maximum for $s = 2$. This fact is illustrated on our next figure.



$E[\alpha, \alpha; f_\theta]$ for $\alpha = 0.8, 0.9, 1.0$ ($\sigma = 1$)

We note that for values $\alpha \neq 1$ the exponential entropies seem to have maximum for other values than $s = 2$ (i.e. for the Gaussian distribution). Compare with our next section and see also [7] and [14].

4. The Differential-entropy Inequality and Sharp Upper Bounds for the Exponential Entropies

We recall that if $f(x)$ is the density of the univariate Gaussian distribution, then the differential-entropy is equal to $\ln(\sigma(2\pi e)^{1/2})$, where σ is the standard deviation (see example 1). Moreover, the **differential-entropy inequality**, which is of great importance in Information Theory (see e.g. [2]), certifies that the differential entropy is less than or equal to $\ln(\sigma(2\pi e)^{1/2})$ for **any** stochastic variable with the standard deviation σ. In view of the illustrations in the previous section it is now tempting to guess that a similar inequality can be true for $E[\alpha; f] = E[\alpha, 1; f]$ also when $\alpha \neq 1$. In fact the following precise generalization of the differential entropy inequality holds:

Theorem 5. *Consider any continuous probability distribution on the real line with the density $f(x)$ and with finite standard deviation σ_0.*

a) If $\alpha > 1$, then

$$E(\alpha; f) \leq \left(\frac{2\alpha}{3\alpha - 1}\right)^{\frac{1}{1-\alpha}} \sqrt{\frac{3\alpha - 1}{\alpha - 1}} B\left(\frac{\alpha}{\alpha - 1}, \frac{1}{2}\right) \sigma_0.$$

Equality is obtained if and only if $f(x)$ coincides (almost everywhere) with

$$f^*(x) = \begin{cases} \frac{c}{a}\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{\alpha-1}} & , |x| < a, \\ 0 & , |x| \geq a, \end{cases}$$

where $c = \frac{1}{B\left(\frac{\alpha}{\alpha-1}, \frac{1}{2}\right)}$ and $a = \sqrt{\frac{3\alpha-1}{\alpha-1}}\sigma_0$.

b) If $1/3 < \alpha < 1$, then

$$E(\alpha; f) \leq \left(\frac{2\alpha}{3\alpha - 1}\right)^{\frac{1}{1-\alpha}} \sqrt{\frac{3\alpha - 1}{1 - \alpha}} B\left(\frac{1 + \alpha}{2(1 - \alpha)}, \frac{1}{2}\right) \sigma_0.$$

Equality is obtained if and only if $f(x)$ coincides (almost everywhere) with

$$f^*(x) = \frac{c}{a}\left(1 + \frac{x^2}{a^2 r}\right)^{-\frac{r+1}{2}}, -\infty < x < \infty,$$

where $r = \frac{1+\alpha}{1-\alpha}$, $c = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\Gamma(\frac{r}{2})}$ *and* $a = \sqrt{\frac{r-2}{r}}\sigma_0$.

A similar but somewhat more complicated estimate holds also for the remaining case $0 \leq \alpha \leq 1/3$. Detailed proofs of these statements can be found in [7]. See also [14].

Remark. By making some straightforward calculations we find that the constants on the right hand side of a) and b) both converge to $(2\pi\varepsilon)^{1/2}\sigma_0$ if $\alpha \to 1+$ and $a \to 1-$, respectively. Moreover, some other elementary calculations show that the corresponding distributions converge to the distribution function of the normal distribution (see [7]).

Remark. The optimal distributions discussed in this section are of the types classified in the usual Pearson's system of frequency curves.

5. Concluding Remarks

1. The classical differential-entropy has already before been generalized in various directions and we refer to the review article [4] where a systematic overview of numerous concepts of generalized entropies and their relations are given. However, also some criticism on such formal generalizations of the fundamental differential entropy can be found e.g. in [1] and [5].

2. By replacing the integrals in the definition of $E[\alpha, \beta; f]$ by more general isotone functionals on suitable classes of functions we obtain the inverted value of the generalized Gini means studied by J. Peetre - L.E. Persson [13] (here we can also permit negative α and β). In particular for the discrete case when these functionals are sums we obtain the classical scale of Gini means, which, in its turn, generalizes the geometric, the arithmetic, the harmonic mean, power means of order α, etc.

3. All entropies discussed in this paper can be described in terms of generalized Gini means via invertions and logarithms. Moreover, new information concerning continuous scales of means between an *arbitrary mean* and for example a fixed Gini mean has recently been obtained by L.E. Persson - S. Sjöstrand [16]. By taking logarithms and/or making invertions of these scales of means we can obtain more general entropies than those discussed here. Many results in this paper can be generalized in this direction too

4. *The Gauss-Laplace mixture* was introduced by J.H. Miller - J.B. Thomas [10] as a model for Gaussian bursts of Laplace impulsive noise 100ε percent of the time ($0 \leq \varepsilon \leq 1$). The Gauss-Laplace mixture density $f(x)$ is

defined by $f(x) = (1-\varepsilon)f_1(x) + \varepsilon f_2(x)$, where $f_1(x)$ is the Gaussian density with $\sigma = \sigma_1$ and $f_2(x)$ is the Laplace density with $\sigma = \sigma_2$. By using Theorem 1(g) we find that the inequality

$$E[\alpha,\beta;f] \geq \frac{E[\alpha,\beta;f_1]E[\alpha,\beta;f_2]}{\varepsilon E[\alpha,\beta;f_1] + (1-\varepsilon)E[\alpha,\beta;f_2]}$$

holds if $0 \leq \alpha \leq 1 \leq \beta$. Thus, according to examples 1 and 2, we can estimate $E[\alpha,\beta;f]$ also for the Gauss-Laplace mixture.

5. The concepts of *ε-entropy and ε-capacity* was early introduced by A.N. Kolmogorov and later on studied by J. Peetre [12]. Also these quantities are important in the theory of data compression (see [8] and the references given there). By using modern interpolation theory it is possible to generalize and complement some results in [12]. We finish this paper by presenting such a generalization which is proved in [8], where also some new applications and examples are included.

Theorem 6. *Let $\lambda = \lambda(t)$ be an interpolation parameter function, let (A_0, A_1) be a compatible Banach couple, where A_1 is continuously imbedded in A_0. If the intermediate Banach space is of the class $C_K(\lambda, c_0)$ then*

$$N(\varepsilon; A_0, A) \leq N(\mu^L; A_0, A_1),$$

where μ^L is Legendre transform of μ.

Concerning definitions and basic concepts in this connection we only refer to [12] and the review article [15].

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