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Global Solutions of Semilinear Wave Equation with Quadratic Nonlinearity

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1. Introduction

In the present paper we give natural sufficient conditions for the existence of a global classical solution of the Caushy problem when the of the spatial variables is $n = 3$. More precisely, consider the following problem

(1) $\quad \Box u(t,x) = F(t,x,u,Du)$ in $R_+ \times R^3$,

(2_ε) $\quad u(0,x) = \varepsilon g_1(x), \quad \partial_0 u(0,x) = \varepsilon g_2(x)$ for $x \in R^3$,

where $\partial_\mu = \frac{\partial}{\partial x_\mu}$, $\mu = 0,1,2,3$; $\quad x_0 = t$;
$\Box = \partial_0^2 - (\partial_1^2 + \partial_2^2 + \partial_3^2)$; $\quad D = (\partial_0, \partial_1, \partial_2, \partial_3)$; $\quad \varepsilon$ is a positive parameter and for $j = 1,2$ the functions $g_j \in C_0^\infty(R^3)$ with $\quad$ supp $g_j \subset \{x : |x| \leq B\}$.

It is well known that if $F(t,x,\zeta,\xi) = O(|\zeta|^3 + |\xi|^3)$ for $(\zeta,\xi) \in R \times R^3$ and the parameter ε is small enough, then problem (1), (2_ε) has a global solution (see for instance [7], [10], [18]). The situation changes radically when nonlinearity is quadratic, i.e. $F(t,x,\zeta,\xi) = O(\zeta^2 + |\xi|^2)$. In this case the smallness of ε is not sufficient to guarantee the global solvability of problem (1), (2_ε). F. John in [4] considered the case when $F = |u|^p$. He proved that if $p > 1 + \sqrt{2}$, problem (1), (2_ε) admits global solvability for small ε, and if $1 < p < 1 + \sqrt{2}$, then for appropriately initial conditions $g_1(x)$ and $g_2(x)$ the solution blows up for a finite time for any $\varepsilon > 0$. In his next paper F. John proved a general result about "blow-up" of the solutions of quasilinear hiperbolic equations. Applying this result to problem (1), (2_ε) with a nonlinearity $F = q(\partial_0 u)^2$ or $F = \partial_0(qu^2)$, we check that if $q =$ const. > 0 and $\varepsilon > 0$ is small enough, then the solution blows up in a finite time. On the other hand, investigating the global solvability of the Cauchy problem for nonlinear hyperbolic systems, S. Klainerman [9] introduced a simple algebraic condition on the quadratic form of the nonlinearity, the so called "null-condition". He proved that this condition leads to global solvability for small initial data. For problem (1), (2_ε) the "null-condition" of S. Klainerman means that the quadratic form of $F(t,x,\zeta,\xi)$ is proportional to $\xi_0^2 - (\xi_1^2 + \xi_2^2 + \xi_3^2)$.

The study of wave processes in nonhomogeneous media leads naturally to problems in which the nonlinearity F depends on x or on (t,x). Recently Mitsuhiro Nakao [15] investigated the existence of a global classical solution of the boundary value problem for equation (1) with a nonlinearity F containing a dissipative term. In the present paper the Cauchy problem with nonlinearities depending on x or (t,x) is considered, without assuming the presence of a dissipative term.

The conditions (H1), (H2) and (H3) given below guarantee the global solvability of problem (1), (2_ε) for small ε without the "null-condition" of Klainerman being fulfilled. For the precise formulation of the conditions (H1), (H2) and (H3) it is convinient to introduce some notation.

For arbitrary real numbers a and b define the norms

$$|u|_{a,b,t} = \sup_{0\le s\le t}\sup_{x\in R^3}\{(1+s+|x|)^{-a}(1+|s-|x||)^{-b}|u(s,x)|\},$$

$$\|u\|_{a,b,t} = \sum_{|\alpha|\le 2}|\partial^\alpha u|_{a,b,t},$$

where $\partial = (\partial_1,\partial_2,\partial_3)$. Moreover, in the paper the following vector fields are used:

$D = (\partial_0,\partial_1,\partial_2,\partial_3)$; $L = \{L_j = x_j\partial_0 + t\partial_j : j = 1,2,3\}$;
$\Omega = \{\Omega_{ij} = x_i\partial_j - x_j\partial_i : 1\le i<j\le k\}$; $S = \sum_{\mu=0}^{3} x_\mu\partial_\mu$;
$\Gamma = \{\Gamma_1,\dots\Gamma_{10}\} = \{D,\Omega,L\}$ and $Z = \{Z_1,\dots,Z_8\} = \{D,\Omega,S\}$.

If $A = \{A_1,\dots,A_r\}$ is any of the above vector fields, then for any nonnegative integer k define

$$|A^k u(t,x)| = \sum_{|\alpha|\le k}|A^\alpha u(t,x)| \quad \text{and}$$

$$\|A^k u(t,.)\| = \left(\int_{R^3}|A^k u(t,x)|^2 dx\right)^{1/2},$$

where $\alpha = (\alpha_1,\dots,\alpha_r)$ is a multiindex with $|\alpha| = \alpha_1+\dots+\alpha_r$ and $A^\alpha u(t,x) = A^{\alpha_1}\dots A^{\alpha_r}u(t,x)$. For two of the cases the most frequently met it is convinient to introduce a special notation:

(a) When A coincides with the generators Γ of Poincare's group,

$$|\Gamma^k u(t,x)| = |u(t,x)|_k \quad \text{and} \quad \|\Gamma^k u(t,\cdot)\| = \|u(t,\cdot)\|_k.$$

(b) When A coincides with the generators Z of the conformal group,

$$|Z^k u(t,x)| = |u(t,x)|_k \quad \text{and} \quad \|Z^k u(t,\cdot)\| = \|u(t,\cdot)\|_k.$$

We should underline that in both cases we use the same notation since the formulations and proofs given below are not influenced by the definition of $|\cdot|_k$ and $\|\cdot\|_k$ we use.

By means of the notation we introduced we can formulate the assumptions on the nonlinearity F:

(H1)
$$F(t,x,u,Du) = \sum_{\mu,\nu=0}^{3} q_{\mu\nu}(t,x)\partial_\mu u(t,x)\partial_\nu u(t,x)$$

and there exist constants $k \geq 2$ and $\delta > 0$ such that

$$\sum_{\mu,\nu=0}^{3} \sup_{t\geq 0} \sup_{|x|\leq B+t} \left\{(1+t)^\delta |q_{\mu\nu}(t,x)|_{k+3}\right\} < \infty;$$

(H2)
$$F(t,x,u,Du) = \sum_{\mu=0}^{3} \partial_\mu(q_\mu(t,x)u^2(t,x))$$

and there exist constants $k \geq 2$ and $\delta > 0$ such that

$$\sum_{\mu=0}^{3} \sup_{t\geq 0} \sup_{|x|\leq B+t} \left\{(1+t)^\delta |q_\mu(t,x)|_{k+3}\right\} < \infty;$$

(H3)
$$F(t,x,u,Du) = q(t,x)u^2(t,x)$$

and there exist constants $a > 0$ and $b \geq 0$ such that $2a + b > 1$ and

$$\sup_{t\geq 0} \{\|q\|_{-a,-b,t}\} < \infty.$$

The main result of the present paper is the following.

Theorem 1. *Let one of conditions* (H1), (H2) *or* (H3) *holds. Then there exist constant* $\varepsilon_0 > 0$ *such that problem (1), (*2_ε*) has unique smooth solution for any* $\varepsilon \in [0, \varepsilon_0)$.

The precision of conditions (H1) and (H2) is illustrated by the following examples.

Let nonlinearity $F = q(t,x)(\partial_0 u(t,x))^2$ or $F = \partial_0(q(t,x)u^2(t,x))$. Let $q(t,x)$ be one of the following functions

$$C(1+t^2)^{-\delta}, \quad C(1+|x|^2)^{-\delta} \quad \text{or} \quad C(1+t^2+|x|^2)^{-\delta},$$

where $C = \text{const.} > 0$ and $\delta > 0$. For such nonlinearities F condition (H1) (or (H2)) is met. Then from Theorem 1 there follows the existence of a global solution of problem (1), (2_ε) for ε small enough. On the other hand, the corollaries from the results of F. John given above show that for $\delta = 0$ the solution of problem (1), (2_ε) blows up for a finite time .

In the same sense condition (H3) is optimal too. The following theorem is valid.

