Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol.8, 1994, Fasc.2–3

Preconditioning Elliptic Problems on Grids with Multilevel Local Refinement*

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Presented by Bl. Sendov

We consider an algebraic derivation of multilevel preconditioners of $BEPS-$ type for solving finite element elliptic problems on grids with local refinement. The multilevel iterations are of a hybrid $V-$cycle type. That is, at every group of k_0 steps of simple block elimination of the nodes on the corresponding refinement level, we perform a correction by certain polynomial approximation based on properly scaled and shifted Chebyshev matrix polynomial. This hybrid multilevel $V-$cycle $BEPS-$ preconditioner is shown to be of optimal order of complexity for $2-D$ and $3-D$ problem domains. It performs without any restrictions on the shape of the subregions as well as with respect to their number and sizes, and to the way of refining for properly chosen parameters. However, if certain regularity of the refinement is assumed the pure $V-$cycle multilevel $BEPS-$ preconditioner is shown to be optimal. This is also illustrated by a number of numerical experiments.

1. Introduction

Consider a sequence of finite element stiffness matrices $\{A^{(k)}\}_{k=1}^{l}$, that is

$$A^{(k)} = \left(a(\phi_i^{(k)}, \phi_j^{(k)})\right)_{x_i, x_j \in N^{(k)}}.$$

* This research has been supported in part by Bulgarian Committee for Science under Grant No. 55-26-3-87 and by National Science Foundation under Grant INT-8914472.

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$\left\{\phi_i^{(k)}\right\}$ are nodal basis functions in the finite element spaces V_k, $V_1 \subset V_2 \subset \ldots \subset V_l$. The bilinear form

$$a(u,\phi) = \int_\Omega \sum k_{i,j}(x,y) \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j}\, dx$$

is supposed symmetric, elliptic and bounded.

Here $N^{(k)}$ is the set of the nodes of the corresponding triangulation τ_k of the considered domain Ω, a two dimensional polygon or a 3-dimensional parallelipiped. The set $\{\tau_k\}$ is obtained by successive local refinement. That is we have at every discretization level the following partitioning of the nodes $N^{(k)}$,

$$N^{(k)} = N_1^{(k)} \bigcup N_2^{(k)},$$

where $N_1^{(k)}$ corresponds to the nodes, vertices, of the elements to be refined at the next level $k+1$.

Then any matrix $A^{(k)}$ admits the following two by two block form

$$A^{(k)} = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix} \begin{array}{l} \} N_1^{(k)} \\ \} N_2^{(k)} \end{array} \tag{1.1}$$

Note that the major block-matrix $A_{11}^{(k)}$ corresponds to the discretization of the considered problem on a grid which is obtained by uniform refinement, that is on a regular grid. We denote by $\tilde{A}^{(k)}$ the matrix $A^{(k-1)}$ but partitioned into the following block form

$$\tilde{A}^{(k)} = \begin{bmatrix} \tilde{A}_{11}^{(k)} & \tilde{A}_{12}^{(k)} \\ \tilde{A}_{21}^{(k)} & \tilde{A}_{22}^{(k)} \end{bmatrix} \begin{array}{l} \} N^{(k-1)} \setminus N_2^{(k)} \\ \} N_2^{(k)} \end{array} . \tag{1.2}$$

Recall that by construction $N_2^{(k)} \subset N^{(k-1)}$ ($N_2^{(k)}$ is the set of nodes on the level k, which corresponds to vertices of elements which are not refined, that is they also are nodes from $N^{(k-1)}$).

The following two-grid preconditioner was proposed in Bramble, Ewing, Pasciak, and Schatz [4] (in a matrix form it was derived in Ewing, Lazarov, and Vassilevski [6])

$$B^{(k)} = \begin{bmatrix} A_{11}^{(k)} & 0 \\ A_{21}^{(k)} & \tilde{S}^{(k)} \end{bmatrix} \begin{bmatrix} I & A_{11}^{(k)^{-1}} A_{12}^{(k)} \\ 0 & I \end{bmatrix} \tag{1.3}$$

where $\tilde{S}^{(k)} = \tilde{A}_{22}^{(k)} - \tilde{A}_{21}^{(k)} \tilde{A}_{11}^{(k)^{-1}} \tilde{A}_{12}^{(k)}$ is the Schur complement of $\tilde{A}^{(k)}$, (1.2).

When solving systems with $B^{(k)}$ one needs two solvings with $A_{11}^{(k)}$, that is on a regular grid and one solving with $\tilde{S}^{(k)}$, which is based on a solving with $\tilde{A}^{(k)}$, that is on a coarser grid.

The scope of this paper is to derive certain algebraic extensions of the above two-grid $BEPS-$ preconditioner (1.3) to the multilevel case. In Ewing, Lazarov, and Vassilevski [7] versions of algebraic multilevel $BEPS-$type preconditioners were proposed and investigated. One of them was of $V-$cycle type, whereas the second one was of pure $W-$cycle type and it was proved in [7] that the $V-$cycle $BEPS-$ preconditioner is nearly optimal but the second one, of $\nu-$fold, $\nu \geq 2$, $V-$ cycle type, is of optimal order. However, in the latter case there was certain restriction on the speed of refining the subregions. In the present paper we weakened this restriction by allowing the multilevel iteration to be of $V-$cycle type at every k_0 succesive steps and only after that to use a $\nu-$fold, $\nu > 1$, $V-$cycle iteration which is based on a certain polynomial acceleration involving properly scaled and shifted Chebyshev polynomials. We were able to show that the optimality property of the hybrid $V-$cycle $BEPS-$ preconditioner thus derived is preserved.

Earlier results on optimal iterative refinement methods were investigated in Widlund [18], Dryja and Widlund [5], Mandel and McCormick [9]. The results of the present paper are a natural extension of the algebraic multilevel theory proposed in Vassilevski [16] and Axelsson and Vassilevski [1] and mainly of the recent report, Vassilevski [17]. Two-grid and multilevel iterative refinement algorithms were studied in McCormick [11], McCormick and Thomas [12], also see the monograph, McCormick [13].

