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# The Free Products of Weak Hamiltonian $l$-Groups

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In this paper we prove that the set of all weak Hamiltonian $l$-groups is a subproduct quasi-torsion class. The construction for free product of weak Hamiltonian $l$-groups is given.
For the definitions and the standard terminologies concerning $l$-groups, the reader is referred to [1,2,3]. We use additive group notation. An $l$-homomorphism is called complete if it preserves all (not necessarily finite) meets and joints. Let $\mathcal{L}$ be the variety of all $l$-groups. Let $G$ be an $l$-group and $K(G)$ the set of all closed convex $l$-subgroups of $G$.

## 1. Weak Hamiltonian $l$-groups.

An $l$-group $G$ is said to be weak Hamiltonian if each closed convex $l$-subgroup of $G$ is normal. It is clear that each abelian $l$-group and each Hamiltonian $l$-group are weak Hamiltonian. Let $WH$ be the set of all weak Hamiltonian $l$-groups.

**Proposition 1.1.** *$WH$ is closed under taking $l$-subgroups.*

Proof. Suppose that an $l$-group $G'$ is weak Hamiltonian and $G$ is an $l$-subgroup of $G$. For each $L \in K(G)$ let

$$\tilde{L} = \{g' \in G' | 0 \leq |g'| \leq g \text{ for some } g \in L\},$$

$$\bar{L} = \{g' \in G' | |g'| \text{ is the join in } G' \text{ of some subset of } L\}$$

Then $L' = \bar{\tilde{L}} \in K(G')$ and $L'$ is normal in $G'$. By Proposition 1.2 (i) of [4] we have $L' \cap G = L$. Let $g \in G$. Then $g + L' - g = L'$, that is for any $l_1 \in L'$

there exists $l_2 \in L'$ such that $g + l_1 - g = l_2$. If $l_1 \in L$, then $g, l_1 \in G$ implies $l_2 \in G$. Hence $l_2 \in L' \cap G = L$. This means $g + L - g = L$ and $L$ is normal in $G$. Therefore $G$ is weak Hamiltonian. ∎

**Proposition 1.2.** *WH is closed under taking direct products.*

Proof. Suppose that $\{G_\lambda | \lambda \in \Lambda\} \subseteq WH$. Let $L$ be a closed convex $l$-subgroup of $\prod_{\lambda\in\Lambda} G_\lambda$. For each $\lambda \in \Lambda$, let $\bar{G}_\lambda = \{g \in \prod_{\lambda\in\Lambda} G_\lambda | \lambda' \neq \lambda \Rightarrow g_{\lambda'} = 0\}$. Then for each $\lambda \in \Lambda$, $\bar{G}_\lambda \simeq G_\lambda$ and $L \cap \bar{G}_\lambda$ is a closed convex $l$-subgroup of $\bar{G}_\lambda$. Hence $L \cap \bar{G}_\lambda$ is normal in $\bar{G}_\lambda$. Let $g = (..., g_\lambda, ...) \in \prod_{\lambda\in\Lambda} G_\lambda$ and $0 < l = (..., l_\lambda, ...) \in L$. Then for each $\lambda \in \Lambda$ there exists $l' \in G_\lambda$ such that $g_\lambda + l_\lambda - g_\lambda = l'_\lambda$. Let $l' = (..., l'_\lambda, ...)$, then $g + l - g = l'$. Since $L$ is closed and $l > 0$, $l' = \bigvee_{\lambda\in\Lambda} \bar{l'_\lambda}$ with $\bar{l'_\lambda} = (0, ..., 0, l'_\lambda, 0, ..., 0) \in L \cap \bar{G}_\lambda$, so $l' \in L$. Therefore $L$ is normal in $\prod_{\lambda\in\Lambda} G_\lambda$ and $\prod_{\lambda\in\Lambda} G_\lambda$ is weak Hamiltonian. ∎

**Proposition 1.3.** *WH is closed under taking complete l-homomorphic images.*

Proof. Suppose that $\varphi$ is a complete $l$-homomorphism from an $l$-group $G$ onto an $l$-group $G'$ and $G$ is weak Hamiltonian. Let $L' \in K(G')$ and $L = \varphi^{-1}(L')$. It is easy to see that $L \in K(G)$. Let $g' \in G'$ and $g \in G$ such that $g' = \varphi(g)$. For any $l' \in L'$ take $l \in L$ such that $l' = \varphi(l)$. Since $G \in WH$ and $L \in K(G)$, there exist $l_1 \in L$ such that $g + l - g = l_1$. So

$$\begin{aligned} g' + l' - g' &= \varphi(g) + \varphi(l) - \varphi(g) = \varphi(g + l - g) \\ &= \varphi(l_1) \in L'. \end{aligned}$$

Hence $L'$ is normal in G. ∎

In [5] P. Gonrad proved that the set Ham of all Hamiltonian $l$-groups is a torsion class. Similarly to Proposition 1.4 of [5] we can show the following proposition. But we omit the proof.