Theorem 2. *Let* $|g_1(x)|^2 + |g_2(x)|^2 \neq 0$, $F = q(t,x)u^2(t,x)$ *and nonnegative constants* C_1, a *and* b *exist such that*

$$0 < C_1 \leq q(t,x)(1+t+|x|)^a(1+|t-|x||)^b. \tag{3}$$

Then, if $2a+b<1$, *then for any* $\varepsilon>0$ *the solution of problem* (1), (2_ε) *blows up.*

Definition. The soluttion of problem (1), (2_ε) is said to blow up if there exists $t_0 < \infty$ and a function $u(t,x)$ such that $u(t,x)$ is a solution of problem (1), (2_ε) in the strip $S(t_0) = [0,t_0) \times R^3$ and

$$\lim_{t\to t_0}\left(\sum_{|\alpha|\leq 1}\sup_{x\in R^3}\{|D^\alpha u(t,x)|\}\right) = \infty.$$

In the proof of Theorem 1, when condition (H1) (or (H2)) is valid, we use the idea of S. Klainerman. For this purpose the validity of appropriate $L^\infty - L^2$ and $L^2 - L^2$ estimates of the solution of the linear nonhomogeneous wave equation is necessary. In the case considered the principal difficulty is in the proof of the $L^\infty - L^2$ estimates. To overcome this difficulty we use the approach of [1], representing the solution of the linear problem as an oscillating integral (see Lemma 2.1). After an integration by parts along suitable vector fields in the domains $2|x\ t$ and $2|x| < t$, by means of the properties of the generators Γ and Z in the dual space and respective Sobolev inequalities on the unit sphere we prove the $L^\infty - L^2$ estimates. These estimates are given in Theorems 2.1 and 2.2 and are of independent interest. In them the participation of the generators L_j $(j = 1,2,3)$ and S is fixed, which turns out to be useful in the investigation of the nonlinear problem. We shall note that the presence of the aggregate $q(t,x)u^2$ in the nonlinearity F leads to a change of the decay of the solution of problem (1), (2_ε) (see [11] and also [4] and [9]). This requires the case $F = q(t,x)u^2$ to be analysed separately. In this case we use the approach of H. P e c h e r [16] for obtaining suitable $L^\infty - L^\infty$ estimates. Afterwards, by means of the "continuation principle" (see [12] and [14]) we prove Theorem 1 if (H3) is valid. For nonlinearities F of the form $q(t,x)u^2(t,x)$ Theorems 1 and 2 prove that the critical value of the parameters a and b which determines the behaviour of the coefficient $q(t,x)$ at infinity is $2a+b=1$.

The plan of the paper is as follows. In Section 2 we prove $L^\infty - L^2$ estimates for the solution of the linear Cauchy problem. In Section 3 we prove Theorem 1 and in Section 4 – Theorem 2.

A short communication of the above results is contained in [13].

2. A priori estimates for wave equation

The main goal of the present section is the proof of $L^\infty - L^2$ for the solution of the Cauchy problem

$$\square u(t,x) = G(t,x) \text{ in } R^4_+, \tag{4}$$

$$u(0,x) = f_1(x), \quad \partial_0 u(0,x) = f_2(x) \text{ on } R^3, \tag{5}$$

where G, f_1 and f_2 are smooth functions,

$$\operatorname{supp} G \subset K(B) = \{(t,x) : |x| \le t+B\},$$
$$\operatorname{supp} f_j \subset \{x : |x| \le B\} \text{ and } B = \text{const.} < \infty.$$

First we shall prove the following theorem.

Theorem 2.1 *Let $u(t,x)$ be a smooth solution of problem* (4), (5). *Then the following estimate is valid*

$$\sum_{|\alpha|=1} |D^\alpha u(t,x)|_k \le \frac{C}{1+t}\{\|D^{k+3}u(0,\cdot)\| + A(G;t)\}, \tag{6}$$

where $C = \text{const.} > 0$, $\alpha = (\alpha_0,\alpha_1,\alpha_2,\alpha_3)$, $A(G;t)$ *is any of the functions*

$$\sum_{r=0}^{\infty}\sum_{|\gamma|\le 1}\left[\sum_{|\beta|\le 2}\sup_{s\in I(r,t)}\{2^r\|\partial^\gamma \Omega^\beta G(s,\cdot)\|_k\} + \sum_{|\beta|\le 1}\sup_{s\in I(r,t)}\{2^r\|\partial^\gamma L^\beta G(s,\cdot)\|_k\}\right]$$

or

$$\sum_{r=0}^{\infty}\sum_{|\gamma|\le 1}\left[\sum_{|\beta|\le 2}\sup_{s\in I(r,t)}\{2^r\|\partial^\gamma\Omega^\beta G(s,\cdot)\|_k\}+\sum_{j=0}^{1}\sup_{s\in I(r,t)}\{2^r\|\partial^\gamma S^j G(s,\cdot)\|_k\}\right],$$

$I(r,t)=[2^{r-1},2^{r+1}]\cap[0,t]$, $r=1,2,\dots$ *and* $I(0,t)=[0,2]\cap[0,t]$.

Before we begin the proof, we shall note that without loss of generality in Theorem 2.1 we can assume that

$$\left\{\begin{array}{l} k=0, \quad f_1(x)\equiv f_2(x)\equiv 0 \quad \text{and} \\ \operatorname{supp} u\cup\operatorname{supp} G\subset\{(t,x):|x|\le t-1\}. \end{array}\right. \tag{7}$$

The verification of this fact relies on the a priori estimate of W. von Wahl [19] and the circumstance that the norms considered are equivalent under bounded in t transformations. Moreover, the commutation proprties of the generators Γ and Z with the d'Alembertian make possible the assumption $k=0$. The fact supp $u\subset K(-1)$ is an obvious consequence of the uniqueness theorem (see [5]).

In the proof of Theorem 2.1 it is convinient to represent the solution $u(t,x)$ by an osculating integral. The following lemma is valid.

Lemma 2.1 *Let $u(t,x)$ be a smooth solution of problem* (4), (5) *and let assumption* (7) *holds. Then*

$$u(t,x)=(2n)^{-3}\operatorname{Im}J(t,x), \tag{8}$$

where

$$J(t,x)=\int_{R^3}\widehat{G}(|\xi|,\xi;t)e^{i(t|\xi|+x\xi)}|\xi|^{-1}d\xi,$$

$$\widehat{G}(\tau,\xi;t)=\int_0^t e^{-is\tau}\widetilde{G}(s,\xi)ds \tag{9}$$

and

$$\widetilde{G}(s,\xi)=\int_{R^3}G(s,y)e^{-iy\xi}dy. \tag{10}$$

Proof. See for instance [1].

The proof of the $L^\infty-L^2$ estimates of $u(t,x)$ is based on the following three technical lemmas.

Lemma 2.2 *There exists a constant $C>0$ such that for any rapidly decreasing function $g(x)$ the following inequalities are valid:*

$$\int_0^\infty \rho\int_{|\omega|=1}|e_3\times\omega|^{-k}|\tilde{g}(\rho\omega)|d\sigma d\rho\le C\sum_{\gamma\le 1}\sum_{\beta\le k}\left(\int_{R^3}|\partial^\gamma\Omega^\beta g|^2dx\right)^{1/2}, \tag{11}$$

$$(12)\qquad \int_0^\infty \rho \sup_{|\omega|=1} |\tilde{g}(\rho\omega)| d\rho \leq C \sum_{\gamma\leq 1}\sum_{\beta\leq 2}\left(\int_{R^3} |\partial^\gamma \Omega^\beta g|^2 dx\right)^{1/2},$$

where $k = 0, 1$; $\tilde{g}(\xi)$ *is the Fourier transform of the function* $g(x)$; $\xi = \rho\omega$, $\omega = (\omega_1, \omega_2, \omega_3)$ *is a vector of the unit sphere;* $\mathbf{e}_3 = (0, 0, 1)$; $\mathbf{e}_3 \times \omega$ *is the vector product of* $\mathbf{e}_3$ *and* ω.