Our approach is however purely algebraic and it is based on certain relations between the Schur complements $S^{(k)}$ of $A^{(k)}$ partitioned into the block form (1.1) and the corresponding Schur complement $\tilde{S}^{(k)}$ of $\tilde{A}^{(k)}$, (1.2). Namely, by a local analysis technique, one can show that $S^{(k)}$ and $\tilde{S}^{(k)}$ are spectrally equivalent and to estimate the extreme eigenvalues of ${\tilde{S}^{(k)}}^{-1}S^{(k)}$ and based only on this information the method is constructed. Hence we do not need any regularity of the bilinear form $a(.,.)$. In particular the corresponding relative condition number is bounded independent of possible jumps of the coefficients of the bilinear form if these coefficients are continuous within each element from the initial triangulation τ_1. However, if we assume certain regularity of the refinement our proof can be modified to show that the pure $V-$cycle multilevel $BEPS-$ preconditioner is optimal, similarly to the results of Widlund [18].

The remainder of the paper is organized as follows. In Section sec2 the construction of the algebraic multilevel $BEPS-$ preconditioners is given and the optimality properties are proved. In Section sec3 in a case of regular local refinement the optimality of the pure $V-$cycle $BEPS-$ preconditioner is proved. In Section sec4 some computational aspects of the method are outlined. Finally, in Section sec5, a number of test examples that demonstrate the practical behaviour of the proposed multilevel $BEPS-$ preconditioners are presented.

Throughout the present paper for a given square matrix C by $\lambda[C]$ we mean any eigenvalue of C.

2. The Algebraic Multilevel BEPS-Preconditioners

Let $\hat{B}^{(k)}$ be an approximation to the stiffness matrix $A^{(k)}$ that is let

$$\lambda\left[A^{(k)^{-1}}\hat{B}^{(k)}\right] \in [1, \lambda_k]. \tag{2.1}$$

Let $k_0 \geq 1$ be a given integer parameter. Then define, $B^{(k)} = \hat{B}^{(k)}$ and for $s = k+1, k+2, \ldots, k+k_0$

$$B^{(s)} = \begin{bmatrix} A_{11}^{(s)} & 0 \\ A_{21}^{(s)} & \tilde{C}^{(s)} \end{bmatrix} \begin{bmatrix} I & A_{11}^{(s)^{-1}} A_{12}^{(s)} \\ 0 & I \end{bmatrix} \tag{2.2}$$

where $\tilde{C}^{(s)}$ is the Schur complement of $\tilde{B}^{(s)}$, that is the Schur complement of $B^{(s-1)}$ but partitioned into the block form,

$$\tilde{B}^{(s)} = \begin{bmatrix} \tilde{A}_{11}^{(s)} & \tilde{A}_{12}^{(s)} \\ \tilde{A}_{21}^{(s)} & \tilde{C}^{(s)} + \tilde{A}_{21}^{(s)}\tilde{A}_{11}^{(s)^{-1}}\tilde{A}_{12}^{(s)} \end{bmatrix} \begin{matrix} \} N^{(s-1)} \setminus N_2^{(s)} \\ \} N_2^{(s)} \end{matrix} \tag{2.3}$$

The blocks $\left\{A_{i,j}^{(s)}\right\}_{i,j=1}^{2}$, $\left\{\tilde{A}_{i,j}^{(s)}\right\}_{i,j=1}^{2}$ are from the block partitionings (1.1), (1.2) of $A^{(s)}$ and $\tilde{A}^{(s)}$ respectively.

We study next the condition number of $A^{(s)^{-1}}B^{(s)}$, $s = k+1, \ldots, k+k_0$. For this we need the following auxiliary fact used in Vassilevski [17].

Lemma 2.1. *There exists an increasing function $\eta = \eta(k_0)$ such that*

$$v_2^t A^{(k)} v_2 \leq \eta(k_0) \begin{bmatrix} cv_1 \\ v_2 \end{bmatrix}^t A^{(k+k_0)} \begin{bmatrix} cv_1 \\ v_2 \end{bmatrix}, \tag{2.4}$$

where $\begin{bmatrix} cv_1 \\ v_2 \end{bmatrix} \begin{array}{l} \} N^{(k+k_0)} \setminus N^{(k)} \\ \} N^{(k)} \end{array}$, *all* v_1, v_2, *and* $k = 1, 2, \ldots, l$.

Proof. (see [15]). Estimate (2.4) is a matrix analogue of the discrete Sobolev inequality. For two-dimensional problems as shown in Yserentant [20] we can set $\eta(k_0) = C\, k_0 \log \mu$, $C = const$ and for three dimensional problem domains as is well known we can define $\eta(k_0) = C\, \mu^{k_0/3}$, if we define the edges of the elements into $\mu^{1/3}$ parts in the process of refining the elements. □

Lemma 2.2. *The following inequalities are valid*

$$\frac{1}{\eta(1)} v_2^t \tilde{S}^{(s)} v_2 \le v_2^t S^{(s)} v_2 \le v_2^t \tilde{S}^{(s)} v_2, \quad \text{all } v_2,$$

that is the Schur complements $\tilde{S}^{(s)}$ *and* $S^{(s)}$ *of* $\tilde{A}^{(s)}$ *and* $A^{(s)}$, *respectively, see (1.2), (1.1), are spectrally equivalent.* ($\eta(1)$ *is form (2.4)).*

Proof. We have by (2.4) for any vector $v = \begin{bmatrix} cw \\ \tilde{v} \end{bmatrix} \begin{array}{l} \} N^{(s)} \setminus N^{(s-1)} \\ \} N^{(s-1)} \end{array}$,

$$\tilde{v}^t \tilde{A}^{(s)} \tilde{v} \le \eta(1) \inf_w v^t A^{(s)} v$$

Let $\tilde{v} = \begin{bmatrix} c\tilde{v}_1 \\ v_2 \end{bmatrix} \begin{array}{l} \} N^{(s-1)} \setminus N_2^{(s)} \\ \} N_2^{(s)} \end{array}$, $v = \begin{bmatrix} cv_1 \\ v_2 \end{bmatrix}$, $v_1 = \begin{bmatrix} cw \\ \tilde{v}_1 \end{bmatrix}$. Then

$$v_2^t \tilde{S}^{(s)} v_2 = \inf_{\tilde{v}_1} \tilde{v}^t \tilde{A}^{(s)} \tilde{v} \le \eta(1) \inf_{\tilde{v}_1, w} v^t A^{(s)} v = \eta(1) v_2^t S^{(s)} v_2.$$

