Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol.8, 1994, Fasc.2–3

On the Limit Distribution of Maxima of Random Number of Bivariate Random Vectors*

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Limit distributions of partial maxima of random number of bivariate random vectors are obtained using random as well as non-random normalization. Normalizations considered here are more general than just linear.

1. Introduction.

Let $\left\{X_n = (X_n^{(1)}, X_n^{(2)}),\ n \geq 1\right\}$ be a sequence of independent, identically distributed (i.i.d.) random pairs defined on a probability space $(\Omega, \mathcal{F}, P)$ with common distribution function (df) F. Let F_i denote the df of $X_n^{(i)}$ and

$$M_n^{(i)} = \max_{1\leq k\leq n} X_k^{(i)}, \quad i = 1, 2.$$

Suppose that there exist sequences $\left\{G_n^{(i)}(.),\ n \geq 1\right\}$, $i = 1, 2$, of strictly increasing, continuous functions defined on R and a nondegenerate df H on R^2 such that

$$(1.1) \qquad \lim_{n\to\infty} P\left(M_n^{(1)} \leq G_n^{(1)}(x_1), M_n^{(2)} \leq G_n^{(2)}(x_2)\right) = H(x_1, x_2)$$

* Research supported by University Grants Comission Junior Research Fellowship at University of Mysore, Mysore

at all continuity points $x = (x_1, x_2)$ of H. Then H is max stable (P a n c h e v a (1988)). Further, from P a n c h e v a (1984) the marginals H_i, $i = 1, 2$ are necessarily continuous and hence H is continuous. For any nondecreasing function $K : R \to R$ we define $K^{-}(y) = \inf\{u : K(u) > y\}$, infimum over an empty set is taken to be ∞. Let $\{m_n,\ n \geq 1\}$ be a sequence of positive integers such that m_n increases to ∞ with n and $\frac{m_n}{n} \to \lambda$, $0 < \lambda < \infty$. As it will be seen from Corollary 2.3, (1.1) implies that there exists $g_{\lambda}^{(i)}(.)$ such that

$$\lim_{n\to\infty} G_n^{(i)-} \circ G_n^{(i)}(.) = g_{\lambda}^{(i)}(.),\ i = 1, 2, \tag{1.2}$$

where $f \circ h = f(h)$. Let $\left\{(N_n^{(1)}, N_n^{(2)}),\ n \geq 1\right\}$ be a sequence of pairs of positive integer valued random variables (rv's) defined on $(\Omega, \mathcal{F}, P)$ such that

$$\left(\frac{N_n^{(1)}}{n}, \frac{N_n^{(2)}}{n}\right) \xrightarrow{P} (N^{(1)}, N^{(2)}) \tag{1.3}$$

where $\xrightarrow{P}$ denotes convergence in probability and $P(N^{(i)} \leq 0) = 0$, $i = 1, 2$. Denote by $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. For any df L on R let $S(L) = \{x : 0 < L(x) < 1\}$ and $r(L) = \sup S(L)$. Let $[x]$ denote the greatest integer less than or equal to x. The purpose of this article is to prove the following two theorems.

Theorem 1.1. *Suppose that (1.1) holds with H_i strictly increasing over $S(H_i)$, $i = 1, 2$, and that (1.3) holds. Then for $x = (x_1, x_2) \in R^2$,*

$$\lim_{n\to\infty} P\left(M_{N_n^{(1)}}^{(1)} \leq G_{N_n^{(1)}}^{(1)}(x_1),\ M_{N_n^{(2)}}^{(2)} \leq G_{N_n^{(2)}}^{(2)}(x_2)\right)$$

$$= \begin{cases} 0 & \text{if } x \in B_0 \\ \int_{y=0}^{\infty}\int_{z=0}^{\infty} H^{y\wedge z}\left(g_{1/y}^{(1)}(x_1), g_{1/z}^{(2)}(x_2)\right) . (H_1(x_1))^{\frac{(y-z)\vee 0}{y}} . & \\ (H_2(x_2))^{\frac{(z-y)\vee 0}{z}} . dP\left(N^{(1)} \leq y,\ N^{(2)} \leq z\right) & \text{if } x \in B_1, \\ H_1(x_1) & \text{if } x \in B_2, \\ H_2(x_2) & \text{if } x \in B_3, \\ 1 & \text{if } x \in B_4; \end{cases}$$