Proof. A fundamental role in the proof of the above inequalities is played by the respective analogue of Sobolev's inequality on the uniit sphere $W = \{\omega : |\omega| = 1\}$. More prcisely, taking into account the fact that the vector fields on W can be represented by means of the generators $\Omega_{jk}, 1 \leq j \leq k \leq 3$, we check that for $1 < \rho < \infty$

$$(13)\qquad \left(\int_{|\omega|=1} |f(\omega)|^\rho d\sigma\right)^{1/\rho} \leq C\left(\sum_{\beta\leq 1}\int_{|\omega|=1} |\Omega^\beta f(\omega)|^2 d\sigma\right)^{1/2},$$

$$(14)\qquad |f(\omega)| \leq C\left(\sum_{|\beta|\leq 2}\int_{|\omega|=1} |\Omega^\beta f|^2 d\sigma\right)^{1/2} \qquad \text{for} \quad \omega \in W,$$

where $C = \text{const.} > 0$ and f is a function for which the right–hand sides if the inequalities have sense. Moreover, for an arbitrary $\rho > 2$, by means of Holder's inequality and (13) we prove that

$$(15)\int_{|\omega|=1} \frac{|f(\omega)|}{|\mathbf{e}_3 \times \omega|} d\sigma \leq \left(\int_{|\omega|=1} |\mathbf{e}_3 \times \omega|^{\frac{\rho}{1-\rho}} d\sigma\right)^{\frac{\rho-1}{\rho}} \left(\int_{|\omega|=1} |f(\omega)|^\rho d\sigma\right)^{1/\rho}$$
$$\leq C\left(\sum_{|\beta|\leq 1}\int_{|\omega|=1} |\Omega^\beta f(\omega)|^2 d\sigma\right)^{1/2}$$

By means of (15) or (14) we estimate above the left–hand sides of inequalities (11) and (12) by

$$\int_0^\infty \rho F(\rho) d\rho, \quad \text{where} \quad F(\rho) = \left(\sum_{|\beta|\leq j}\int_{|\omega|=1} |(\Omega^\beta g)\tilde{\ }(\rho\omega)|^2 d\sigma\right)^{1/2}.$$

and j is equal respectively to k or 2. Note that from the form of the function F it follows that

$$\int_0^\infty \rho F(\rho) d\rho \leq C\left(\sum_{|\beta|\leq j}\int_{R^3} (1 + |\xi|^2)|(\Omega^\beta g)\tilde{\ }(\xi)|^2 d\xi\right)^{1/2}.$$

Then by Plancherel's theorem we complete the proof of (11) and (12). ∎

By means of analogous arguments the following lemma is proved.

Lemma 2.3 *There exists a constant $C > 0$ such that for any rapidly decreasing function $g \in C_0^\infty(R^3)$ the following estimates are valid:*

$$\int_{|\omega|=1} |R(g)(\tau,\omega)||e_3 \times \omega|^{-k} d\sigma \tag{16}$$
$$\leq C(1+B)^{1/2} \left(\sum_{|\beta|\leq k} \int_{R^3} |(\Omega^\beta g(x)|^2 dx \right)^{1/2},$$

$$|R(g)(\tau,\omega)| \leq C(1+B)^{1/2} \left(\sum_{|\beta|\leq 2} \int_{R^3} |(\Omega^\beta g(x)|^2 dx \right)^{1/2}, \tag{17}$$

where $B = \sup\{|x| : g(x) \neq 0\}$; $k = 0, 1$ and $|R(g)(\tau,\omega)| = \int_{x.\omega=\tau} g(x) d\sigma_x$ is the Radon transform of the function g.

Proof. First we shall note that supp $g \subset \{x : |x| \leq B\}$ it follows immediately that the Radon transform $R(g)(\tau,\omega)$ has a compact support in τ, i.e.

$$\text{supp } R(g)(.,\omega) \subset \{\tau : |\tau| \leq B\}.$$

Consequently,

$$|R(g)(\tau,\omega)| \leq C(1+B)^{1/2} \left(\int_{-\infty}^{+\infty} |\partial_s R(g)(s,\omega)|^2 ds \right)^{1/2}.$$

Then by means of Plancherel's equality for the Radon transform (see [2]) we check that

$$\begin{aligned} \int_{|\omega|=1} R(g)(\tau,\omega)| &\leq C(1+B)^{1/2} \left(\int_{-\infty}^{+\infty} \int_{|\omega|=1} |\partial_s R(g)(s,\omega)|^2 d\sigma ds \right)^{1/2} \\ &= C(1+B)^{1/2} \left(\int_{R^3} |g(x)|^2 dx \right)^{1/2}, \end{aligned}$$

which completes the proof of (16) for $k = 0$.

In the proof of (16) for $k = 1$ and (17), first by means respectively of (15) and (14) we estimate above their left–hand sides by

$$C(1+B)^{1/2} \left(\sum_{|\beta|\leq j} \int_{-\infty}^{+\infty} \int_{|\omega|=1} |\partial_s R(\Omega^\beta g)(s,\omega)|^2 d\sigma ds \right)^{1/2},$$

where j is equal respectively to 1 or 2. Then Plancherel's equality completes the proof of the lemma. ∎

In the strict proof of $L^\infty - L^2$ estimates Lemmas 2.2 and 2.3 are repeatedly used. For the sake of brevity of the exposition, in a separate lemma, we shall analyse the cases met the most frequently.

Lemma 2.4 *For any real-valued smooth function* $F(t,x)$ *such that* supp $F(t,.) \subset \{x : |x| \leq t-1\}$ *the following inequalities are valid:*

$$\left| \int_0^\infty \int_0^{2\pi} \int_0^\pi \rho \left| \frac{x}{|x|} \times \omega \right|^k f(\omega) \widetilde{F}(s,\rho\omega) e^{i\rho\tau} d\theta d\varphi d\rho \right| \leq CM \sum_{|\beta|\leq 1-k} \sum_{|\gamma|\leq 1} \|\partial^\gamma \Omega^\beta F(s,.)\|, \tag{18}$$

$$\left| \int_0^\infty \rho \widetilde{F}(s,\rho\omega) f(\omega) e^{i\rho\tau} d\rho \right| \leq CM \sum_{|\beta|\leq 2} \sum_{|\gamma|\leq 1} \|\partial^\gamma \Omega^\beta F(s,.)\|, \tag{19}$$

$$\left| \mathrm{Re} \int_0^\infty \int_0^{2\pi} \int_0^\pi \left| \frac{x}{|x|} \times \omega \right|^k f(\omega) \widetilde{F}(s,\rho\omega) e^{i\rho\tau} d\theta d\varphi d\rho \right| \leq CM \sum_{|\beta|\leq 1-k} (1+s)^{1/2} \|\Omega^\beta F(s,.)\|, \tag{20}$$

$$\left| \mathrm{Re} \int_0^\infty f(\omega) \widetilde{F}(s,\rho\omega) e^{i\rho\tau} d\rho \right| \leq CM \sum_{|\beta|\leq 2} (1+s)^{1/2} \|\Omega^\beta F(s,.)\|, \tag{21}$$

where $\omega = (\sin\theta \sin\varphi, \sin\theta\cos\varphi, \cos\theta)$; $k = 0,1$; $C =$ const. > 0; $f(\omega)$ *is a real-valued function such that* $\sup |f(\omega)| \leq M$ *and* $\widetilde{F}(s,\rho\omega)$ *is the partial Fourier transform defined in* (10).

Proof. a) The left–hand side of inequalities (18) and (19) are estimated by an integral of the module of function under the sign of the integral. Then, using the obvious inequality $|e^{i\rho\tau} f(\omega)| \leq M$ and Lemma 2.2 we prove (18) and (19).

b) In the proof of (20) and (21) we transform the integrand by means of the equality

$$\begin{aligned} \mathrm{Re} \int_0^\infty \widetilde{F}(s,\rho\omega) e^{i\rho\tau} d\rho &= \frac{1}{2} \int_{-\infty}^{+\infty} \widetilde{F}(s,\rho\omega) e^{i\rho\tau} d\rho \\ &= \int_{x.\omega=\tau} F(s,x) d\sigma_x = R(F(s,.))(\tau,\omega). \end{aligned}$$

This equality is a consequence of the definitions of the Fourier and Radon transforms and of the assumption that F is a real–valued (see [2]). Then the left–hand side of (20) is estimated above by

$$CM\int_{|\omega=1}\left|\frac{x}{|x|}\times\omega\right|^{k-1}|R(F(s,.))(\tau,\omega)|d\sigma.$$

Estimating the above integral by means of Lemma 2.3 [inequality (16)] we prove (20). Inequality (21) is proved quite analoguously by means of (17). Lemma 2.4 is proved. ∎

Proof of Theorem 2.1. Suppose that assumption (7) is valid. Then for $t\leq 1$ inequality (6) is obvious since $u(t,x)\equiv 0$.

Consider the case $t\geq 1$. Since the respective norms are equivalent under rotations around the zero, then it suffices to prove (6) for $x=(0,0,|x|)$. Using the representation of Lemma 2.1, we obtain that

$$D^{\alpha}u(t,x)=(2\pi)^{-3}\mathrm{Re}J_{\gamma}(G;t,x), \tag{22}$$

where $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$, $\gamma=(\alpha_1,\alpha_2,\alpha_3)$, $|\alpha|=1$ and

$$J_{\gamma}(G;t,x)=\int_{R^3}\widehat{G}(|\xi|,\xi;t)\left(\frac{\xi}{|\xi|}\right)^{\gamma}e^{i(t|\xi|+x\xi)}d\xi.$$

In fact, we obtain

$$D^{\alpha}u(t,x)=(2\pi)^{-3}\left[\mathrm{Re}J_{\gamma}(G;t,x)+\alpha_0\mathrm{Im}I_0(G;t,x)\right],$$

where

$$I_0(G;t,x)=\int_{R^3}\widetilde{G}(t,\xi)|\xi|^{-1}e^{ix\xi}d\xi.$$

Since $G(t,x)$ is real–valued, then

$$\overline{I_0(G;t,x)}=\int_{R^3}\widetilde{G}(t,-\xi)|\xi|^{-1}e^{-ix\xi}d\xi=I_0(G;t,x)$$

and, consequently, $\mathrm{Im}I_0(G;t,x)=0$.