The other required inequality follows from the following identity for any $\phi \in V_s$ with a coefficient vector v with respect to the standard nodal basis in V_s

$$v^t A^{(s)} v = a(\phi, \phi).$$

Then using the partitioning $v = \begin{bmatrix} cw \\ \tilde{v}_1 \\ v_2 \end{bmatrix} \begin{array}{l} \} N^{(s)} \setminus N^{(s-1)} \\ \} N^{(s-1)} \setminus N_2^{(s)} \\ \} N_2^{(s)} \end{array}$, we have

$$\begin{aligned} v_2^t S^{(s)} v_2 &= \inf_{\tilde{v}_1, w} v^t A^{(s)} v = \inf_{\substack{\phi|_{N^{(s)} \setminus N^{(s-1)}} \\ \phi|_{N^{(s-1)} \setminus N_2^{(s)}}}} a(\phi, \phi) \\ &\le \inf_{\substack{\tilde{\phi}|_{N^{(s-1)} \setminus N_2^{(s)}} = \phi|_{N^{(s-1)} \setminus N_2^{(s)}} \\ \tilde{\phi} \in V_{s-1} \subset V_s}} a(\tilde{\phi}, \tilde{\phi}) \\ &= \inf_{\tilde{v}_1} \tilde{v}^t \tilde{A}^{(s)} \tilde{v}, \qquad \tilde{v} = \begin{bmatrix} c\tilde{v}_1 \\ v_2 \end{bmatrix} \begin{array}{l} \} N^{(s-1)} \setminus N_2^{(s)} \\ \} N_2^{(s)} \end{array} \\ &= v_2^t \tilde{S}^{(s)} v_2. \end{aligned}$$

The last inequality is the desired result. □

Lemma 2.3. *The following bounds for the spectrum of* $A^{(s)^{-1}}B^{(s)}, s = k+1,\dots,k+k_0$, *are valid*

$$\lambda\left[A^{(s)^{-1}}B^{(s)}\right] \in \left[1, 1 + [\hat{\lambda}_k - 1]\eta(s-k) + \sum_{r=1}^{s-k}\eta(r)\right].$$

Proof. By the definition (2.2) we have

$$v^t(B^{(s)} - A^{(s)})v = v_2^t(\tilde{C}^{(s)} - S^{(s)})v_2.$$

Choose now $\tilde{v} = \begin{bmatrix} c\tilde{v}_1 \\ v_2 \end{bmatrix} \begin{matrix} \} N^{(s-1)} \setminus N_2^{(s)} \\ \} N_2^{(s)} \end{matrix}$. such that

$$\tilde{B}^{(s)}\tilde{v} = \begin{bmatrix} c0 \\ \tilde{C}^{(s)}v_2 \end{bmatrix},$$

that is

$$\tilde{A}_{11}^{(s)}\tilde{v}_1 + \tilde{A}_{12}^{(s)}v_2 = 0,$$

which implies

$$\tilde{A}^{(s)}\tilde{v} = \begin{bmatrix} c0 \\ \tilde{S}^{(s)}v_2 \end{bmatrix}.$$

Hence

$$v^t(B^{(s)} - A^{(s)})v = \tilde{v}^t(\tilde{B}^{(s)} - \tilde{A}^{(s)})\tilde{v} + v_2^t(\tilde{S}^{(s)} - S^{(s)})v_2 \geq \tilde{v}^t(\tilde{B}^{(s)} - \tilde{A}^{(s)})\tilde{v}, \tag{2.5}$$

by Lemma 2.2 and by induction on $s \geq k$ (note that $\lambda\left[\hat{B}^{(k)^{-1}}A^{(k)}\right] \leq 1$, (2.1)) the left bound 1 of $\lambda\left[A^{(s)^{-1}}B^{(s)}\right]$ is proved.

Using (2.5) recursively we have

$$\begin{aligned} v^t(B^{(s)} - A^{(s)})v &= \tilde{v}^t(\tilde{B}^{(s)} - \tilde{A}^{(s)})\tilde{v} + v_2^t(\tilde{S}^{(s)} - S^{(s)})v_2 \\ &\leq \tilde{v}^t(\tilde{B}^{(s)} - \tilde{A}^{(s)})\tilde{v} + \tilde{v}^t\tilde{A}^{(s)}\tilde{v} \\ &\leq \sum_{r=k+1}^{s} \tilde{v}^{(r)^t}\tilde{A}^{(r)}\tilde{v}^{(r)} + \tilde{v}^{(k+1)^t}(\tilde{B}^{(k+1)} - \tilde{A}^{(k+1)})\tilde{v}^{(k+1)} \end{aligned} \tag{2.6}$$

where $\tilde{v}^{(r)}$ are defined as follows:

(a) $\tilde{v}^{(s)} = v$,

(b) for $r = s-1$ downto k first partition $v^{(r+1)}$ into a domain decomposition manner

$$v^{(r+1)} = \begin{bmatrix} cv_1^{(r+1)} \\ v_2^{(r+1)} \end{bmatrix} \begin{matrix} \} N_1^{(r+1)} \\ \} N_2^{(r+1)} \end{matrix}$$

then extend $v_2^{(r+1)}$ on $N^{(r)} \setminus N_2^{(r+1)}$ to $\tilde{v}^{(r+1)}$ such that

$$\tilde{A}^{(r+1)}\tilde{v}^{(r+1)} = \begin{bmatrix} c0 \\ \tilde{S}^{(r+1)}v_2^{(r+1)} \end{bmatrix}$$

(that is $\tilde{A}_{11}^{(r+1)}\tilde{v}_1^{(r+1)} + \tilde{A}_{12}^{(r+1)}v_2^{(r+1)} = 0$) and finally set $v^{(r)} = \tilde{v}^{(r+1)}$.