where

$$B_0 = \{x \ : \ H_i(x_i) = 0 \ \textit{for at least one } i, \ i = 1,2\};$$
$$B_1 = \{x \ : \ 0 < H_i(x_i) < 1 \ i = 1,2\};$$
$$B_2 = \{x \ : \ 0 < H_1(x_1) < 1, \ H_2(x_2) = 1\};$$
$$B_3 = \{x \ : \ H_1(x_1) = 1, \ 0 < H_2(x_2) < 1\};$$

and

$$B_4 = \{x \ : \ H_i(x_i) = 1, \ i = 1,2\}.$$

Theorem 1.2. *Suppose (1.1) and (1.3) hold. Then*

$$\lim_{n\to\infty} P\left(M^{(1)}_{N^{(1)}_n} \le G^{(1)}_n(x_1), \ M^{(2)}_{N^{(2)}_n} \le G^{(2)}_n(x_2)\right)$$
$$= \int_{y=0}^{\infty}\int_{z=0}^{\infty} (H(x))^{y\wedge z}.(H_1(x_1))^{(y-z)\vee 0}$$
$$\times (H_2(x_2))^{(z-y)\vee 0} dP\left(N^{(1)} \le y, \ N^{(2)} \le z\right), \qquad x \in R_2.$$

The above theorems are generalizations of the theorem in Barndorff-Nielsen (1964) in that univariate case with linear normalization has been considered.

In section 2 some preliminary results are presented and the proofs of the theorems are given in section 3. In the last section some remarks are made.

2. Some preliminary results.

In this section four lemmas are presented. The proof of the first is similar to that of (ii) of the theorem of Barndorff-Nielsen (1964) and the proof of the third is on the same lines as that of Lemma 3 of Blum et al. (1963). The second can be proved using Proposition 0.1, Resnick (1987).

Lemma 2.1. *If (1.1) and (1.3) hold than for $l = 1,2$ we have*

$$\lim_{n\to\infty} P\left(M^{(l)}_{N^{(l)}_n} \le G^{(l)}_{N^{(l)}_n}(x_l)\right) = H_l(x_l), \quad x_l \in R.$$

Lemma 2.2. *Let $\{K_n, \ n \ge 1\}$ be a sequence of df's on R such that*

$$K_n(L_n(.)) \xrightarrow{W} U(.) \text{ and } K_n(G_n(.)) \xrightarrow{W} V(.),$$

where $\xrightarrow{W}$ denotes weak convergence; L_n and G_n are strictly increasing and continuous functions on R; and U and V are nondegenerate df's. If U and V are continuous and U is strictly increasing over $S(U)$, then $\lim_{n\to\infty} G_n^- \circ L_n(.)$ exists and is equal to $V^- \circ U(.)$.

The following corollary follows from the above lemma.

Corollary 2.3. *If $\{m_n,\ n \geq 1\}$ is a sequence of positive integers such that m_n increases to ∞ with n, $\frac{m_n}{n} \to \lambda$, $0 < \lambda < \infty$, and if (1.1) holds, then $\lim_{n\to\infty} (G_{m_n}^{(i)})^- \circ G_n^{(i)}(.)$ exists and is equal to $H_i^- \circ H_i^{\lambda}(.)$, $i = 1, 2$.*

Lemma 2.4. *Let $\{k_n,\ n \geq 1\}$ and $\{m_n,\ n \geq 1\}$, $k_n \leq m_n$, be two sequences of positive integers increasing to ∞ with n. Let $A_n \in \mathcal{F}$, $n \geq 1$, be such that A_n depends only on $\{x_k,\ k_n \leq k \leq m_n\}$. Then for any $A \in \mathcal{F}$,*

$$\lim_{n\to\infty} \{P(A_n \mid A) - P(A_n)\} = 0,$$

where $P(A_n \mid A)$ is defined as equal to $P(A_n)$ if $P(A) = 0$.

Lemma 2.5. *Let $\{k_n,\ n \geq 1\}$ and $\{m_n,\ n \geq 1\}$ be as in Lemma 2.4. Let $\{\alpha_n,\ n \geq 1\}$ and $\{\beta_n,\ n \geq 1\}$ be two sequences of real numbers such that $\alpha_n < r(F_1)$, $\beta_n < r(F_2)$, $\alpha_n \to r(F_1)$, and $\beta_n \to r(F_2)$ as $n \to \infty$. Then for any $A \in \mathcal{F}$,*

$$\lim_{n\to\infty} \left\{P\left(M_{k_n}^{(1)} \leq \alpha_n,\ M_{m_n}^{(2)} \leq \beta_n \mid A\right) - P\left(M_{k_n}^{(1)} \leq \alpha_n,\ M_{m_n}^{(2)} \leq \beta_n\right)\right\} = 0.$$