**Proposition 1.4.** *WH is closed under taking joins of convex l-subgroups.*

A family $\mathcal{U}$ of $l$-groups is called a sub-product quasi-torsion class, if it closed under taking (1) $l$-subgroups, (2) direct products, (3) joins of convex $l$-subgroups and (4) complete $l$-homomorphic images. All our classes of $l$-groups are always assumed to contain along with a given $l$-group all its $l$-isomorphic copies. It follows from Proposition 1.1, 1.2, 1.3 and 1.4 that

**Theorem 1.5.** *$WH$ is a sub-product quasi-torsion class.*

## 2. WH-free products

Let $\mathcal{U}$ be a class of $l$-groups and $\{G_\lambda | \lambda \in \Lambda\}$ be a family of $l$-groups in $\mathcal{U}$. The $\mathcal{U}$-free product of $G_\lambda$ is an $l$-group $G$, denoted by ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$, together with a family of injective $l$-homomorphisms $\alpha_\lambda : G_\lambda \to G$ (called coprojections) such that

(1) $\bigcup_{\lambda\in\Lambda} \alpha_\lambda(G_\lambda)$ generates ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ as an $l$-group;

(2) if $K \in \mathcal{U}$ and $\{\beta_\lambda : G_\lambda \to K | \lambda \in \Lambda\}$ is a family of $l$-homomorphisms, then there exists a unique $l$-homomorphism $\gamma : G \to K$ satisfying $\beta_\lambda = \gamma\alpha_\lambda$ for all $\lambda \in \Lambda$.

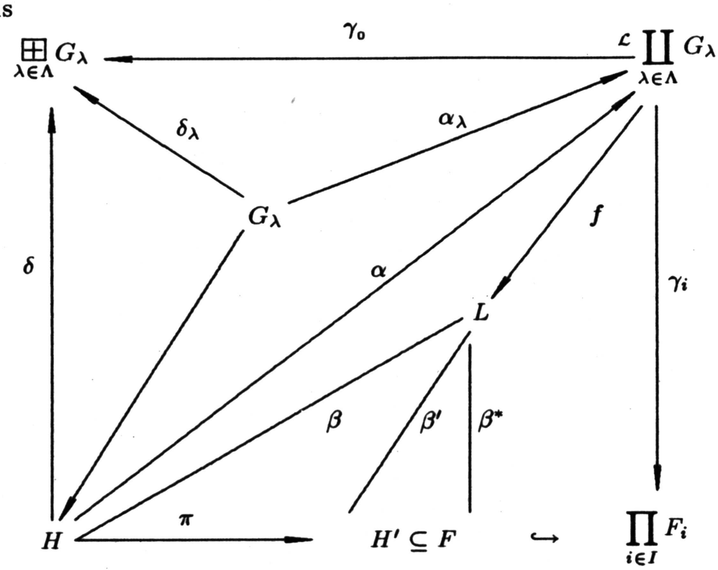
A class $\mathcal{U}$ of $l$-groups is called a sub-product class if it is closed under taking (1) $l$-subgroups and (2) direct products. Let $\mathcal{U}$ be a sub-product class of $l$-groups and $\{G_\lambda | \lambda \in \Lambda\}$ be a family of $l$-groups in $\mathcal{U}$. By Corollary 2 of Theorem 2 of [6] $\mathcal{U}$-free product ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ always exists. Since $\mathcal{L}$ and $WH$ are sub-product classes, so there exist $\mathcal{L}$-free products and $WH$-free products. In [7,8] W.B. Powell and C. Tsinakis have given several constructions of free products in the variety of abelian $l$-groups and in the variety of representable $l$-groups. In this section we will give construction of $WH$-free products.

Let $\{G_\lambda | \lambda \in \Lambda\} \subseteq WH$. Then there exists $\mathcal{L}$-free product ${}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$ with the coprojections $\alpha_\lambda$. (In [9] W.C. Holland and E. Scrimger have given a description for $\mathcal{L}$-free products.) It is clear that the cardinal sum $\boxplus G_\lambda$ is an $l$-group in $WH$ and each $G_\lambda(\lambda \in \Lambda)$ can be naturally embedded into $\boxplus_{\lambda\in\Lambda} G_\lambda$ as an $l$-group with embedding $\delta_\lambda$. Let $H$ be the group free product of $\{G_\lambda | \lambda \in \Lambda\}$. Then there exists a group homomorphism $\delta : H \to \boxplus_{\lambda\in\Lambda} G_\lambda$ which extends each $\delta_\lambda(\lambda \in \Lambda)$, and there exists a group homomorphism $\alpha : H \to {}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$ which extends each $\alpha_\lambda(\lambda \in \Lambda)$. On the other hand, there exists an $l$-homomorphism $\gamma_0 : \to {}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda \to \boxplus_{\lambda\in\Lambda} G_\lambda$ such that $\gamma_0\alpha_\lambda = \delta_\lambda$ for each $\lambda \in \Lambda$. Let $D = \{F_i | i \in I\}$ be the set of all $l$-homomorphic images in $WH$ of ${}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$ with the