By construction we have

$$\tilde{v}^{(r+1)^t}\tilde{A}^{(r+1)}\tilde{v}^{(r+1)} = v_2^{(r+1)^t}\tilde{S}^{(r+1)}v_2^{(r+1)} = \inf_{\mathbf{w}_1 : \mathbf{w} = \begin{bmatrix} cw_1 \\ v_2^{(r+1)} \end{bmatrix} \begin{matrix} \} N^{(r)} \setminus N_2^{(r+1)} \\ \} N_2^{(r+1)} \end{matrix}} w^t A^{(r)} w$$

Using Lemma 2.1 we get, $r < s$,

$$\tilde{v}^{(r)^t}\tilde{A}^{(r)}\tilde{v}^{(r)} \le \eta(s - r + 1)v^{(s)^t}A^{(s)}v^{(s)}$$

and substituting in (2.6) we obtain, see (2.1)

$$v^t(B^{(s)} - A^{(s)})v \le \sum_{r=k+1}^{s} \eta(s - r + 1)v^{(r)^t}A^{(r)}v^{(r)} + (\hat{\lambda}_k - 1)\eta(s - k)v^t A^{(s)}v,$$

which implies the desired result,

$$\lambda_s \le 1 + [\hat{\lambda}_k - 1]\eta(s - k) + \sum_{r=1}^{s-k} \eta(r).$$

□

Now we define the algebraic multilevel $BEPS-$ type preconditioner which combines the two versions proposed in Ewing, Lazarov, and Vassilevski [7]. See also a corresponding definition in the hierarchical multilevel case, Vassilevski [17].

Let $p_\nu = p_\nu(t)$ be a given polynomial of degree $\nu \ge 1$ such that

$$\text{(2.7)} \qquad p_\nu(0) = 1, \qquad ,0 \le p_\nu(t) < 1, \qquad ,t \in (0, 1].$$

For a given integer $k_0 \ge 1$ we define,

(1) $B^{(1)} = A^{(1)}$,

(2) for $k = 2, 3, \ldots, l$ set

$$B^{(k)} = \begin{bmatrix} A_{11}^{(k)} & 0 \\ A_{21}^{(k)} & C^{(k)} \end{bmatrix} \begin{bmatrix} I & A_{11}^{(k)^{-1}}A_{12}^{(k)} \\ 0 & I \end{bmatrix}$$

where

$$C^{(k)} = \tilde{C}^{(k)},$$

for $k \neq sk_0 + 1$ and $\tilde{C}^{(k)}$ is the Schur complement of $\tilde{B}^{(k)}$, that is the Schur complement of $B^{(k-1)}$ partitioned into the block form (2.3);

and for $k = sk_0 + 1,\ s = 1, 2, \ldots, l/k_0 - 1$ we first correct $\tilde{B}^{(k)}$ to $\hat{B}^{(k-1)}$ as follows

$$\hat{B}^{(k-1)^{-1}} = \left[I - p_\nu\left(\tilde{B}^{(k)^{-1}}\tilde{A}^{(k)}\right)\right]\tilde{A}^{(k)^{-1}} \tag{2.8}$$

and then $C^{(k)}$ is defined as the Schur complement of $\hat{B}^{(k)}$, partitioned into the two by two block form with respect to the partitioning of the nodes $N^{(k-1)} = (N^{(k-1)} \setminus N_2^{(k)}) \bigcup N_2^{(k)}$

Remark 2.4 Definition (1), (2) for $k_0 = 1$ reduces to the multilevel $BEPS-$ preconditioner studied in Ewing, Lazarov and Vassilevski [7]

Note that by Lemma 2.3 the condition number of $B^{(s)^{-1}}A^{(s)}$, for $s = k + 1, k+2, \ldots, k+k_0$ can increase unacceptably. That is why we make a correction in order to stabilize the possible growth of *cond* $\left(B^{(s)^{-1}}A^{(s)}\right)$.

We choose the following polynomial

$$p_\nu(t) = \frac{1 + T_\nu\left(\frac{1+\alpha-2t}{1-\alpha}\right)}{1 + T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)}, \qquad t \in [\alpha, 1] \tag{2.9}$$

T_ν is the Chebyshev polynomial of degree ν, that is $T_0(x) = 1,\ T_1(x) = x,\ T_{s+1}(x) = 2x.T_s(x) - T_{s-1}(x),\ s = 1, 2, \ldots$ and $\alpha \in (0, 1)$ is a properly chosen parameter.

The following estimate is valid for any polynomial $p_\nu(t)$ satisfying (2.7):

$$\hat{\lambda}_{(k-1)} = \sup\left\{\frac{1}{1 - p_\nu(t)}, \qquad t \in [1/\lambda_{k-1}, 1]\right\}.$$

Let for a given $k = sk_0$ the parameter α satisfy

$$0 < \alpha \leq 1/\lambda_{k-1}, \qquad \lambda_{k-1} = \lambda_{\max}\left[\tilde{A}^{(k)^{-1}}\tilde{B}^{(k)}\right],$$

Then by correction (2.8) we get the following improvement of $\hat{\lambda}_{k-1} =$

$\lambda_{\max}\left[\tilde{A}^{(k)^{-1}}\hat{B}^{(k-1)}\right]$,

$$\hat{\lambda}_{k-1} \leq \sup\left\{\frac{1}{1-p_\nu(t)}, \qquad t \in [\alpha, 1]\right\}, \qquad (\alpha \leq 1/\lambda_{k-1})$$

$$= \frac{1}{1-\frac{2}{1+T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)}} = \frac{T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)+1}{T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)-1}$$

$$= \frac{(1+q^\nu)^2}{(1-q^\nu)^2}, \qquad q = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}$$

$$= \left[\frac{(1+\sqrt{\alpha})^\nu + (1-\sqrt{\alpha})^\nu}{2\sqrt{\alpha}\sum\limits_{s=1}^{\nu}(1+\sqrt{\alpha})^{\nu-s}(1-\sqrt{\alpha})^{s-1}}\right]^2, \tag{2.10}$$

Since $q^\nu \longrightarrow 0$, when $\nu \longrightarrow \infty$, it is seen that $\hat{\lambda}_{k-1}$ can be done arbitrary close to 1 if ν is sufficiently large.