Proof. The claim follows trivially if $P(A) = 0$. Let $P(A) > 0$. Choose $\{\theta_n,\ n \geq 1\}$ such that $\theta_n \leq k_n$, $\lim_{n\to\infty} \theta_n = \infty$, and $\lim_{n\to\infty} F^{\theta_n}(\alpha_n, \beta_n) = 1$. Note that θ_n may be chosen as $k_n \wedge \eta_n$, where $\eta_n = [-\log(1 - F(\alpha_n, \beta_n))]$. Let $\bar{M}_{k_n}^{(1)} = \max_{\theta_n < k \leq k_n} M_k^{(1)}$, $\bar{M}_{k_n}^{(2)} = \max_{\theta_n < k \leq m_n} M_k^{(2)}$. From Lemma 2.4, it follows that for any event A,

(2.1)
$$\lim_{n\to\infty} \left\{P\left(\bar{M}_{k_n}^{(1)} \leq \alpha_n,\ \bar{M}_{m_n}^{(2)} \leq \beta_n \mid A\right) - P\left(\bar{M}_{k_n}^{(1)} \leq \alpha_n,\ \bar{M}_{m_n}^{(2)} \leq \beta_n\right)\right\} = 0,$$

and

$$\begin{aligned} &\lim_{n\to\infty} \left\{P\left(\bar{M}_{k_n}^{(1)} \leq \alpha_n,\ \bar{M}_{m_n}^{(2)} \leq \beta_n\right) - P\left(M_{k_n}^{(1)} \leq \alpha_n,\ M_{m_n}^{(2)} \leq \beta_n\right)\right\} \\ &= \lim_{n\to\infty} P\left(\bar{M}_{k_n}^{(1)} \leq \alpha_n,\ \bar{M}_{m_n}^{(2)} \leq \beta_n\right) \cdot \left(1 - F^{\theta_n}(\alpha_n, \beta_n)\right) = 0. \end{aligned} \tag{2.2}$$

Finally,

$$\overline{\lim}_{n\to\infty}\left\{P\left(\bar{M}_{k_n}^{(1)}\le\alpha_n,\ \bar{M}_{m_n}^{(2)}\le\beta_n\mid A\right)\right.$$
$$(2.3)\qquad \left.-P\left(M_{k_n}^{(1)}\le\alpha_n,\ M_{m_n}^{(2)}\le\beta_n\mid A\right)\right\}$$
$$\le\lim_{n\to\infty}\frac{(1-F^{\theta_n}(\alpha_n,\beta_n))}{P(A)}=0.$$

The lemma now follows from (2.1), (2.2) and (2.3).

3. Proofs of theorems

Proof of Theorem 1.1. We essentially follow Barndorff–Nielsen (1964) in a similar context. We prove the theorem when $x\in B_1$ only, as the other cases are simple.

For $(y_1,y_2)\in R^2$, arbitrarily fixed integer $k\ge 1$ and any integer $n\ge 1$, we have

$$P\left(M_{N_n^{(1)}}^{(1)}\le y_1,\ M_{N_n^{(2)}}^{(2)}\le y_2\right)=A_n+B_n+C_n$$
$$(3.1)\qquad +\sum_{l=2}^{\infty}D_l(n)+\sum_{l=2}^{\infty}\sum_{j=2}^{\infty}E_{lj}(n),$$

where

$$A_n=P\left(M_{N_n^{(1)}}^{(1)}\le y_1,\ M_{N_n^{(2)}}^{(2)}\le y_2,\ \left|\frac{N_n^{(1)}}{n}-N^{(1)}\right|>\frac{1}{k}\right)$$
$$\le P\left(\left|\frac{N_n^{(1)}}{n}-N^{(1)}\right|>\frac{1}{k}\right),$$
$$B_n=P\left(M_{N_n^{(1)}}^{(1)}\le y_1,\ M_{N_n^{(2)}}^{(2)}\le y_2,\ \left|\frac{N_n^{(1)}}{n}-N^{(1)}\right|\le\frac{1}{k},\ \left|\frac{N_n^{(2)}}{n}-N^{(2)}\right|>\frac{1}{k}\right)$$
$$\le P\left(\left|\frac{N_n^{(2)}}{n}-N^{(2)}\right|>\frac{1}{k}\right),$$
$$C_n=P\left(M_{N_n^{(1)}}^{(1)}\le y_1,\ M_{N_n^{(2)}}^{(2)}\le y_2,\ \left|\frac{N_n^{(1)}}{n}-N^{(1)}\right|\le\frac{1}{k},\ \left|\frac{N_n^{(2)}}{n}-N^{(2)}\right|\le\frac{1}{k},\right.$$
$$\left.N^{(1)}\le\frac{2}{k}\right)$$

