Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol.9, 1995, Fasc.1

New Representation of a Special Non-Symmetric Homogeneous Domain in C^n, n=8

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1. Introduction.

We consider a homogeneous bounded domain D in C^n. The classification of D is a basic problem. If D is symmetric, then the classification is known. It remains open the problem when D is not symmetric. The non-symmetric homogeneous bounded domains in C^4, C^5 and C^6 have been classified ([10], [12]). This problem has also been solved only for one case in C^n $\forall$ $n \geq 7$. The aim of the present paper is to classify some other non-symmetric homogeneous bounded domains in C^8.

The paper contains three paragraphs. Each of them is analysed as follows. The second paragraph includes basic properties and theorems about homogeneous bounded domains and normal j-algebras. The last paragraph deals with the classification of a cathegory of non-symmetric homogeneous bounded domains in C^8.

2. A homogeneous bounded domain D in C^n can be written:

$$D = G(D)/H \tag{3.1}$$

where $G(D)$ is the group of holomorphic transformations on D and H is the isotropy subgroup of $G(D)$ at the point $z_0 \in C^n$. The relation (2.1) can also be writen as follows:

$$D = G_0(D)/H_0$$

where $G_0(D)$ is the identity component of $G(D)$ and H_0 is the isotropy subgroup S of $G_0(D)$ at $z_0 \in D$.

It is known that there exists a solvable Lie subgroup S of $G(D)$ which can be identified with D.

Therefore S is a Kähler Manifold on which there exists a complex structure on it denoted by J.

Let s be the Lie algebra of S which can be identified with the tangent space of S at its identity element e.

The almost complex structure J on D defines an endomorphism J_0 on s with the following properties:

$$J_0 : s \to s \ J_0 : X \to J_0(X) \ J_0^2 = -Id\ . \tag{3.2}$$

This endomorphism J_0 satisfies the following relation

$$[X,Y] + J_0[J_0(X),Y] + J_0[X,J_0(X)] - [J_0(X),J_0(Y)] = 0 \tag{3.3}$$

which is obtained from the fact that almost complex structure on D is integrable. The Kähler metric g on D induces a Hermitian positive definite symmetric bilinear form B on S. From B we obtain a linear form ω defined by:

$$\omega \ : \ s \to R\ ,\ \omega \ : \ X \to \omega(x) = B(x, J_0(x)) \tag{3.4}$$

satisfying the following conditions:

$$\omega([J_0(X),J_0(Y)] = \omega([X,Y]) \tag{3.5}$$

$$\omega([J_0(X),x] > 0 \qquad x \neq 0 \tag{3.6}$$

Therefore from the homogeneous bounded domain $D = G/H$ we obtain the $\{s, J_0, \omega\}$, where s is a special solvable Lie algebra, J_0 is an endomorphism on s having the properties (2.2) and (2.3) and ω is a linear form on s with the properties (2.5) and (2.6).

This set $\{s, J_0, \omega\}$ is called normal J-algebra.

Every normal J-algebra has also the property that the operator:

$$\alpha d\tau_0 \ : \ s \to s\ ,\ \alpha d\tau_0 \ : \ \tau \to \alpha d\tau_0(\tau) = [\tau_0, \tau] \tag{3.7}$$

has only real characteristic roots $\forall\ \tau_0 \in s$, that is, $\alpha d\tau_0$, as a matrix, is R-triangular.

The inverse is also true. Let $(s, J_0, \omega$ be a triple, where s is a solvable Lie algebra having the property (2.7), J_0 is an endomorphism on s having the properties (2.2) and (2.3) and ω is a linear form on s having the properties (2.5) and (2.6).

Then there exists a unique solvable Lie group S whose algebra is s which can be identified with the tangent space of S at its identity e. The endomorphism J_0 on s gives the complex structure on S and finally the linear form ω on s induces a Hermitian inner product on s defined by:

$$\langle\ X, Y\ \rangle\ =\ ([J_0X, Y]) \tag{3.8}$$

which determines the Kähler metric G on S. The couple (S,g) is a Kähler manifold beholomorphically isomorphic onto homogeneous bounded domain in C^n. In the next paragraph we shall give one triplet (s, J_0, ω) and the Kähler manifold (S,g) which is obtained by this triple.