In the integrals J_{γ}, $|\gamma|\leq 1$, we perform a polar change of the coordinates

$$\begin{cases}\xi=\rho\omega, & \omega=(\sin\theta\sin\varphi,\sin\theta\cos\varphi,\cos\theta),\\ 0\leq\theta<\pi, & 0\leq\varphi<2\pi,\quad 0\leq\rho<\infty\end{cases} \tag{23}$$

and obtain

$$(24)\quad J_\gamma(G;t,x)=\int_0^\infty\int_0^{2\pi}\int_0^\pi \widehat{G}(\rho,\rho\omega;t)\omega^\gamma e^{i\rho(t+|x|\cos\theta)}\rho^2\sin\theta d\theta d\varphi d\rho.$$

In the estimate of $|\mathrm{Re}J_\gamma|$ consider separately the cases $2|x|\geq t$ and $2|x|<t$.

a) Let $2|x|\geq t$. After an inntegration by parts in (24) with respect to the variable θ we obtain

$$(25)\qquad J_\gamma(G;t,x)=\frac{1}{i|x|}(-I_1(t,x)+I_2(t,x)+I_3(t,x)),$$

where $I_3(t,x)$ is equal to

$$\int_0^\infty\int_0^{2\pi}\int_0^\pi \rho e^{i\rho(t+x\omega)}\left[\widehat{G}\partial_\theta(\omega^\gamma)+\omega^\gamma\eta.(\Lambda G)^\frown\right]d\theta d\varphi d\rho,$$

$$I_j(t,x)=2\pi\left(\frac{\varepsilon_j x}{|x|}\right)^\gamma\int_0^\infty \rho\widehat{G}(\rho,\frac{\varepsilon_j\rho x}{|x|};t)e^{i\rho(t+\varepsilon_j|x|)}d\rho,$$

$\eta=\left(\frac{x}{|x|}\times\omega\right)\left|\frac{x}{|x|}\times\omega\right|^{-1}=(\omega_1^2+\omega_2^2)(-\omega_2,\omega_1;0)$; $j=1,2$; $\varepsilon_1=-1$; $\varepsilon_2=1$; and $\Lambda=(\Omega_{23},-\Omega_{13},\Omega_{12})$. We shall stress that in $I_3(t,x)$ we have used the identity

$$\frac{\partial}{\partial\theta}\widehat{G}(\rho,\rho\omega;t)=\left|\frac{x}{|x|}\times\omega\right|^{-1}\left(\frac{x}{|x|}\times\omega\right).(\Lambda G)^\frown(\rho,\rho\omega;t)$$

which is verified immediately.

By means of Lemma 2.4 we estimate $|I_k(t,x)|$ and obtain that

$$(27)\quad |J_\gamma(G;t,x)|\leq\frac{1}{|x|}\sum_{k=1}^3|I_k(t,x)|\leq\frac{C}{|x|}\sum_{|\gamma|\leq1}\sum_{|\beta|\leq2}\int_0^t\|\partial^\gamma\Omega^\beta G(s,.)\|ds.$$

By means of (22) and (27) we prove that if $2|x|\geq t$, then

$$(28)\qquad \sum_{|\alpha|=1}|D^\alpha u(t,x)|\leq\frac{C}{t}\sum_{|\gamma|\leq1}\sum_{|\beta|\leq2}\int_0^t\|\partial^\gamma\Omega^\beta G(s,.)\|ds$$

$$\leq\frac{C}{t}\sum_{r=1}^\infty\sum_{|\gamma|\leq1}\sum_{|\beta|\leq2}\sup_{s\in I(r,t)}\{2^r\|\partial^\gamma\Omega^\beta G(s,.)\|\},$$

where $I(r,t) = [2^{r-1}, 2^{r+1}] \cap [0,t]$ and $C = \text{const.} > 0$. In the last inequality we use the obvious relation

$$\int_0^t \|G(s,.)\|ds \le \sum_{r=1}^{\infty} \int_{I(r,t)} \|G(s,.)\|ds \le 2\sum_{r=1}^{\infty} \sup_{I(r,t)} 2^r \|G(s,.)\|.$$

(b) Let $2|x| \le t$. In the expression for J_γ integrate by parts with respect to the variable ρ and obtain

$$(29)\quad J_\gamma(G;t,x) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{i\omega^\gamma e^{i\rho(t+|x|\cos\theta)}\sin\theta}{t+|x|\cos\theta}\rho[2\widehat{G} + \rho\partial_\rho\widehat{G}]d\rho d\theta d\varphi.$$

Note that $\rho\partial_\rho\widehat{G}$ can be represented in the following two ways:

(i) by the boosts $L = (L_1, L_2, L_3)$,

$$(30)\quad \begin{aligned} \rho\partial_\rho\widehat{G}(\rho,\rho\omega;t) &= \sum_{j=1}^{3}\omega_j L_j\widehat{G}(\rho,\rho\omega;t) \\ &= \sum_{j=1}^{3}\omega_j\left[e^{-it\rho}(x_jG)\widetilde{\ }(t,\rho\omega) - (L_jG)\widehat{\ }(\rho,\rho\omega;t)\right]; \end{aligned}$$

(ii) by the scaling S,

$$(31)\quad \begin{aligned} \rho\partial_\rho\widehat{G}(\rho,\rho\omega;t) &= S\widehat{G}(\rho,\rho\omega;t) \\ &= te^{-it\rho}\widetilde{G}(t,\rho\omega) - (4G+SG)\widehat{\ }(\rho,\rho\omega;t). \end{aligned}$$

We continue (29) by means of (30) or (31). The modules of the integrals obtained can be estimated by means of Lemma 2.4 since

$$\left|\frac{\omega^\gamma}{t+|x|\omega_3}\right| \le 2/t.$$

As a result we obtain

$$(32)\quad |J_\gamma(G;t,x)| \le C\left\{\sum_{|\gamma|\le 1}\|\partial^\gamma G(t,.)\| + \frac{1}{t}\int_0^t f(s)ds\right\},$$

where $f(s)$ is any of the functions

$$(33)\quad \sum_{|\gamma|\le 1}\sum_{|\beta|\le 1}\|\partial^\gamma L^\beta G(s,.)\| \quad \text{or} \quad \sum_{|\gamma|\le 1}\sum_{j\le 1}\|\partial^\gamma S^j G(s,.)\|.$$

The estimate obtained enables us, analogously to point (a), to prove that for $2|x| < t$

$$\sum_{|\alpha|=1} |D^\alpha u(t,x)| \le \frac{C}{t} \sum_{r=1}^{\infty} \sup_{s\in I(t,r)} \{2^r f(s)\}, \tag{34}$$

where $f(s)$ is any of the functions (33).

Inequalities (28) and (34) prove (6) in the case $t \ge 1$ as well, thus Theorem 2.1 is proved. ∎

The technique elaborated above enables us to make precise Theorem 2.1. The following theorem is valid.

Theorem 2.2 *Let $u(t,x)$ be a smooth solution of problem* (4), (5). *Then the following estimates hold*

$$(1+t+|x|)|u(t,x)|_k \le C(\|D^{2+k}u(0,.)\| + A_0(G;t)), \tag{35}$$

$$(1+t+|x|)(1+|t-|x||) \sum_{|\alpha|=1} |D^\alpha u(t,x)|_k \le C\left[\|D^{3+k}u(0,.)\| + A_1(G;t)\right], \tag{36}$$

where $C = \text{const.} > 0$, $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $A_0(G,t)$ *is any of the functions*

$$\sum_{r=0}^{\infty} \sum_{|\beta|\le 1, |\gamma|\le 2} \sup_{s\in I(r,t)} \left\{2^{3r/2}(\|L^\beta G(s,.)\|_k + \|\Omega^\gamma G(s,.)\|_k)\right\},$$

$$\sum_{r=0}^{\infty} \sum_{j\le 1, |\gamma|\le 2} \sup_{s\in I(r,t)} \left\{2^{3r/2}(\|S^j G(s,.)\|_k + \|\Omega^\gamma G(s,.)\|_k)\right\},$$

and $A_1(G,t)$ *is any of the functions*

$$\sum_{r=0}^{\infty} \sum_{\substack{|\gamma|+|\beta|+|\delta|\le 3 \\ |\gamma|\le 2, |\beta|\le 2, |\delta|\le 1}} \sup_{s\in I(r,t)} \left\{2^{3r/2}(\|\partial^\delta \Omega^\beta L^\gamma G(s,.)\|_k\right\}$$