$$D_l(n) = P\left(M^{(1)}_{N_n^{(1)}} \le y_1,\ M^{(2)}_{N_n^{(2)}} \le y_2,\ \left|\frac{N_n^{(1)}}{n} - N^{(1)}\right| \le \frac{1}{k},\ \left|\frac{N_n^{(2)}}{n} - N^{(2)}\right| \le \frac{1}{k},\right.$$
$$\left.\frac{1}{k} < N^{(1)} \le \frac{l+1}{k},\ N^{(2)} \le \frac{2}{k}\right)$$

$$E_{lj}(n) = P\left(M^{(1)}_{N_n^{(1)}} \le y_1,\ M^{(2)}_{N_n^{(2)}} \le y_2,\ \left|\frac{N_n^{(1)}}{n} - N^{(1)}\right| \le \frac{1}{k},\ \left|\frac{N_n^{(2)}}{n} - N^{(2)}\right| \le \frac{1}{k},\right.$$
$$\left.\frac{1}{k} < N^{(1)} \le \frac{l+1}{k},\ \frac{j}{k} < N^{(2)} \le \frac{j+1}{k}\right),\ 2 \le l, j < \infty.$$

Observe that

$$\sum_{l=2}^{\infty} D_l(n) \le P\left(N^{(2)} \le \frac{2}{k}\right). \tag{3.2}$$

For $2 \le l,\ j < \infty$, denote by

$$S_l = \left(\frac{l}{k} < N^{(1)} \le \frac{l+1}{k}\right),\ R_j = \left(\frac{j}{k} < N^{(2)} \le \frac{j+1}{k}\right)$$

$$Q_{lj} = S_l \cap R_j,\ \Pi_{lj}(k) = P(Q_{lj}),\ n_{0p} = \left[\frac{n(p-1)}{k}\right],$$

$$n_{1p} = \left[\frac{n(p+1)}{k}\right],\ n_{2p} = \left[\frac{n(p+2)}{k}\right],\quad p = l, j.$$

Let $G^{(1)}_{p_n}(x_1)$ and $G^{(1)}_{u_n}(x_1)$ respectively denote the maximum and minimum of $\left\{G^{(1)}_t(x_1) : n_{0l} < t \le n_{2l}\right\}$; and $G^{(2)}_{q_n}(x_2)$ and $G^{(2)}_{v_n}(x_2)$ denote respectively the maximum and minimum of $\left\{G^{(2)}_t(x_2) : n_{0j} < t \le n_{2j}\right\}$.Substituing $y_i = G^{(i)}_{N_n^{(i)}}(x_i)$ in (3.1),

$$\sum_{l=2}^{\infty}\sum_{j=2}^{\infty} E_{lj}(n) \le \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} P\left(M^{(1)}_{n_{0l}} \le G^{(1)}_{p_n}(x_1),\ M^{(2)}_{n_{0j}} \le G^{(2)}_{q_n}(x_2) \mid Q_{lj}\right)\Pi_{lj}(k)$$
$$= \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} a^{(1)}_{lj}(n).\Pi_{lj}(k) + \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} P\left(M^{(1)}_{n_{0l}} \le G^{(1)}_{p_n}(x_1),\ M^{(2)}_{n_{0j}} \le G^{(2)}_{q_n}(x_2)\right)\Pi_{lj}(k)$$
$$= \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} a^{(1)}_{lj}(n).\Pi_{lj}(k) + \sum_{l=2}^{\infty}\sum_{l=2}^{\infty} a^{(2)}_{lj}(n)\Pi_{lj}(k)$$
$$+ \sum_{l=2}^{\infty}\sum_{l=2}^{\infty} P\left(M^{(1)}_{n_{0l}} \le G^{(1)}_{n_{1l}}(x_1),\ M^{(2)}_{n_{0j}} \le G^{(2)}_{n_{1j}}(x_2)\right)\Pi_{lj}(k),$$