3. We consider the solvable Lie algebra s, which can be described by the set of matrices

$$s = \{A = \begin{bmatrix} 0 & X_1 & X_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_4 & X_5 & X_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X_9 & X_{10} & X_{11} & X_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & X_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{16} \end{bmatrix}\} \tag{5.9}$$

$X_1, X_2, X_3, \ldots, X_{16} \in R^*$.

From this construction of s we conclude that the endomorphism J_0 has the form:

$$J_0 = (\beta_{*1}),\ \beta_{*1} \in R \qquad k = 1, 2, \ldots, 16,\ l = 1, 2, \ldots, 16 \tag{5.10}$$

which must satisfy the relation (2.2) and (2.3).

From this conditions and after a lot of estimates we obtain:

$$J_0 = \begin{bmatrix} J_1 & J_3 \\ J_3 & J_2 \end{bmatrix}, \tag{5.11}$$

where

$$J_1 = \begin{pmatrix} p_1 & 0 & \xi_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_2 & 0 & \xi_2 & 0 & 0 & 0 & 0 \\ -\frac{1+p_1^2}{\xi_1} & 0 & -p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1+p_2^2}{\xi_2} & 0 & -p_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3 & 0 & \xi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_4 & 0 & \xi_4 \\ 0 & 0 & 0 & 0 & -\frac{1+p_3^2}{\xi_3} & 0 & -p_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1+p_4^2}{\xi_4} & 0 & -p_4 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} p_5 & 0 & 0 & 0 & \xi_5 & 0 & 0 & 0 \\ 0 & p_6 & 0 & 0 & 0 & \xi_6 & 0 & 0 \\ 0 & 0 & p_7 & 0 & 0 & 0 & \xi_7 & 0 \\ 0 & 0 & 0 & p_8 & 0 & 0 & 0 & \xi_8 \\ -\frac{1+p_5^2}{\xi_5} & 0 & 0 & 0 & -p_5 & 0 & 0 & 0 \\ 0 & -\frac{1+p_6^2}{\xi_5} & 0 & 0 & 0 & -p_6 & 0 & 0 \\ 0 & 0 & -\frac{1+p_7^2}{\xi_5} & 0 & 0 & 0 & -p_7 & 0 \\ 0 & 0 & 0 & -\frac{1+p_8^2}{\xi_5} & 0 & 0 & 0 & -p_8 \end{pmatrix}$$

and J_3 is (8×8)–matrix with all elements zero.

The linear form ω, on this Lie algebra s, is defined by:

$$\omega(X) = \langle X_0, X \rangle \tag{5.12}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on s and $X_0 = (K_1, K_2, \ldots, K_8, K_9, \ldots, K_{16})$ is a fixed vector. In order that ω satisfies the conditions (2.5) and (2.6) we must have:

$$K_1\xi_1 > 0, K_2\xi_2 > 0, \ldots, K_{12}\xi_{12} > 0 \tag{5.13}$$

$$i = 1, 2, 5, 6, 9, 10, 11, 12$$

Now, we have proved the following theorem:

Theorem 3.1. *There exists a homogeneous bounded domain in C^n, $n = 8$ having (s, J_0, ω) normal J-algebra, where s, J_0 and ω are given by (3.1), (3.3) and (3.4) respectively.*

Now, we determine the solvable Lie group S which corresponds to the solvable Lie algebra s.

We denote by $GL(s)$ the group of all non-singular endomorphisms on s. The Lie algebra $gl(s)$ of $GL(s)$ consists of all endomorphisms of s with the stand bracket operation:

$$[X, Y] = XY - YX \tag{5.14}$$

The mapping:

$$\alpha d : s \leftarrow gl(s), \qquad \alpha d : B \leftarrow \alpha dB \tag{5.15}$$

where

$$\alpha dB : s \to s, \qquad \alpha dB : T \to \alpha dB(T) = [T, B] \tag{5.16}$$

is a homomorphism of s onto a subalgebra $\alpha d(s)$ of $gl(s)$. Let Int(s) be the analytic subgroup of GL(s) whose Lie algebra is $\alpha d(s)$ which is called adjoint group of s. The group Aut(s) has a unique analytic structure under which it becomes a topological Lie subgroup of GL(s).

We denote by d(s) the Lie algebra of Aut(s). Now, the group Int(s) is connected, so it is generated by elements $e^{\alpha dX}$, $X \in s$. Therefore Int(s) is a normal subgroup of Aut(s).