Now we prove the main result on the relative condition number of $B^{(k)}$ with respect to $A^{(k)}$.

Theorem 2.5. *Let $\nu > \sqrt{\eta(k_0)}$, see (2.4) and $\alpha \in (0,1)$ satisfy the inequality*

$$\frac{(1-\alpha)^\nu}{\alpha\left[\sum_{s=1}^{\nu}(1+\sqrt{\alpha})^{\nu-s}(1-\sqrt{\alpha})^{s-1}\right]^2} \leq \left[1/\alpha - \sum_{s=0}^{k_0}\eta(s)\right]/\eta(k_0) \tag{2.11}$$

Then the algebraic multilevel preconditioner $B^{(k)}$ of hybrid $V-$cycle type (1), (2), with the properly scaled and shifted Chebyshev polynomials, (2.9), are spectrally equivalent to the corresponding stiffness matrices $A^{(k)}$ and the following estimate is valid

$$\lambda\left[A^{(k)^{-1}}B^{(k)}\right] \in [1, 1/\alpha].$$

Proof. The result is essentially given by Lemma 2.3. Let for some $s \geq 1$ and $k = (s-1)k_0 + 1$

$$\lambda_{k-1} = \lambda_{\max}\left[\tilde{A}^{(k)^{-1}}\tilde{B}^{(k)}\right] \leq 1/\alpha.$$

Consider

$$\hat{\lambda}_{k-1} = \lambda_{\max}\left[\tilde{A}^{(k)^{-1}}\hat{B}^{(k-1)}\right], \text{ see (2.8)}.$$

Then we have

$$\hat{\lambda}_{k-1} \leq \left[\frac{(1+\sqrt{\alpha})^{\nu}+(1-\sqrt{\alpha})^{\nu}}{(1+\sqrt{\alpha})^{\nu}-(1-\sqrt{\alpha})^{\nu}}\right]^2 .$$

This estimate and Lemma 2.3 imply

$$\begin{aligned}\lambda_{sk_0} &\leq 1+[\hat{\lambda}_{k-1}-1]\eta(k_0)+\sum_{r=1}^{k_0}\eta(r)\\ &\leq \eta(k_0)\frac{(1-\alpha)^{\nu}}{\alpha\left[\sum_{\sigma=1}^{\nu}(1+\sqrt{\alpha})^{\nu-\sigma}(1-\sqrt{\alpha})^{\sigma-1}\right]^2}+\sum_{r=1}^{k_0}\eta(r)+1, \quad \text{see (2.10)},\\ &\leq \frac{1}{\alpha}, \qquad \text{by (2.11)},\end{aligned}$$

which completes the proof. □

Remark 2.6 Note that (2.11) for $\nu > \sqrt{\eta(k_0)}$ can be ensured if $\alpha \in (0,1)$ is sufficiently small since for $\alpha \longrightarrow 0$ the left-hand side of (2.11) behaves like $1/\nu^2$ whereas the right-hand side tends to $1/\eta(k_0)$.

3. Regular Refinement Case

In this section we consider a case of local refinement which we call a regular refinement. Starting with a subdomain $\Omega^{(2)}$ of $\Omega^{(1)} = \Omega$ which consists of the coarse-grid elements that are to be refined, we generate a sequence of subdomains $\{\Omega^{(k)}\}$, $\Omega^{(k-1)} \supset \Omega^{(k)}$. That is, each $\Omega^{(k)}$ contains those elements from τ_{k-1} that are refined to define the uniformly refined elements of τ_k.

Our main assumption on the subdomain sequence $\{\Omega^{(k)}\}_{k=1}^{l}$ is the following:

(A) extension property:

For any $v \in V_{k-1}(\Omega^{(k-1)} \setminus \Omega^{(k)})$, i.e. a function from V_{k-1} restricted to $\Omega^{(k-1)} \setminus \Omega^{(k)}$, there exists an extension $Pv \in V_{k-1}(\Omega^{(k-1)})$ such that

$$Pv = v \qquad \text{in } \Omega^{(k-1)} \setminus \Omega^{(k)},$$

and

$$a_{\Omega^{(k)}}(Pv, Pv) \leq C\, a_{\Omega^{(k-1)}\setminus\Omega^{(k)}}(v, v)$$

with a constant C independent of k and v. Here by $a_D(.,.)$ for any subdomain D of Ω we mean the restriction of $a(.,.)$ to D.

This means that the energy norm of the extension function is about the same from both sides ot $\Gamma^{(k)}$, the interface between $\Omega^{(k)}$ and $\Omega^{(k-1)}\backslash\Omega^{(k)}$. This is for example the case when $\Omega^{(k-1)}\backslash\Omega^{(k)}$ is not very thin part of $\Omega^{(k-1)}$. For more details about extensions of finite element functions we refer to Widlund [19], Nepomnyaschikh [14], and Nepomnyaschikh and Widlund [15].

Then we can prove the following results which is a refinement of Lemma 2.3.

Theorem 3.1. *Consider the multilevel BEPS– preconditioner defined by (2.1), (2.2) starting from $k = 1$ with $\hat{B}^{(1)} = A^{(1)}$. Then the following uniform bounds for the spectrum of $B^{(s)^{-1}}A^{(s)}$ are valid*

$$\lambda\left[B^{(s)^{-1}}A^{(s)}\right] \in [1, 1 + C],$$

where C is the constant from assumption **(A)**.