$$\begin{aligned}
&\sum_{l=2}^{\infty}\sum_{j=2}^{\infty} E_{lj}(n) \le \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} a_{lj}^{(1)}(n).\Pi_{lj}(k) + \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} a_{lj}^{(2)}(n).\Pi_{lj}(k) \\
&+ \sum_{l=2}^{\infty}\sum_{j=2}^{l} F^{n_{0j}}\left(G_n^{(1)} \circ (G_n^{(1)})^- \circ G_{n_{1l}}^{(1)}(x_1), G_n^{(2)} \circ (G_n^{(2)})^- \circ G_{n_{1j}}^{(2)}(x_2)\right) \\
&\times \left(F_1\left(G_{n_{1l}}^{(1)}(x_1)\right)\right)^{n_{0l}-n_{0j}} \Pi_{lj}(k) \\
&+ \sum_{l=2}^{\infty}\sum_{j=l+1}^{\infty} F^{n_{0l}}\left(G_n^{(1)} \circ (G_n^{(1)})^- \circ G_{n_{1l}}^{(1)}(x_1), G_n^{(2)} \circ (G_n^{(2)})^- \circ G_{n_{1j}}^{(2)}(x_2)\right) \\
&\times \left(F_2\left(G_{n_{1j}}^{(2)}(x_2)\right)\right)^{n_{0j}-n_{0l}} \Pi_{lj}(k),
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
a_{lj}^{(1)}(n) =& P\left(M_{n_{0l}}^{(1)} \le G_{p_n}^{(1)}(x_1),\ M_{n_{0j}}^{(2)} \le G_{q_n}^{(2)}(x_2) \mid Q_{lj}\right) \\
&- P\left(M_{n_{0l}}^{(1)} \le G_{p_n}^{(1)}(x_1),\ M_{n_{0j}}^{(2)} \le G_{q_n}^{(2)}(x_2)\right),
\end{aligned}$$

and

$$\begin{aligned}
a_{lj}^{(2)}(n) =& P\left(M_{n_{0l}}^{(1)} \le G_{p_n}^{(1)}(x_1),\ M_{n_{0j}}^{(2)} \le G_{q_n}^{(2)}(x_2)\right) \\
&- P\left(M_{n_{0l}}^{(1)} \le G_{n_{1l}}^{(1)}(x_1),\ M_{n_{0j}}^{(2)} \le G_{n_{1j}}^{(2)}(x_2)\right).
\end{aligned}$$

Note that $|a_{lj}^{(1)}(n)| \le 2$ and by Lemma 2.5, we have $\lim_{n\to\infty} a_{lj}^{(1)}(n) = 0$. Hence

$$\lim_{n\to\infty} \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} a_{lj}^{(1)}(n).\Pi_{lj}(k) = 0.$$

Also,

$$\begin{aligned}
a_{lj}^{(2)}(n) \le& P\left(G_{n_{1l}}^{(1)}(x_1) < M_{n_{0l}}^{(1)} \le G_{p_n}^{(1)}(x_1)\right) \\
&+ P\left(G_{n_{1j}}^{(2)}(x_2) < M_{n_{0j}}^{(2)} \le G_{q_n}^{(2)}(x_2)\right) \\
\le& \left(\left(F_1^{p_n}\left(G_{p_n}^{(1)}(x_1)\right)\right)^{\frac{n_{0l}}{n_{2l}}} - \left(F_1^{n_{1l}}\left(G_{n_{1l}}^{(1)}(x_1)\right)\right)^{\frac{n_{0l}}{n_{1l}}}\right) \\
&+ \left(\left(F_2^{q_n}\left(G_{q_n}^{(2)}(x_2)\right)\right)^{\frac{n_{0j}}{n_{2j}}} - \left(F_2^{n_{1j}}\left(G_{n_{1j}}^{(2)}(x_2)\right)\right)^{\frac{n_{0j}}{n_{1j}}}\right)
\end{aligned}$$

and hence

$$\overline{\lim}_{n\to\infty} \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} a_{lj}^{(2)}(n).\Pi_{lj}(k)$$

(3.4)
$$\leq \sum_{l=2}^{\infty}\left\{(H_1(x_1))^{\frac{l-1}{l+2}} - (H_1(x_1))^{\frac{l-1}{l+1}}\right\}.\Pi_{l.}(k)$$
$$+\sum_{j=2}^{\infty}\left\{(H_2(x_2))^{\frac{j-1}{j+2}} - (H_2(x_2))^{\frac{j-1}{j+1}}\right\}.\Pi_{.j}(k)$$
$$= a^{(3)}(k),$$

say, where

$$\Pi_{l.}(k) = \sum_{j=2}^{\infty}\Pi_{lj}(k),\ \Pi_{.j}(k) = \sum_{l=2}^{\infty}\Pi_{lj}(k).$$