$$+(1+t)^{3/2} \sum_{|\beta|=1} \||x|\partial^\beta G(t,.)\|_k,$$

$$\sum_{r=0}^{\infty} \sum_{\substack{j+|\beta|+|\delta|\leq 3 \\ j\leq 2,|\beta|\leq 2,|\delta|\leq 1}} \sup_{s\in I(r,t)} \left\{2^{3r/2}(\|\partial^{\delta}\Omega^{\beta}S^{j}G(s,.)\|_k\right\}$$

$$+(1+t)^{3/2}\left[\sum_{|\beta|=1} \||x|\partial^{\beta}G(t,.)\|_k + t^{1/2}\sum_{|\beta|\leq 1}\|\partial^{\beta}G(t,.)\|_k\right],$$

$$\sum_{r=0}^{\infty} \sum_{\substack{j+|\beta|+|\gamma|+|\delta|\leq 3 \\ j\leq 1,|\beta|\leq 2,|\delta|\leq 1,|\gamma|\leq 1}} \sup_{s\in I(r,t)} \left\{2^{3r/2}(\|\partial^{\delta}\Omega^{\beta}S^{j}L^{\gamma}G(s,.)\|_k\right\}$$

$$+(1+t)^{3/2}\sum_{|\beta|=1}\||x|\partial^{\beta}G(t,.)\|_k,$$

$I(r,t) = [2^{r-1}, 2^{r+1}] \cap [0,t], \quad r = 1,2,\ldots$ *and* $I(0,t) = [0,2] \cap [0,t]$.

Proof. First we shall note that without loss of generality we can assume that assumption (7) is valid. The verification of this fact is based on the equivalence of the respective norms under bounded wuth respect to t translations and the $L^{\infty} - L^{1}$ estimate of S. Klainerman [8]. Moreover, the respective norms are equivalent under rotations around the zero. All this shows that it suffices to consider the case $x = (0,0,|x|)$ and $t \geq 1$.

(a) First we shall prove (35) (see [1], Theorem 2). Passing to polar coordinates (23) in the representation (8) of the solution $u(t,x)$, we have,

$$u(t,x) = (2\pi)^{-3}\mathrm{Im}J(t,x),$$

where $J(t,x)$ is equal to

$$\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \widehat{G}(\rho,\rho\omega;t)e^{i\rho(t+|x|\cos\theta)}\rho\sin\theta d\theta d\varphi d\rho.$$

Analogously to Theorem 2.1, integrating by parts witrh respect to the variable ρ and the variable θ, we check that

(a1) for $2|x| < t$

$$(37) \qquad J(t,x) = i\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \frac{\left|\frac{x}{|x|}\times\omega\right| e^{i\rho(t+x.\omega)}}{t+x.\omega}[\widehat{G}+\rho\partial_{\rho}\widehat{G}]d\theta d\varphi d\rho.$$

(a2) for $2|x| \geq t$

$$J(t,x) = \frac{1}{i|x|}[-J_1(t,x) + J_2(t,x) + J_3(t,x)],$$

where

$$J_j(t,x) = 2\pi \int_0^t \int_0^\infty \widetilde{G}(s, \frac{\varepsilon_j \rho x}{|x|}) e^{i\rho(t-s+\varepsilon_j|x|)} d\rho ds,$$

$$J_3(t,x) = \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^\pi \eta \Lambda \widetilde{G}(s, \rho\omega) e^{i\rho(t-s+\varepsilon_j|x|)} d\theta d\varphi d\rho ds,$$

$\eta = (\omega^2 + \omega^2)^{-1/2}(-\omega_2, \omega_1, 0)$; $j = 1,2$; $\varepsilon_1 = -1$; $\varepsilon_2 = 1$ and $\Lambda = (\Omega_{23}, -\Omega_{13}, \Omega_{12})$.

Estimating by means of Lemma 2.4 the integrals $\mathrm{Re} J_j$ we prove that for $2|x| \geq t$

$$\begin{aligned} (38) \qquad |\mathrm{Im} J(t,x)| &\leq \frac{1}{|x|} \sum_{j=1}^{3} |\mathrm{Re} J_j(t,x)| \\ &\leq \frac{C}{|x|} \sum_{|\beta| \leq 2} \int_0^t (1+s)^{1/2} \|\Omega^\beta G(s,.)\| ds \\ &\leq \frac{C}{|x|} \sum_{r=1}^{\infty} \sup_{s \in I(r,t)} \left\{ 2^{3r/2} \sum_{|\beta| \leq 2} \|\Omega^\beta G(s,.)\| ds \right\}. \end{aligned}$$

When $2|x| < t$, we transform (37) by means of relations (30) or (31) and then by means of Lemma 2.4 we prove

$$\begin{aligned} (39) \qquad |\mathrm{Im} J(t,x)| &\leq C\{(1+t)^{1/2} \|G(t,.)\| + \\ &\quad + \sum_{|\beta| \leq 1} \frac{2}{t} \int_0^t (1+t)^{1/2} \|L^\beta G(t,.)\|\} \\ &\leq \frac{C}{t} \sum_{r=1}^{\infty} \sup_{s \in I(r,t)} \left\{ 2^{3r/2} \sum_{|\beta| \leq 1} \|L^\beta G(s,.)\| ds \right\} \end{aligned}$$

and, analogously,

$$(40) \qquad |\mathrm{Im} J(t,x)| \leq \frac{C}{t} \sum_{r=1}^{\infty} \sup_{s \in I(r,t)} \left\{ 2^{3r/2} \sum_{j \leq 1} \|S^j G(s,.)\| ds \right\}.$$

Then inequalities (38), (39) and (40) complete the proof of (35).

(b) We shall prove (36) using the results and notations of Theorem 2.1. First note that if $|t-|x|| \leq 1$, then (36) is an immediate consequence of Theorem 2.1. In the study of the case $|t-|x||>1$ we consuder separately the cases (b1) $2|x|<t$ and (b2) $2|x| \geq t$. In view of equality (22), we estimate $|\mathrm{Re}J_\gamma(G;t,x)|$.

(b1) Let $2|x|<t$. In equality (29) we integrate by parts with respect to ρ. The integral obtained has an integrand

$$\frac{\omega^\gamma}{(t+x.\omega)^2}\left|\frac{x}{|x|}\times\omega\right| e^{i\rho(t+x.\omega)}[2\widehat{G}+3\rho\partial_\rho\widehat{G}+(\rho\partial_\rho)^2\widehat{G}].$$

By means of (30) and (31) we represent the function
$H=2\widehat{G}+3\rho\partial_\rho\widehat{G}+(\rho\partial_\rho)^2\widehat{G}$:

(i) by the boosts L; (ii) by the scaling S; (iii) by boosts L and the scaling S of degree not greater than 1.

Analogously to Theorem 2.1, we estimate the integrals obtained by means of Lemma 2.4 and get

$$|\mathrm{Re}J_\gamma(G;t,x)| \leq C|t-|x||^{-2}A_{11}(t), \tag{41}$$

where $A_{11}(t)$ is any of the following three functions

$$\sum_{|\beta|\leq 2}\int_0^t (1+s)^{1/2}\|L^\beta G(s,.)\|ds+$$
$$+(1+t)^{1/2}\left[\sum_{|\beta|\leq 1}\||x|L^\beta G(t,.)\|+\sum_{j=1}^{2}\sum_{|\beta|=1}\||x|^j\partial^\beta G(t,.)\|+\|G(t,.)\|\right],$$

$$\sum_{j=0}^{2}\int_0^t (1+s)^{1/2}\|S^j G(s,.)\|ds+t^2\sum_{|\beta|\leq 1}\|\partial^\beta G(t,.)\|+$$
$$+(1+t)^{3/2}\left(\sum_{|\beta|\leq 1}\||x|^{|\beta|}\partial^\beta G(t,.)\|+\|SG(t,.)\|\right),$$

$$\sum_{j+|\beta|\leq 1}\int_0^t (1+s)^{1/2}\|S^j L^\beta G(s,.)\|ds$$
$$+t\sum_{j=0}^{1}\sum_{|\beta|\leq 1}\||x|^j\partial^\beta G(t,.)\| \quad + \quad (1+t)^{3/2}\sum_{|\beta|\leq 1}\|L^\beta G(t,.)\|+$$
$$+(1+t)^{1/2}\sum_{|\beta|\leq 1}\||x|^{1+|\beta|}\partial^\beta G(t,.)\|.$$

Taking into account the assumptions $2|x| < t$ and supp $G \subset= \{(t,x); |x| < t-1\}$, by means of (41) we conclude the verification of (36).