Proof. Recall the proof of Lemma 2.3. We have

$$v^t\left(B^{(s)} - A^{(s)}\right)v = \sum_{r=2}^{s} v_2^{(r)^t}\left(\tilde{S}^{(r)} - S^{(r)}\right)v_2^{(r)}, \tag{3.1}$$

since $\tilde{B}^{(2)} = \tilde{A}^{(2)} = A^{(1)} = \tilde{A}^{(2)}$. Let v be the coefficient vector of some $v \in V_s$. We note first that by construction

$$v_2^{(r)^t}\tilde{S}^{(r)}v_2^{(r)} = \inf_{w_1 : w = \begin{bmatrix} cw_1 \\ v_2^{(r)} \end{bmatrix} \begin{matrix} \}\,\Omega^{(r)} \\ \}\,\Omega\setminus\Omega^{(r)} \end{matrix}} w^t\tilde{A}^{(r)}w \le a(\tilde{w}, \tilde{w}), \tag{3.2}$$

where $\tilde{w} \in V_{r-1}$ is the extension of $v|_{\Omega\setminus\Omega^{(r)}}$ from assumption **(A)**. Next we use the following inequalities,

$$\begin{aligned} v_2^{(r)^t}S^{(r)}v_2^{(r)} &= \inf\left\{a(w, w) :\ w \in V_r,\ w = v \text{ in } \Omega\setminus\Omega^{(r)}\right\} \\ &\ge a_{\Omega\setminus\Omega^{(r)}}(v, v) = a_{\Omega\setminus\Omega^{(r)}}(\tilde{w}, \tilde{w}), \end{aligned} \tag{3.3}$$

since $\tilde{w} = v$ in $\Omega\setminus\Omega^{(r)}$.

Hence by (3.2) and (3.3) we obtain

$$\begin{aligned} v_2^{(r)^t}\left(\tilde{S}^{(r)} - S^{(r)}\right)v_2^{(r)} &\le a(\tilde{w}, \tilde{w}) - a_{\Omega\setminus\Omega^{(r)}}(\tilde{w}, \tilde{w}) \\ &= a_{\Omega^{(r)}}(\tilde{w}, \tilde{w}) \le C\, a_{\Omega^{(r-1)}\setminus\Omega^{(r)}}(\tilde{w}, \tilde{w}), \end{aligned}$$

by assumption **(A)**. Then the desired result follows from (3.1) and the last inequality. The lower bound 1 for the spectrum of $B^{(s)^{-1}}A^{(s)}$ was shown in Lemma 2.3. □

R e m a r k 3.2. We note that the constant C in assumption **(A)** may depend in general on possible jumps in the coefficients of the bilinear form $a(.,.)$, whereas the constants in the Theorem 2.5 and the preceding results are bounded with respect to possible jumps of the coefficients in $a(.,.)$ as long as they are continuous in the interior of the elements from the initial triangulation $\mathcal{T}_1$.

4. Computational Aspects

We consider a sequence of nested grids with nodes $N^{(k)}$, $k = 1, 2, \ldots, l$, obtained by succesive steps of local refinement of the initial (coarsest) triangulation. Let $N^{(k)} = N_1^{(k)} \bigcup N_2^{(k)}$ be the corresponding domain decomposition partitioning of the nodes on level k where the set of nodes $N_1^{(k)}$ corresponds to the triangulation in the subdomain $\Omega^{(k)}$ where we have regular, that is k steps of uniform refinement. We also have, by construction, $N_2^{(k)} \subset N^{(k-1)}$.

Let the number of nodes in $N_1^{(k)}$ and $N_2^{(k)}$ be $n_1^{(k)}$, $n_2^{(k)}$ respectively, and in $N^{(k)}$, $n^{(k)} = n_1^{(k)} + n_2^{(k)}$.

We assume that the problems on the regular set of nodes $N_1^{(k)}$ with block matrices $A_{11}^{(k)}$ from the corresponding domain decomposition partitioning of $A^{(k)}$, (1.1) can be solved efficiently with $O(n_1^{(k)})$ or more precisely $O(n_1^{(k)} \log 1/\varepsilon)$ arithmetic operations. For example, this can be an ordinary multigrid solver, e.g. H a c k b u s c h [8], since in the subdomain $\Omega^{(k)}$ we use regular, that is uniform grid refinement. Another possible choice is any fast elliptic solver, if $\Omega^{(k)}$ is a rectangle and the coefficients of the differential operator are constant or allow separation of variables. Then one can use, for example, the generalized marching algorithm, cf. B a n k and R o s e [2], B a n k [3].

We now consider the asymptotic work estimate of one multilevel cycle, which can be used as a preconditioning step in the preconditioned conjugate gradient method. Let $\omega(k)$, $k = sk_0$ be the amount of arithmetic work on level k. Then we have, per iterative step, see (2.7) in definition (1), (2).

$$\omega(sk_0) = C \sum_{r=0}^{k_0-1} n_1^{(sk_0-r)} + \nu\omega((s-1)k_0) \tag{4.1}$$

The last term corresponds to the polynomial correction (2.8) on the level $(s-1)k_0$. The constant C depends, in general, on the accuracy within the subdomain problems are solved. Note that the problems corresponding to $A_{11}^{(k)}$ are defined on uniform grids.

In order to have the number of arithmetic operations proportional to the number of the nodes in $N^{(l)}$ on the final triangulation τ_l we need an estimate of the form

$$\omega(l) \le C\, n^{(l)}.$$

Let us have for some $\beta \in (0,1]$

$$n_2^{(k+1)} = n_2^{(k)} + (1-\beta)n_1^{(k)},$$

that is $(1-\beta)$-th part of the regular elements on the level k are not further refined.