Note that $\lim_{k\to\infty} a^{(3)}(k) = 0$. Now from Corollary 2.3, and (3.1) through (3.4)

$$\overline{\lim}_{n\to\infty} P\left(M^{(1)}_{N_n^{(1)}} \leq G^{(1)}_{N_n^{(1)}}(x_1), M^{(2)}_{N_n^{(2)}} \leq G^{(2)}_{N_n^{(2)}}(x_2)\right)$$

(3.5)
$$\leq \sum_{i=1}^{2}\left(N^{(i)} \leq \frac{2}{k}\right) + a^{(3)}(k)$$
$$+\sum_{l=0}^{\infty}\sum_{j=0}^{l}\left(H\left(g^{(1)}_{1/\frac{(l+1)}{k}}(x_1), g^{(2)}_{1/\frac{(j+1)}{k}}(x_2)\right)\right)^{\frac{(j+1)}{k}}.(H_1(x_1))^{\frac{l-j}{l+1}}.\Pi_{lj}(k)$$
$$+\sum_{l=0}^{\infty}\sum_{j=l+1}^{\infty}\left(H\left(g^{(1)}_{1/\frac{(l+1)}{k}}(x_1), g^{(2)}_{1/\frac{(j+1)}{k}}(x_2)\right)\right)^{\frac{(l+1)}{k}}.(H_2(x_2))^{\frac{j-l}{j+1}}.\Pi_{lj}(k),$$

where the fact that the convergence in (1.1) is uniform is used. Again, we have

$$\sum_{l=2}^{\infty}\sum_{j=2}^{\infty} E_{lj}(n) \geq \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} P\left(M^{(1)}_{n_{2l}} \leq G^{(1)}_{u_n}(x_1),\ M^{(2)}_{n_{2j}} \leq G^{(2)}_{v_n}(x_2) \mid Q_{lj}\right)\Pi_{lj}(k)$$
$$-\sum_{l=2}^{\infty}\sum_{j=2}^{\infty} P\left(\left\{\left|\frac{N_n^{(1)}}{n} - N^{(1)}\right| \vee \left|\frac{N_n^{(2)}}{n} - N^{(2)}\right| > \frac{1}{k}\right\}\cap Q_{ij}\right)$$
$$\geq \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} P\left(M^{(1)}_{n_{2l}} \leq G^{(1)}_{u_n}(x_1),\ M^{(2)}_{n_{2j}} \leq G^{(2)}_{v_n}(x_2)\right)\ .\ \Pi_{lj}(k)$$
$$-b^{(1)}(n) - \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} b_{lj}^{(2)}(n)\Pi_{lj}(k)$$

$$= \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} P\left(M_{n_{2l}}^{(1)} \le G_{n_{1l}}^{(1)}(x_1),\ M_{n_{2j}}^{(2)} \le G_{n_{1j}}^{(2)}(x_2)\right)\Pi_{lj}(k) - b^{(1)}(n)$$
$$- \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} b_{lj}^{(2)}(n)\Pi_{lj}(k) - \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} b_{lj}^{(3)}(n)\Pi_{lj}(k)$$
$$\ge \sum_{l=2}^{\infty}\sum_{j=2}^{l} F^{n_{2j}}\left(G_n^{(1)} \circ (G_n^{(1)})^- \circ G_{n_{1l}}^{(1)}(x_1), G_n^{(2)} \circ (G_n^{(2)})^- \circ G_{n_{1j}}^{(2)}(x_2)\right)$$
$$\times\left(F_1\left(G_{n_{1l}}^{(2)}(x_2)\right)\right)^{n_{2l}-n_{2j}}\Pi_{lj}(k)$$
$$+\sum_{l=2}^{\infty}\sum_{j=l+1}^{\infty} F^{n_{2l}}\left(G_n^{(1)} \circ (G_n^{(1)})^- \circ G_{n_{1l}}^{(1)}(x_1), G_n^{(2)} \circ (G_n^{(2)})^- \circ G_{n_{1j}}^{(2)}(x_2)\right)$$
$$\times\left(F_2\left(G_{n_{1j}}^{(2)}(x_2)\right)\right)^{n_{2j}-n_{2l}}\Pi_{lj}(k)$$
$$-b^{(1)}(n) - \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} b_{lj}^{(2)}(n).\Pi_{lj}(k) - \sum_{l=2}^{\infty}\sum_{j=2}^{\infty} b_{lj}^{(3)}(n).\Pi_{lj}(k),$$