(b2) Let $2|x| \geq 1$. In equality (25) we integrate by parts with respect to the variable ρ and obtain

$$J_\gamma(G;t,x) = \frac{1}{|x|}(-J_1(t,x) + J_2(t,x) + J_3(t,x)),$$

where $J_3(t,x) = \displaystyle\int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{e^{i\rho(t+x.\omega)}}{t+x.\omega} H_1(t,x,\rho,\omega) d\theta d\varphi d\rho,$
$H_1(t,x,\rho,\omega) = (\partial_\theta \omega^\gamma)(\widehat{G} + \rho\partial_\rho \widehat{G}) + \omega^\gamma \eta.((\Lambda G)\widehat{} + \rho\partial_\rho(\Lambda G)\widehat{}),$
$\eta = (\omega_1^2 + \omega_2^2)^{-1/2}(-\omega_2, \omega_1, 0), \quad \Lambda = (\Omega_{23}, -\Omega_{13}, \Omega_{12})$
and for $j = 1,2$ the function $J_j(t,x)$ is equal to

$$2\pi \left(\frac{\varepsilon_j x}{|x|}\right)^\gamma \frac{1}{t+\varepsilon_j|x|} \int_0^\infty e^{i\rho(t+\varepsilon_j|x|)} \partial_\rho \left(\rho \widehat{G}(\rho, \frac{\varepsilon_j \rho x}{|x|}; t)\right) d\rho,$$

where $\varepsilon_1 = -1; \quad \varepsilon_2 = 1.$

As above, we express the integrands in J_j $(j = 1,2,3)$ by: (i) the boosts; (ii) the scaling. We estimate the integrals obtained by means of Lemma 2.4 and prove

$$|\mathrm{Re} J_\gamma(G;t,x)| \leq \frac{C}{|x||t-|x||} A_{12}(t), \tag{42}$$

where $A_{12}(t)$ is any of the following fuctions

$$t(1+t)^{1/2} \sum_{|\beta|\leq 2} \|\Omega^\beta G(t,.)\| + \sum_{|\zeta|\leq 2, |\beta|\leq 1 |\zeta|+|\beta|\leq 3} \int_0^t (1+s)^{1/2} \|\Omega^\zeta L^\beta G(s,.)\| ds,$$

$$t(1+t)^{1/2} \sum_{|\beta|\leq 2} \|\Omega^\beta G(t,.)\| + \sum_{\substack{|\gamma|+j\leq 3 \\ |\gamma|\leq 2, j\leq 1}} \int_0^t (1+s)^{1/2} \|\Omega^\gamma S^j G(s,.)\| ds.$$

Using the obvious inequality

$$t(1+t)^{1/2}|f(t)| + \int_0^t (1+s)^{1/2}|f(s)| ds \leq C \sum_{r=0}^\infty \sup_{I(r,t)} 2^{3r/2}|f(s)|,$$

we complete the proof of (36).

Theorem 2.2 is proved. ∎

Corollary 2.5 *Let the assumptions of Theorem 2.2 be valid. If* $G(t,x) = \sum_{\mu=0}^{3} \partial_\mu G^\mu(t,x)$ *and* supp $G^\mu(t,x) \subset \{(t,x); |x| \le t+B\}$, *then*

$$|u(t,x)|_k \le C(1+t)^{-1}\left\{\|D^{k+3}u(0,.)\| + \sum_{\mu=0}^{3} A(G^\mu;t)\right\}, \tag{43}$$

$$|u(t,x)|_k \le C\left[(1+t+|x|)(1+t-|x|)\right]^{-1}\left[\sum_{\mu=0}^{3} A_1(G^\mu;t) + \|D^{k+3}u(0,.)\|\right], \tag{44}$$

where $C =$ const. > 0 *and* $A(G^\mu;t)$ *are defined respectively in Theorem 2.1 and Theorem 2.2.*

Proof. Without loss of generality we can assume that assumptionn (7) is valid. Represent the solution of (4), (5) by the equality

$$u(t,x) = \sum_{\mu=0}^{3} \partial_\mu v_\mu(t,x),$$

where $v_\mu(t,x)$ is a solution of the Caushy problem

$$\Box v(t,x) = G^\mu(t,x), \quad v(0,x) \equiv \partial_0 v(0,x) \equiv 0. \tag{45}$$

We estimate $|\partial_\mu v_\mu(t,x)|$ by means of (6) and (36) respectively and complete the proof. ■

In the proof of Theorem 1 we shall use the following well known conformal estimate.

Lemma 2.6 *Let the assumptions of Theorem 2.1 hold. Then the following estimates are valid:*
(1) *If* supp $G \subset \{(t,x); |x| \le t+R\} = K(R)$, *then*

$$\sum_{|\alpha|=1}^{3} \|D^\alpha u(t,.)\|_k \le C\left(\sum_{j=0}^{3} \|\partial_j u(0,.)\|_k + \int_0^t \|G(s,.)\|_k ds\right); \tag{46}$$

(2) *If* $G(t,x) = \sum_{j=0}^{3} \partial_j G^j(t,x)$ *and* $\cup_{j=0}^{3}$supp $G^j \subset K(R)$, *then*

$$\|u(t,.)\|_k \le C\left(\|u(0,.)\|_k + \sum_{j=0}^{3}\int_0^t \|G^j(s,.)\|_k ds\right). \tag{47}$$

Proof. (See [9]). ∎

3. Global solvability of Caushy problem for the nonlinear wave equation

First we shall note that the local solvability and uniqueness of problem (1), (2_ε) is proved (see e.g. [17]). This enables us to construct a family of functions

$$U = \{u_\varepsilon(t,x); 0 < \varepsilon \leq 1\}$$

such that $u_\varepsilon(t,x)$ is a solution of the Caushy problem (1), (2_ε) in a strip $S(s) = [0,s) \times R^3$. For any $\varepsilon \in (0,1]$ we define

$$t_\varepsilon = \max\{s\,;\ u_\varepsilon(t,x) \text{ satisfies (1) in } [0,s) \times R^3\}.$$

From the continuation principle proved by A. Majda in [12] (see also [14]) it follows that just one of the following cases is possible:

(i) $t_\varepsilon < \infty$ and $\displaystyle\lim_{t\to t_\varepsilon} \sum_{|\alpha|\leq 1} \sup_{x\in R^3} \{|D^\alpha u_\varepsilon(t,x)|\} = \infty$

(ii) $t_\varepsilon = \infty$.

Consequently, the proof of Theorem 1 is reduced to the proof of the following assertion:

(P) There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$ from $t_\varepsilon < \infty$ it follows that

$$\sum_{|\alpha|\leq 1} \sup_{(t,x)\in S(t_\varepsilon)} \{|D^\alpha u_\varepsilon(t,x)|\} < \infty.$$

We shall cary out the proof of Theorem 1 in two steps. In step 1 we shall prove assertion (P) if at least one of assumptions (H1) or (H2) are satisfied, and in step 2 we shall consider the case when assumption (H3) is satisfied.

Step 1. The proof of assertion (P) when at least one of the asumptions (H1) or (H2) is fulfilled, is carried out in one and the same way. It is based on the apriory estimates of Section 2 and the following auxiliary lemma.

Lemma 3.1 *For any $\varepsilon \in (0,1]$ let a couple of nonnegative functions $f_\varepsilon(t)$, $e_\varepsilon(t)$ be defined and continuous on the interval $[0,t_\varepsilon)$. Suppose that there exist independent of ε connstants δ, C_j $(j = 1,2,3)$ such that:*

$$e_\varepsilon(0) + f_\varepsilon(0) \leq C_1\varepsilon; \tag{48}$$

(49) *If for any* $s \in [0,t]$ *the inequalities*

$$a(1+s)^a \geq f_\varepsilon(s) \quad \text{and} \quad e_\varepsilon(s) \leq a \quad \text{hold, then}$$

$$e_\varepsilon(s) \leq C_2\varepsilon \quad \text{for} \quad s \in [0,t),$$

where $a \in (0,\delta)$ *and* $t \in (0,t_\varepsilon)$;

(50) *If for any* $s \in [0,t]$ *the inequality*

$$e_\varepsilon(s) \leq a \quad \text{holds, then}$$

$$f_\varepsilon(s) \leq C_3(\varepsilon + e_\varepsilon(1+s)^a) \quad \text{for} \quad s \in [0,t),$$

where $a \in (0,\delta)$ *and* $t \in (0,t_\varepsilon)$;

Then for any $a \in (0,\delta)$ *and any* $\varepsilon \in (0, \left(\frac{a}{2C}\right)^2)$ *we have*

$$e_\varepsilon(t) + (1+t)^{-a} f_\varepsilon(t) \leq 2C\sqrt{\varepsilon}, \tag{51}$$

where $C = C_1 + C_2 + C_3 + C_2C_3$ *and* $t \in [0,t_\varepsilon)$.