Assume now that we have generated $n_1^{(k+1)} = \mu(\beta n_1^{(k)})$, $\mu > 1$, regular nodes on the $k+1$-th refined level. We also assume that

$$\mu\beta > 1,$$

but not very large. In fact, for $2-D$ domain Ω, $\mu\beta$ can be chosen arbitrary close to 1. For $3-D$ domains we have to satisfy the inequality $1 \ge \beta > C\mu^{-5/6}$ as we shall see below,

Then we obtain, by (4.1),

$$\begin{aligned}\omega(sk_0) &= C\sum_{r=0}^{k_0-1} n_1^{(sk_0)}/(\mu\beta)^r + \nu\omega((s-1)k_0)\\ &= C\frac{(1/\mu\beta)^{k_0}-1}{1/\mu\beta-1}n_1^{(sk_0)} + \nu\omega((s-1)k_0), \qquad s = 2,3,\ldots,l/k_0.\end{aligned}$$

Hence

$$\begin{aligned}\omega(l) &= \Delta_{k_0}\left[n_1^{(l)} + \nu n_1^{(l-k_0)} + \cdots + \nu^{[l/k_0]-1}n_1^{(k_0)}\right] + \nu^{[l/k_0]-1}\omega(k_0),\\ &= \Delta_{k_0}n_1^{(l)}\sum_{r=1}^{[l/k_0]}\left(\frac{\nu}{(\mu\beta)^{k_0}}\right)^{r-1} + \nu^{[l/k_0]-1}\omega(k_0). \qquad (4.2)\end{aligned}$$

Here

$$\Delta_{k_0} = C\,\frac{1 - 1/(\mu\beta)^{k_0}}{1 - 1/\mu\beta}.$$

Then

$$\begin{aligned}
n^{(k+1)} &= n_1^{(k+1)} + n_2^{(k+1)} \\
&= \mu\beta n_1^{(k)} + (1-\beta)n_1^{(k)} + n_2^{(k)} \\
&= (\mu - 1)\beta n_1^{(k)} + n^{(k)} \\
&= (\mu - 1)\beta \sum_{r=k_0}^{k} n_1^{(r)} + n^{(k_0)} \\
&= (\mu - 1)\beta n_1^{(k_0)} \sum_{r=0}^{k-k_0} (\mu\beta)^r + n^{(k_0)} \\
&= \frac{\mu - 1}{\mu\beta - 1}\beta[(\mu\beta)^{k-k_0+1} - 1]n_1^{(k_0)} + n^{(k_0)} \qquad (4.3)
\end{aligned}$$

Assume now that

$$\nu < (\mu\beta)^{(k_0)}.$$

Then by (4.2)

$$\begin{aligned}
\omega(l) &= \Delta_{k_0}\frac{1 - (\nu/(\mu\beta)^{(k_0)})^{l/k_0}}{1 - \nu/(\mu\beta)^{(k_0)}} n_1^{(l)} + \nu^{l/k_0 - 1}\omega(k_0) \\
&\le C\, n_1^{(l)} \nu^{l/k_0 - 1}\omega(k_0) \qquad (4.4)
\end{aligned}$$

As, by (4.3),

$$\frac{n^{(l)}}{n^{(k_0)}} \max\left[\frac{\mu\beta - 1}{\mu\beta - \beta}\left(\frac{n^{(k_0)}}{n_1^{(k_0)}}\right), 1\right] \ge (\mu\beta)^{l-k_0} > \nu^{l/k_0 - 1},$$

we get

$$\begin{aligned}
\omega(l) \le &\left[C\,\frac{1 - 1/\tilde{\mu}^{k_0}}{1 - 1/\tilde{\mu}}\;\frac{1 - (\nu/\tilde{\mu}^{(k_0)})^{l/k_0}}{1 - \nu/\tilde{\mu}^{(k_0)}}\;\frac{n_1^{(l)}}{n^{(l)}}\right] n^{(l)} \\
&+ n^{(l)}\frac{\omega(k_0)}{n^{(k_0)}} \max\left[\frac{\tilde{\mu} - 1}{\tilde{\mu} - \beta}\left(\frac{n^{(k_0)}}{n_1^{(k_0)}}\right), 1\right], \qquad \tilde{\mu} = \mu\beta, \\
\le &\left[C_1 + C_2\frac{\omega(k_0)}{n^{(k_0)}}\right] n^{(l)},
\end{aligned}$$

where

$$C_1 = \frac{C}{(1-\nu/\tilde{\mu}^{k_0})(1-1/\tilde{\mu})};$$

$$C_2 = \max\left[\frac{\tilde{\mu}-1}{\tilde{\mu}-\beta}\left(\frac{n^{(k_0)}}{n_1^{(k_0)}}\right), 1\right] \leq \frac{n^{(k_0)}}{n_1^{(k_0)}}, \qquad \tilde{\mu} = \mu\beta.$$

That is $\omega(l)$ is proportional to the number of the nodes $N^{(l)}$ of the finest level l.

Recall that by Theorem 2.5 we also have the restriction $\nu > \sqrt{\eta}(k_0) = C(\log\mu)^{1/2}k_0^{1/2}$ for $2-D$ domain or $\nu > C(\mu^{1/6})^{k_0}$ for $3-D$ domain Ω, by Lemma 2.1.

Thus we have to satisfy the inequalities

$$(\mu\beta)^{k_0} > \nu > \sqrt{\eta}(k_0) = \begin{cases} C\sqrt{k_0}\log^{1/2}\mu, & \text{2-D domain } \Omega \\ C(\mu^{1/6})^{k_0}, & \text{3-D domain } \Omega \end{cases}$$

For $2-D$ domain Ω the above inequality is readily ensured if k_0 is sufficiently large since we have assumed $\mu\beta > 1$, but arbitrary close to 1. For $3-D$ domain Ω it can be ensured if $1 \geq \beta > C\mu^{-5/6}$. The eigenvalues λ_k can be estimated by computing $\lambda_{\min}\left[B^{(k)^{-1}}A^{(k)}\right] = 1/\lambda_k$ using the Lanczos method. The action of $B^{(k)^{-1}}$ on certain vectors can be realized by the $AMLI-$ algorithm from Vassilevski [17]. This information can be used in order to construct the optimal polynomial for the next level $k+1$, etc. Hence we will have $p_\nu(t) = p_{\nu_k}(t)$, $k = 2, 3, \ldots, l$, and most of the ν_k will be 1 guaranteed by Theorem 2.5. For details on this adaptive implementation of the methods, cf. Vassilevski [17]. Such an implementation will be especially useful when we have very irregular refinement.