where

$$b^{(1)}(n) = P\left(\left|\frac{N_n^{(1)}}{n} - N^{(1)}\right| \vee \left|\frac{N_n^{(2)}}{n} - N^{(2)}\right| > \frac{1}{k}\right),$$
$$b_{lj}^{(2)}(n) = P\left(M_{n_{2l}}^{(1)} \le G_{u_n}^{(1)}(x_1),\ M_{n_{2j}}^{(2)} \le G_{v_n}^{(2)}(x_2)\right)$$
$$- P\left(M_{n_{2l}}^{(1)} \le G_{u_n}^{(1)}(x_1),\ M_{n_{2j}}^{(2)} \le G_{v_n}^{(2)}(x_2) \mid Q_{lj}\right)$$

and

$$b_{lj}^{(3)}(n) = P\left(M_{n_{2l}}^{(1)} \le G_{n_{1l}}^{(1)}(x_1),\ M_{n_{2j}}^{(2)} \le G_{n_{1j}}^{(2)}(x_2)\right)$$
$$- P\left(M_{n_{2l}}^{(1)} \le G_{u_n}^{(1)}(x_1),\ M_{n_{2j}}^{(2)} \le G_{v_n}^{(2)}(x_2)\right).$$

Note that $\lim_{n\to\infty} b^{(1)}(n) = 0$. From Lemma 2.5, $\lim_{n\to\infty} b_{lj}^{(2)}(n) = 0$ and hence

$$\overline{\lim}_{n\to\infty}\sum_{l=2}^{\infty}\sum_{j=2}^{\infty} b_{lj}^{(2)}(n)\Pi_{lj}(k) = 0.$$

Similar to (3.4) it can be shown that

$$\overline{\lim}_{n\to\infty}\sum_{l=2}^{\infty}\sum_{j=2}^{\infty} b_{lj}^{(3)}(n)\Pi_{lj}(k) \le \sum_{l=2}^{\infty}\left\{(H_1(x_1))^{\frac{l+2}{l+1}} - (H_1(x_1))^{\frac{l+2}{l-1}}\right\}.\Pi_{l.}(k)$$
$$+\sum_{j=2}^{\infty}\left\{(H_2(x_2))^{\frac{j+2}{j+1}} - (H_2(x_2))^{\frac{j+2}{j-1}}\right\}.\Pi_{.j}(k) = b^{(4)}(k),$$

say, and $\lim\limits_{k\to\infty} b^{(4)}(k) = 0$. Thus

$$
\begin{aligned}
&\underline{\lim}_{n\to\infty} P\left(M^{(1)}_{N^{(1)}_n} \le G^{(1)}_{N^{(1)}_n}(x_1), M^{(2)}_{N^{(2)}_n} \le G^{(2)}_{N^{(2)}_n}(x_2)\right) \\
&\ge \sum_{l=0}^{\infty}\sum_{j=0}^{l} \left(H\left(g^{(1)}_{1/\frac{(l+1)}{k}}(x_1), g^{(2)}_{1/\frac{(j+1)}{k}}(x_2)\right)\right)^{\frac{(j+1)}{k}} .\left(H_1(x_1)\right)^{\frac{l-j}{l+1}} .\Pi_{lj}(k) \\
&+ \sum_{l=0}^{\infty}\sum_{j=l+1}^{\infty} \left(H\left(g^{(1)}_{1/\frac{(l+1)}{k}}(x_1), g^{(2)}_{1/\frac{(j+1)}{k}}(x_2)\right)\right)^{\frac{(l+1)}{k}} .\left(H_2(x_2)\right)^{\frac{j-l}{j+1}} .\Pi_{lj}(k) \\
&- b^{(4)}(k) - b^{(5)}(k),
\end{aligned} \tag{3.6}
$$

where $b^{(5)}(k)$ is such that $\lim\limits_{k\to\infty} b^{(5)}(k) = 0$. Taking limit as $k \to \infty$ in (3.5) and (3.6), the claim is proved.