Proof. Fix arbitrary $a \in (0,\delta)$ and arbitrary $\varepsilon \in (0, \left(\frac{a}{2C}\right)^2)$. Define t_ε^0 equal to

$$\sup\left\{t : e_\varepsilon(s) + (1+s)^{-a} f_\varepsilon(s) \leq 2\varepsilon\sqrt{\varepsilon} \text{ for } s \in [0,t)\right\}.$$

From (48) it follows that $t_\varepsilon^0 > 0$. Suppose that there exists $\varepsilon \in (0, \left(\frac{a}{2C}\right)^2)$ such that $t_\varepsilon^0 < t_\varepsilon$. Then from the continuity of $e_\varepsilon(t)$ and $f_\varepsilon(t)$ it follows that

$$e_\varepsilon(t_\varepsilon^0) + (1+t_\varepsilon^0)^{-a} f_\varepsilon(t_\varepsilon^0) = 2C\sqrt{\varepsilon}. \tag{52}$$

Since $2C\sqrt{\varepsilon} < a$, then from the definition of t_ε^0 and (52) it follows that

$$(1+s)^{-a} f_\varepsilon(s) < a \quad \text{and} \quad e_\varepsilon(s) < a$$

for $s \in [0, t_\varepsilon^0]$. Then from conditions (49) and (50) it follows respectively that

$$e_\varepsilon(s) < C_2\varepsilon \tag{53}$$

and

$$f_\varepsilon(s) \leq C_3(\varepsilon + e_\varepsilon(s)(1+s)^{-a}). \tag{54}$$

We continue (52) by means of (53) and (54) and get a contradiction. Hence $t_\varepsilon^0 = t_\varepsilon$ and lemma is proved. ■

First we shall prove assertion (P) if condition (H1) holds. By means of the constants k of (H1), for any $\varepsilon \in (0,1]$ define the continuous in $[0,t_\varepsilon)$ functions

$$e_\varepsilon(t) = \sum_{j=0}^{3} \sup_{0\le s\le t} \sup_{y\in R^3} (1+s)|\partial_j u_\varepsilon(s,y)|_{[\frac{k+3}{2}]},$$

$$f_\varepsilon(t) = \sum_{j=0}^{3} \sup_{0\le s\le t} \|\partial_j u_\varepsilon(s,.)\|_{k+3}.$$

Note that assertion (P) will follow from Lemma 3.1 if we can prove that for the so defined functions $\{e_\varepsilon, f_\varepsilon\}$ conditions (48), (49) and (50) are met.

Obviously, condition (48) is met with a constant C_1 depending just on $g_1(x)$ and $g_2(x)$. Moreover, from condition (H1) it follows that

$$(55)\|F(s,x,u_\varepsilon,Du_\varepsilon)\|_{k+3} \le C_0(1+s)^{-\delta} \sum_{j,i=0}^{3} |\partial_j u_\varepsilon(s,x)|_{[\frac{k+3}{2}]}\|\partial_i u_\varepsilon(s,.)\|_{k+3},$$

where the constant δ is defined in (H1) and C_0 is an independent of ε positive constant.

Then from Theorem 2.1 it follows that for $t \in [0,t_\varepsilon)$

$$(1+t)\sum_{|\alpha|=1} |D^\alpha u_\varepsilon(t,x)|_k \le C_4 \left\{\varepsilon + \sum_{r=0}^{\infty} \sup_{I(r,t)} 2^r (1+s)^{-(1+\delta)} e_\varepsilon(s) f_\varepsilon(s)\right\}.$$

Connsequently, if $a \in (0,\delta/2)$, $a(1+s)^a \ge f_\varepsilon(s)$ and $e_\varepsilon(s) \le a$ for $s \in [0,t]$, then

$$(1+t)\sum_{|\alpha|=1} |D^\alpha u_\varepsilon(t,x)|_k \le C_4\varepsilon + C_5 e_\varepsilon(t) a,$$

where the constants C_4 and C_5 depend just on $g_1(x)$, $g_2(x)$, $q_{i,j}(t,x)$ and k. The last inequality proves condition (49) with $C_2 = 2C_4$ and $a \in (0,\delta_1)$ for $\delta_1 = \min\{\delta/2, 1/(2C_5)\}$.

Let us check condition (50). From the conformal estimate (46) it follows that

$$\sum_{j=0}^{3} \|\partial_j u_\varepsilon(t,.)\|_{k+3} \le C_6 \left\{\varepsilon + \int_0^t \|F(s,x,u_\varepsilon,Du_\varepsilon)\|_{k+3} ds\right\}.$$

We conntinue the inequality obtained by means of (55) and prove that

$$f_\varepsilon(t) \le C_7 \left\{\varepsilon + \int_0^t (1+s)^{-(1+\delta)} f_\varepsilon(s) e_\varepsilon(s) ds\right\},$$

where C_7 = const. > 0 does not depend on ε. Then condition (50) follows from the inequality obtained and Gronwall's lemma. Thus the applicability of Lemma 3.1 to the family of functions $\{e_\varepsilon(t), f_\varepsilon(t), 0 < \varepsilon \leq 1\}$ is proved, and so is assertion (P).

Suppose that condition (H2) is valid. By means of the constant k of (H2), for any $\varepsilon \in (0,1]$ we define continuous in $[0, t_\varepsilon)$ functions

$$e_\varepsilon(t) = \sup_{0\leq s\leq t} \sup_{y\in R^3} (1+s)|u_\varepsilon(s,y)|_{[\frac{k+3}{2}]},$$

$$f_\varepsilon(t) = \sup_{0\leq s\leq t} \|u_\varepsilon(s,.)\|_{k+3}.$$

The proof of assertion (P) is carried out again by means of Lemma 3.1. For this purpose we prove that there exist constants δ_1 and C_j $(j = 1,2,3)$ such that for the functions e_ε and f_ε conditions (48), (49) and (50) are met. The proof is carried out as above. We shall only note that in the estimation of $|u_\varepsilon(t,y)|_{[\frac{k+3}{2}]}$ and $\|u_\varepsilon(t,.)\|_{k+3}$ we use respectively Corollary 2.5 (inequality (43)) and Lemma 2.6 (inequality (47)).

S t e p 2. Let condition (H3) hold. By means of the functions U define

$$e_\varepsilon(t) = \|u_\varepsilon\|_{-1,-a,t}, \quad 0 < \varepsilon \leq 1,$$

where the constant a is given by condition (H3). From the definition of the norm $\|.\|_{-1,-a,t}$ and the smmothness of the local solutions $u_\varepsilon(t,x)$ it follows that the function $e_\varepsilon(t)$ is continuous in the interval $[0, t_\varepsilon)$ and

$$\sum_{|\alpha|\leq 1} |D^\alpha u_\varepsilon(t,x)| \leq C_0(te_\varepsilon(t) + \varepsilon), \tag{56}$$

where the constant C_0 does not depend on ε. Taking into account inequality (56), we note that assertion (P) can proved by means of the following lemma.

Lemma 3.2 *Let the nonnegative functions* $e_\varepsilon(t) \in C^0([0, t_\varepsilon))$, $0 < \varepsilon \leq 1$ *and satisfy the following conditions:*

$$e_\varepsilon(0) \leq C_1\varepsilon, \tag{57}$$

$$e_\varepsilon(t) \leq C_2\varepsilon + e_\varepsilon^2(t), \tag{58}$$

where C_j = const. > 0 *and* C_j *do not depend on* ε. *Then for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $\varepsilon_0 = \min\{1, C_1^2(2C_1^2 + 2C_2)^{-2}\}$ *the following estimate is valid*

$$\sup_{t\in[0,t_\varepsilon)} e_\varepsilon(t) \leq C_1\sqrt{\varepsilon}. \tag{59}$$

P r o o f. Suppose that (59) is not valid. Hence there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $t^1_{\varepsilon_1} \in (0, t_\varepsilon)$ such that $e_{\varepsilon_1}(t^1_{\varepsilon_1}) = C_1\sqrt{\varepsilon_1}$. But then by means of (58) we get a contradiction with the choise of ε_0. ■

The verification of conditions (57) and (58) for the functions $e_\varepsilon(t)$ constructed at the beginning of Step 2 will be carried out using the approach of H. P e c h e r [16] (see also [3]).