The complexity analysis for the pure $V-$cycle method ($\nu = 1$) is much simpler (cf. e.g. Ewing, Lazarov, and Vassilevski [7]). In this case there is no restriction on β.

5. Numerical Experiments

In this section we present numerical results corresponding to the following bilinear form

$$a(u,\phi) = \int_\Omega k(x,y) \nabla u . \nabla \phi \, dx\, dy, \tag{5.1}$$

where Ω is the unit square and the finite element spaces consist of piecewise linear functions on isosceles triangulations τ_k, so that at each level $k = 2, 3, \ldots, l = 7$ the locally refined subregion $\Omega^{(k)}$ is the north-east quarter of $\Omega^{(k-1)}$.

We refine the coarse-grid triangles in the subregions into a number of congruent ones, so that $h_k = h_{k-1}/n_0$, where $n_0 = 2, 4, 8$ and $k = 2, 3, \ldots, 7$.

The corresponding boundary conditions are of Neumann type on the boundaries $x = 1$ and $y = 1$. On the boundaries $x = 0$ and $y = 0$ Dirichlet boundary conditions are imposed.

We test the sequences of stiffness matrices $\{A^{(k)}\}_{k=1}^{l}$, corresponding to the bilinear form from (5.1) with the following coefficient $k = k(x, y)$:

Problem 1: (a smooth coefficient)

$$k(x, y) = 1 + x^2 + y^2$$

Problem 2: (a discontinuous coefficient)

$$k(x,y) = \begin{cases} 10^{-6}, & x < 1/2 \text{ or } y < 1/2 \\ 10^{-3}, & 1/2 < x < 3/4,\ y > 3/4 \text{ or } 1/2 < x < 1,\ 1/2 < y < 3/4 \\ 1, & x > 3/4 \text{ and } y > 3/4 \end{cases}$$

We consider the pure $V-$cycle multilevel $BEPS-$ preconditioner, i.e. at each level k, $k = 2, 3, \ldots, 7$ the polynomials p_ν from (2.9) are linear, i.e. $p_\nu(t) = 1 - t$, $\nu = 1$.

We estimate the spectrum of $B^{(k)^{-1}} A^{(k)}$ using the Lanczos method where the action of $B^{(k)^{-1}}$ on certain vectors is computed by the algorithm $AMLI$ from [17]. The subregion problems are solved by a preconditioned conjugate gradient method with a high accuracy. The stopping criterion for these problems we used is

$$r^t r < \varepsilon, \qquad \varepsilon = 10^{-24}$$

where r is the residual vector.

The results are listed in Tables 1, 2 corresponding to the Problems 1, 2 respectively. They illustrate that the pure $V-$cycle multilevel $BEPS-$ preconditioner is of optimal order. Since we solve the problems in the subregions exactly we have that $\lambda_{\min}\left(A^{(k)^{-1}} B^{(k)}\right)$ is always 1. Our experiments are done on uniform grids with meshsize on the initial level $h_1 = 1/N$, $N = 4$ and the number of levels equals to 7.

For Problem 2 we see from Table 2 that the eigenvalues are independent of the jump of the diffusion coefficient. In this case the pure $V-$cycle multilevel

$BEPS-$ preconditioner turned out to be also optimal, although the assumption **(A)** from Section sec3 holds with a large constant $C = O(1/\varepsilon)$, $\varepsilon = 10^{-3}$.

In the second test we varied the initial coarse meshsize $h_1 = 1/N$, $N = 8, 16, 32$ whereas $\frac{h_{k-1}}{h_k} = 2$ for both test problems, 1 and 2. The corresponding largest eigenvalues $\lambda_k = \lambda_{\max}\left(A^{(k)^{-1}} B^{(k)}\right)$ are shown in Tables 3-4. Here we also observe that the pure $V-$cycle multilevel $BEPS-$ preconditioner is of optimal order with respect to the coarse-grid size.

n_0	k	2	3	4	5	6	7
2	λ_k	1.2698	1.2716	1.2601	1.2712	1.2709	1.2710
4	λ_k	1.4077	1.4079	1.4080	1.4082	1.4083	1.4084
8	λ_k	1.4602	1.4603	1.4605	1.4606	1.4608	1.4609

Table 1. Largest eigenvalue λ_k of $A^{(k)^{-1}} B^{(k)}$ for Problem 1.

n_0	k	2	3	4	5	6	7
2	λ_k	1.7394	1.7396	1.7398	1.7399	1.7401	1.7403
4	λ_k	2.3217	2.3219	2.3222	2.3224	2.3226	2.3229
8	λ_k	2.6076	2.6079	2.6082	2.6084	2.6087	2.6089

Table 2. Largest eigenvalue λ_k of $A^{(k)^{-1}} B^{(k)}$ for Problem 2.

N	k	2	3	4	5	6	7
8	λ_k	1.2579	1.2529	1.2521	1.2515	1.2510	1.2549
16	λ_k	1.2385	1.2370	1.2361	1.2354	1.2360	1.2358
32	λ_k	1.2368	1.2359	1.2353	1.2348	1.2345	1.2342

Table 3. Largest eigenvalue λ_k of $A^{(k)^{-1}} B^{(k)}$ for Problem 1., $n_0 = 2$.

N	k	2	3	4	5	6	7
8	λ_k	1.7394	1.7396	1.7398	1.7399	1.7401	1.7403
16	λ_k	1.6446	1.6447	1.6449	1.6451	1.6452	1.6454
32	λ_k	1.6436	1.6438	1.6440	1.6443	1.6445	1.6446

Table 4. Largest eigenvalue λ_k of $A^{(k)^{-1}} B^{(k)}$ for Problem 2., $n_0 = 2$.

6. Acknowledgment

The authors are grateful to Prof. Richard Ewing for the hospitality and support given through the Institute for Scientific Computations, University of Wyoming, Laramie, Wyoming, U.S.A.

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Received 15.10.1992