From the above we conclude that the solvable Lie group S is defined:

$$s = \{L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}\}, \tag{5.17}$$

where

$$L_1 = \begin{pmatrix} 1 & \frac{X_1}{X_3}(e^{X_3} - 1) & \frac{X_2}{X_4}(e^{X_4} - 1) & 0 & 0 & 0 \\ 0 & e^{X_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{X_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{X_5}{X_7}(e^{X_7} - 1) & \frac{X_6}{X_8}(e^{X_8} - 1) \\ 0 & 0 & 0 & 0 & e^{X_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{X_8} \end{pmatrix},$$

$$L_4 = \begin{pmatrix} 1 & \frac{X_9}{X_{13}}(e^{X_{13}}-1) & \frac{X_{10}}{X_{14}}(e^{X_{14}}-1) & \frac{X_{11}}{X_{15}}(e^{X_{15}}-1) & \frac{X_{12}}{X_{16}}(e^{X_{16}}-1) \\ 0 & e^{X_{13}} & 0 & 0 & 0 \\ 0 & 0 & e^{X_{14}} & 0 & 0 \\ 0 & 0 & 0 & e^{X_{15}} & 0 \\ 0 & 0 & 0 & 0 & e^{X_{16}} \end{pmatrix},$$

L_2 is (5×5)–matrix,L_3 is (6×6)–matrix both with all elements zero.

The inner product on the solvable Lie algebra is defined by:

$$\langle X, Y\rangle = \omega([J_0X, Y]) \tag{5.18}$$

where ω is given by (3.4). This inner product determines the K'ahler metric on S which is essentiallythe Bergman metric on it.

Now, we can state the following theorem:

Theorem 3.2. *The homogeneous non–symmetric bounded domain in C^n, $n = 8$ is biholorphically isomorphic onto the solvable Lie group defined by (3.9). The Kähler metric g on S is defined by the relation (3.10).*

Let F be a Lie automorphism on s. This F can be represented by the matrix:

$$F_{isom} = \begin{bmatrix} F_1 & J_3 \\ J_3 & F_2 \end{bmatrix}, \tag{5.19}$$

where

$$F_1 = \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & \alpha_{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{55} & 0 & \alpha_{57} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{66} & 0 & \alpha_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} \alpha_{88} & 0 & 0 & 0 & \alpha_{913} & 0 & 0 & 0 \\ 0 & \alpha_{1010} & 0 & 0 & \alpha_{1014} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{1111} & 0 & 0 & \alpha_{1115} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{1212} & 0 & 0 & 0 & \alpha_{1216} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which becomes an isometry with respect to the inner product:

$$\langle X,Y\rangle = \langle X_0,[JX,Y]\rangle = \omega([X,Y]).$$

If we have

$$\alpha_{11} =\pm 1 \qquad i = 1,2,5,6,9,10,11,12$$

$$\begin{array}{lcllcl}
\alpha_{13} & = & -\dfrac{2p_1\xi_1}{1+p_1^2}, & \alpha_{24} & = & -\dfrac{2p_2\xi_2}{1+p_2^2}, \\
\alpha_{57} & = & -\dfrac{2p_3\xi_3}{1+p_3^2}, & \alpha_{68} & = & -\dfrac{2p_4\xi_4}{1+p_4^2}, \\
\alpha_{913} & = & -\dfrac{2p_5\xi_5}{1+p_5^2}, & \alpha_{1014} & = & -\dfrac{2p_6\xi_6}{1+p_6^2}, \\
\alpha_{1115} & = & -\dfrac{2p_7\xi_7}{1+p_7^2}, & \alpha_{1216} & = & -\dfrac{2p_8\xi_8}{1+p_8^2},
\end{array}$$

or

$$\alpha_{13} = \alpha_{24} = \alpha_{57} = \alpha_{68} = \alpha_{913} = \alpha_{1014} = \alpha_{1115} = \alpha_{1216} = 0.$$

From the form F_{isom} we obtain that it has the eigenvalue 1 with multiplicity 8.

Therefore we have proved the following theorem.

Theorem 3.3 *The homogeneous bounded domain in C^n, $n = 8$ described by the theorem (3.2) does not admit any k–symmetric structure.*

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Received 15.01.1993