Taking into account Kirchhoff's formula and by the uniqeness of solutions theorm we prove that

$$u_\varepsilon(t,x) \equiv v_\varepsilon(t,x) + \mathcal{L}(u_\varepsilon)(t,x) \quad \text{in} \quad S(t_\varepsilon),$$

where $v_\varepsilon(t,x)$ is a solution of the Cauchy problem

$$\begin{cases} \Box v_\varepsilon(t,x) = 0 \text{ in } R^4_+, \\ v_\varepsilon(0,x) = \varepsilon g_1(x) \quad \text{and} \quad \partial_0 v_\varepsilon(0,x) = \varepsilon g_2(x) \end{cases} \tag{60}$$

$$\text{and} \quad \mathcal{L}(u_\varepsilon)(t,x) = \frac{1}{4\pi}\int_0^t \frac{1}{t-\tau}\int_{|y-x|=t-\tau} q(\tau,y)u^2_\varepsilon(\tau,y)d\sigma_y d\tau.$$

In [16] (see Lemma 2 and Remark 1) H. P e c h e r proved that there exists $C = \text{const.} > 0$ depending juct on $g_1(x)$ and $g_2(x)$ such that

$$\sum_{|\gamma|\le 2} |\partial^\gamma v_\varepsilon(t,x)| \le \frac{C\varepsilon}{(1+t+|x|)(1+|t-|x||)^a}. \tag{61}$$

Moreover, straightforward calculations yield that for $|\gamma| \le 2$ the following equality is valid

$$\partial^\gamma\left(\mathcal{L}(u_\varepsilon)(t,x)\right) =$$
$$= \frac{1}{4\pi}\int_0^t (t-\tau)\int_{|\omega|=1} \partial^\gamma\left[q(\tau, x+(t-\tau)\omega)u^2_\varepsilon(\tau, x+(t-\tau)\omega)\right] d\sigma d\tau.$$

Estimating the integrand by means of assumption (H3) we prove that

$$|\partial^\gamma(\mathcal{L}(u_\varepsilon)(t,x))| \le C_3.I(t,x).\|u_\varepsilon\|^2_{-1,-a,t}, \tag{62}$$

where the constant C_3 does not depend on ε and

$$I(t,x) = \frac{1}{2\pi}\int_0^t \frac{1}{t-\tau}\int_{|y-x|=t-\tau} \frac{d\sigma_y}{(1+\tau+|y|)^{a+2}(1+|\tau-|y||)^{b+2a}} d\tau.$$

In the right-hand side of this equality we pass to polar coordinates and after obvious transformations obtain

$$(63)\ I(t,x) = \int_0^t \frac{1}{|x|} \int_{||x|-(t-\tau)|}^{|x|+t-\tau} \lambda(1+\tau+\lambda)^{-(a+2)}(1+|\tau-\lambda|)^{-(b+2a)} d\lambda d\tau$$

In the estimate of the integral $I(t,x)$ we shall use the following lemma:

Lemma 3.3 *Let* $\rho > 2$, $q > 1$ *and* $h(\lambda,\tau) = \lambda(1+\lambda+\tau)^{-\rho}(1+|\lambda-\tau|)^{-q}$. *Then there exists* $C = \text{const.} > 0$ *such that for any* $t \geq 0$ *and* $r \geq 0$ *the following inequality is valid*

$$(64)\qquad \int_0^t \left(\int_{|r-t+\tau|}^{r+t-\tau} h(\lambda,\tau) d\lambda \right) d\tau \leq Cr(1+r+t)^{-1}(1+|r-t|)^{2-\rho}.$$

The proof is analogous to the proof of Lemma 1 of [16].

We shall note that from condition (H3) we have $a > 0$ and $2a + b > 1$. Consequently the integral $I(t,x)$ (see equality (63)) can be estimated by Lemma 3.3. Then we obtain

$$(65)\qquad I(t,x) \leq C(1+t+|x|)^{-1}(1+|t-|x||)^{-a}.$$

Inequalities (61), (62) and (65) complete the verification of conditions (57) and (58) for the functions $e_\varepsilon(t)$. Then Lemma 3.2 and inequality (56) complete the proof of assertion (P). ■

At the end of this seqtion we shall note that the proof of Theorem 1 enables us to analise also the case

$$F(t,x,u,Du) = Q(t,x,u,Du) + C(u,Du),$$

where $C(\omega_0,\omega) = O(|\omega_0|^3 + |\omega|^3)$. For instance if Q satisfies one of the conditions (H1), (H2) or (H3) and C equals respectively $C^1(Du)$, $C^2(u,Du) = \sum_{j=0}^3 \partial_j C_j^2(u)$ or $C^3(u)$, then the proof of the global solvability of problem (1), (2_ε) is a repetition of the proof of Theorem 1.

4. Blow-up of the solution of the Caushy problem for the equation $\Box u = q(t,x)u^2(t,x)$

Using a suitable conversation law for the linear wave equation and estimating the nonlinearity, we reduce the proof to the investigation of a differential inequality. This differential inequality is investigated by means of the following technical lemma proved by K a t o [6].

Lemma 4.1 *Let the function* $\varphi(t) \in C^2([a,d))$ *and satisfy the inequality*

(66) $$\varphi''(t) \geq Ct^{-(\rho+1)}\varphi^{\rho}(t), \quad \varphi(a) > 0, \quad \varphi'(a) \geq 0,$$

where $\rho =$ const. > 1, $C =$ const. > 0, *and* $a > 0$. *Then* $d < +\infty$.

The proof of Theorem is carried out supposing that the assertion is not true. Taking into account the continuation principle of A. M a j d a, we assume that:

(H) $\left\{\right.$ There exists $\varepsilon > 0$ such that problem (1), (2_ε) has a smooth global solution $u(t,x)$.

We define:

$$B = \sup\{|x|; x \in \text{supp } g_1 \cup \text{supp } g_2\},$$

$$Q(C) = \{x; |x| \leq C\} \quad \text{and}$$

(67) $$\varphi(t) = \int_{R^3} u(t,x)dx.$$

The function $\varphi(t)$ is correctly defined since from the uniqeness theorem it follows that supp $u(t,.) \subset Q(B+t)$. Moreover, from the assumption (H) it follows that $\varphi(t) \in C^2(0,+\infty)$. We shall show that $\varphi(t)$ enjoys properties contradicting Lemma 4.1. Integrating equation (1) with respect to x, we obtain that for any $t \in (0,+\infty)$ the following equality is valid

(68) $$\varphi''(t) = \int_{Q(B+t)} q(t,x)u^2(t,x)dx.$$

Moreover, from (67) by means of Hőlder's inequality and assumption (3) of Theorem 2 we check that

$$\varphi^2(t) \leq C_1^{-1} I(t)\varphi''(t),$$

where the constant C_1 is defined in condition (3) of Theorem 2 and

$$I(t) = \int_{Q(B+t)} (1+t+|x|)^a(1+|t-|x||)^b dx.$$

We estimate the integral $I(t)$ from above and prove that there exists $C =$ const. > 0 such that

(69) $$\varphi''(t) = C(1+t)^{-(a+b+3)}\varphi^2(t)$$

for $t \in (0, +\infty)$. In order to obtain a contradiction with Lemma 4.1 it is necessary to obtain an estimate of $\varphi(t)$ from below.

By Kirchhoff's formula we check that

$$v(t,x) \leq u(t,x) \quad \text{for} \quad (t,x) \in R^4_+, \tag{70}$$

where $v(t,x)$ is the solution of the Caushy problem

$$\begin{cases} \Box v(t,x) = 0 \quad \text{in} \quad R^4_+ \\ v(0,x) = \varepsilon g_1(x), \quad \partial_0 v(0,x) = \varepsilon g_2(x). \end{cases}$$

By means of the conservation law and the strong Huygens principle we prove that for $t \geq 0$

$$\int_{t-B \leq |x| \leq t+B} v(t,x)dx = C_1 t + C_2,$$

where the constants C_1 and C_2 depend just on the initial conditions (2_ε). We continue the equality obtained by (70). Then by means of Hőlder's inequality and (68) we prove that

$$C_1 t + C_2 \leq \left(C_1^{-1} J(t) \varphi''(t)\right)^{1/2},$$

where $J(t) = \int_{t-B<|x|<t+B} (1+t+|x|)^a (1+|t-|x||)^b \, dx$.

Moreover, the integrand in $J(t)$ is estimated above by $(1+B+2t)^a(1+B)^b$, hence there exists $C = \text{const.} > 0$ such that

$$\varphi''(t) \geq C(1+t)^{-a}.$$

We integrate the inequality obtained and find a sufficiently large constant C_3 for which

$$\varphi(t) \geq C_3(1+t)^{2-a}. \tag{71}$$

From inequalities (69) and (70) it follows that

$$\varphi''(t) \geq C(1+t)^{-(a+b+3)} \left[C_3(1+t)^{2-a}\right]^{\frac{a+b}{1-a}} \varphi(t)^\rho, \tag{72}$$

where $\rho = \frac{2-3a-b}{1-a}$ and $t \in (0, +\infty)$.

From condition (3) of Theorem 2 it follows that $\rho > 1$. Then (72) contradicts Lemma 4.1. The contradiction obtained completes the proof of Theorem 2.

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