Proof of Theorem 1.2. Substituing $y_i = G^{(i)}_n(x_i)$, $i = 1, 2$, in (3.1) and proceeding as in the proof of Theorem 1.1, it can be shown that

$$
\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{j=0}^{l} (H(x))^{\frac{j+2}{k}} .(H_1(x_1))^{\frac{l-j}{k}} .\Pi_{lj}(k) \\
&+ \sum_{l=0}^{\infty}\sum_{j=l+1}^{\infty} (H(x))^{\frac{l+2}{k}} .(H_2(x_2))^{\frac{j-l}{k}} .\Pi_{lj}(k) - b^{(6)}(k) \\
&\le \overline{\lim}_{n\to\infty} P\left(M^{(1)}_{N^{(1)}_n} \le G^{(1)}_{N^{(1)}_n}(x_1), M^{(2)}_{N^{(2)}_n} \le G^{(2)}_{N^{(2)}_n}(x_2)\right) \\
&\le \sum_{i=1}^{2} P\left(N^{(i)} \le \frac{2}{k}\right) + a^{(4)}(k) \\
&+ \sum_{l=0}^{\infty}\sum_{j=0}^{l} (H(x))^{\frac{j-1}{k}} .(H_1(x_1))^{\frac{l-j}{k}} .\Pi_{lj}(k) \\
&+ \sum_{l=0}^{\infty}\sum_{j=l+1}^{\infty} (H(x))^{\frac{l-1}{k}} .(H_2(x_2))^{\frac{j-l}{k}} .\Pi_{lj}(k),
\end{aligned}
$$

where $a^{(4)}(k)$ and $b^{(6)}(k)$ are such that $\lim\limits_{k\to\infty} a^{(4)}(k) = \lim\limits_{k\to\infty} b^{(6)}(k) = 0$. If $H(x) =$

0 then the result follows from this. If $H(x) > 0$ then

$$\sum_{l=0}^{\infty}\sum_{j=0}^{l}(H(x))^{\frac{j+1}{k}}.(H_1(x_1))^{\frac{l-j}{k}}.\Pi_{lj}(k)$$

$$+\sum_{l=0}^{\infty}\sum_{j=l+1}^{\infty}(H(x))^{\frac{l+1}{k}}.(H_2(x_2))^{\frac{j-l}{k}}.\Pi_{lj}(k)-b^{(7)}(k)$$

$$\leq \underline{\lim}_{n\to\infty} P\left(M^{(1)}_{N^{(1)}_n}\leq G^{(1)}_{N^{(1)}_n}(x_1), M^{(2)}_{N^{(2)}_n}\leq G^{(2)}_{N^{(2)}_n}(x_2)\right)$$

$$\leq \sum_{l=0}^{\infty}\sum_{j=0}^{l}(H(x))^{\frac{j+1}{k}}(H_1(x_1))^{\frac{l-j}{k}}.\Pi_{lj}(k)$$

$$+\sum_{l=0}^{\infty}\sum_{j=l+1}^{\infty}(H(x))^{\frac{l+1}{k}}.(H_2(x_2))^{\frac{j-l}{k}}.\Pi_{lj}(k)+\sum_{i=1}^{2}P\left(N^{(i)}\leq\frac{2}{k}\right)+a^{(5)}(k),$$

where $a^{(5)}(k)$ and $b^{(7)}(k)$ are such that $\lim_{k\to\infty} a^{(5)}(k) = \lim_{k\to\infty} b^{(7)}(k) = 0$. Taking limit as $k \to \infty$, the result follows.

4. Some remarks.

(i) When $G_n^{(i)}(x_i) = a_n(i).x_i + b_n(i)$, $a_n(i) > 0$, then the marginals H_i in (1.1) belong to the class of extreme value df's of Gnedenko. When

$$G_n^{(i)}(x_i) = \alpha_n(i).|x_i|^{\beta_n(i)}.sgn(x_i),\ \alpha_n(i) > 0,\ \beta_n(i) > 0,$$

then the marginals H_i in (1.1) belong to the class of extreme value df's given in Pancheva (1984). The hypotheses of Theorem 1.1 are automatically satisfied in both cases.

(ii) If $N^{(1)} = N^{(2)}$ a.s., then the limit in Theorem 1.1 reduces to $H(x)$, $x \in R^2$ and the limit in Theorem 1.2 reduces to $\int_{y=0}^{\infty}(H(x))^y.dp(N^{(1)} \leq y)$. Note that $H\left(g_s^{(1)}(x_1), g_s^{(2)}(x_2)\right) = H^s(x)$, $0 < s < \infty$.

(iii) If the limit random pair in (1.1) has independent components then the limit in Theorem 1.1 is again H. If, in addition, $N^{(1)}$ and $N^{(2)}$ are independent then the limit random pair in Theorem 1.2 will have independent components.

5. References

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Received 09.02.1993