$l$-homomorphisms $\gamma_i$ $(i \in I)$. Thus $\boxplus_{\lambda\in\Lambda} G_\lambda \in D$ and $D$ is not empty. For each $\lambda \in \Lambda$ and each $i \in I$, $\gamma_i\alpha_\lambda$ is an $l$-homomorphism of $G_\lambda$ into $F_i$. By proposition 1.2 the direct product $\prod_{i\in I} F_i$ is an $l$-group in $WH$. For each $\lambda \in \Lambda$, let $\pi_\lambda$ be the natural $l$-homomorphism of $G_\lambda$ onto the $l$-subgroup $G'_\lambda$ of $\prod_{i\in I} F_i$. That is

$$\pi_\lambda(g_\lambda) = (\ldots, \gamma_i\alpha_\lambda(g_\lambda), \ldots)$$

for $g_\lambda \in G_\lambda$. Let $H'$ be the subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} G'_\lambda$. Let $\pi$ be the group homomorphism of $H$ onto $H'$ which extends each $\pi_\lambda(\lambda \in \Lambda)$. That is




$$\pi(h) = (\ldots, \gamma_i\alpha(h), \ldots)$$

for $h \in H$. Since $\boxplus_{\lambda\in\Lambda} G_\lambda \in D$ and each $\delta_\lambda(\lambda \in \Lambda)$ is an $l$-isomorphism, $\pi_\lambda$ is an $l$-isomorphism for each $\lambda \in \Lambda$. Let $F$ be the sublattice of $\prod_{i\in I} F_i$ generated by $H'$. For each $h \in H$, put $h' = \pi(h)$. Since $\prod_{i\in I} F_i$ is a distributive lattice,

$$F = \{\bigvee_{j\in J} \bigwedge_{k\in K} h'_{jk} \mid \; h_{jk} \in H, J \text{ and } K \text{ finite }\}.$$

Then we have the following construction theorem for ${}^{WH}\prod_{\lambda\in\Lambda} G_\lambda$.

**Theorem 2.1.** *Suppose that* $\{G_\lambda | \lambda \in \Lambda\} \subseteq WH$. *Then the* $WH$*-free product* ${}^{WH}\coprod_{\lambda\in\Lambda} G_\lambda$ *is the sublattice* $F$ *of the direct product* $\prod_{i\in I} F_i$ *generated by the group homomorphic image* $H'$ *of the group free product* $H$ *of* $G_\lambda$, *where* $\{F_i | i \in I\}$ *are all homomorphic images in* $WH$ *of the* $\mathcal{L}$*-free product* ${}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$.

Proof. We have seen that $F \in WH$ and each $G_\lambda (\lambda \in \Lambda)$ can be embedded into $F$ as an $l$-group. Suppose that $L \in WH$ and $\{\beta_\lambda : G_\lambda \to L | \lambda \in \Lambda\}$ is a family of $l$-homomorphisms. We must show that there exists a unique $l$-homomorphism $\beta^* \colon F \to L$ such that $\beta^* \pi_\lambda = \beta_\lambda$ for each $\lambda \in \Lambda$. By the universal property of group free product, there exists a group homomorphism $\beta \colon H \to L$ which extends each $\beta_\lambda (\lambda \in \Lambda)$. For any $h' = \pi(h) \in H'$, put

$$\beta'(h') = \beta(h).$$

By the universal property of $\mathcal{L}$-free product, there exists a unique $l$-homomorphism $f : {}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda \to L$ such that $\beta_\lambda = f\alpha_\lambda$ for each $\lambda \in \Lambda$. Then

$$f\alpha = \beta'\pi = \beta.$$

By Lemma 11.3.1 of [3] we need only to show that for each finite subset $\{h_{jk} | j \in J, k \in K\} \subseteq H$, $\bigvee_{j\in J} \bigwedge_{k\in K} \beta'\pi(h_{jk}) = 0$ implies $\bigvee_{j\in J} \bigwedge_{k\in K} \pi(h_{jk}) = 0$. In fact,

$$\bigvee_{j\in J} \bigwedge_{k\in K} f\alpha(h_{jk}) = \bigvee_{j\in J} \bigwedge_{k\in K} \beta'\pi(h_{jk}) \neq o.$$

Because $f({}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda) \in D$, $\bigvee_{j\in J} \bigwedge_{k\in K} \gamma_i\alpha(h_{jk}) \neq 0$ for some $i \in I$. So

$$\begin{aligned} \bigvee_{j\in J} \bigwedge_{k\in K} \pi(h_{jk}) &= \bigvee_{j\in J} \bigwedge_{k\in K} (\ldots, \gamma_i\alpha(h_{jk}), \ldots) \\ &= (\ldots, \bigvee_{j\in J} \bigwedge_{k\in K} \gamma_i\alpha(h_{jk}), \ldots) \\ &\neq 0. \end{aligned}$$

Therefore $\beta'$ can be uniquely extended to an $l$-homomorphism $\beta^* \colon F \to L$. ∎

Some results concerning $l$-groups, the reader is also referred to [10-17].

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