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A Generalization of the Fixed Point Theorem of Bhola and Sharma[1]

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A generalization of a theorem of Bhola and Sharma for two mappings on a complete metric space is given.

The following theorem was proved by Bhola and Sharma [1].

Theorem 1. *Let T be a self-mapping of a complete metric space (X,d) satisfying*

$$d(Tx,Ty) \leq a\sqrt{d(x,Tx)d(y,Ty)} + b\sqrt{d(x,Ty)d(y,Tx)} \tag{1}$$

for all x, y in X, where $0 < a < 1$ and $b \geq 0$. Then T has a unique fixed point, and the iterate $T^n x$ converges to the fixed point for every x in X.

Notice that Theorem 1 is incorrect as stated. If T is the identity mapping, then T satisfies inequality (1) with $b \geq 1$ and every point is a fixed point of T. For the fixed point to be unique it is necessary that $0 \leq b < 1$.

We now note that

$$\begin{aligned}\sqrt{d(x,Tx)d(y,Ty)} &\leq \tfrac{1}{2}[d(x,Tx)+d(y,Ty)] \\ &\leq \max\{d(x,y), d(x,Tx), d(y,Ty), \tfrac{1}{2}[d(x,Ty)+d(y,Tx)]\}.\end{aligned}$$

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It follows that if inequality (1) holds, then the inequality

$$d(Tx,Ty) \le a\max\{d(x,y),d(x,Tx),d(y,Ty),\tfrac{1}{2}[d(x,Ty)+d(y,Tx)], c\sqrt{d(x,Ty)d(y,Tx)}\} \tag{2}$$

holds, where $ac = b$.

We now prove the following generalization of Theorem 1.

Theorem 2. *Let S and T be self-mappings of a complete metric space (X,d) satisfying the inequality*

$$d(Sx,Ty) \le a\max\{d(x,y),d(x,Sx),d(y,Ty),\tfrac{1}{2}[d(x,Ty)+d(y,Sx)], b\sqrt{d(x,Ty)d(y,Sx)}\} \tag{3}$$

for all x,y in X, where $0 < a < 1$ and $b \ge 0$. Then S and T have a common fixed point. Further, if $ab < 1$, then the fixed point is unique.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n : n = 1,2,\dots\}$ by

$$x_1 = Sx_0,\ x_2 = Tx_1,\ \dots,\ x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ \dots\ .$$

Using inequality (3), we have

$$\begin{aligned} d(x_{2n+1},x_{2n}) &= d(Sx_{2n},Tx_{2n-1}) \\ &\le a\max\{d(x_{2n},x_{2n-1}),d(x_{2n},x_{2n+1}),\tfrac{1}{2}d(x_{2n-1},x_{2n+1})\} \\ &\le a\max\{d(x_{2n},x_{2n-1}),\tfrac{1}{2}[d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})]\} \end{aligned}$$

and it follows that

$$d(x_{2n+1},x_{2n}) \le ad(x_{2n},x_{2n-1}),$$

since $0 < a/(2-a) < a$. Similarly, we have

$$d(x_{2n},x_{2n-1}) \le ad(x_{2n-1},x_{2n-2})$$

and so

$$d(x_{2n+1},x_{2n}) \le ad(x_{2n},x_{2n-1}) \le a^{2n}d(x_1,x_0).$$

Since $a < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in the complete metric space X and so has a limit z.

Using inequality (3) we have

$$\begin{aligned} d(Sz, x_{2n}) &= d(Sz, Tx_{2n-1}) \\ &\le a \max\{d(z, x_{2n-1}), d(z, Sz), d(x_{2n-1}, x_{2n}), \tfrac{1}{2}[d(z, x_{2n}) + d(x_{2n-1}, Sz)], \\ &\qquad b\sqrt{d(z, x_{2n})d(x_{2n-1}, Sz)}\}. \end{aligned}$$

Letting n tend to infinity, we see that $Sz = z$. Similarly, $Tz = z$ and so z is a common fixed point of S and T.

To prove the uniqueness of z when $ab < 1$, we suppose that w is a second fixed point of T. Then, using inequality (3) we have

$$d(z, w) = d(Sz, Tw) \le a \max\{d(z, w), bd(z, w)\},$$

and it follows that z is the unique fixed point of T. Similarly, z is the unique fixed point of S.

The corollary follows immediately.

Corollary. *Let T be a self-mapping of a complete metric space (X, d) satisfying*

$$\begin{aligned} d(Tx, Ty) \le\ & a \max\{d(x, y), d(x, Ty), d(y, Ty), \tfrac{1}{2}[d(x, Ty) + d(y, Tx)], \\ & b\sqrt{d(x, Ty)d(y, Tx)}\} \end{aligned}$$

for all x, y in X, where $0 < a < 1$ and $b \ge 0$. Then T has a fixed point. Further, if $b < 1$, then the fixed point is unique.

Theorem 3. *Let S and T be continuous self-mappings of a compact metric space (X, d) satisfying*

$$\begin{aligned} d(Sx, Ty) <\ & \max\{d(x, y), d(x, Sx), d(y, Ty), \tfrac{1}{2}[d(x, Ty) + d(y, Sx)], \\ & b\sqrt{d(x, Ty)d(y, Sx)}\} \end{aligned}$$

for all x, y in X for which the right hand side of the inequality is positive, where $b > 0$. Then S and T have a common fixed point. Further, if $b < 1$, then the common fixed point is unique.

Proof. Let $a = \inf \alpha$ taken over all α for which

$$\begin{aligned} d(Sx, Ty) \le\ & \alpha\{d(x, y), d(x, Sx), d(y, Ty), \tfrac{1}{2}[d(x, Ty), +d(y, Sx)], \\ & b\sqrt{d(x, Ty)d(y, Sx)}\}. \end{aligned}$$

If $a < 1$, then the result follows from Theorem 2.

If $a = 1$, then since S and T are continuous and X is compact, there exist points z, w in X such that

$$d(Sz, Tw) = \max\{d(z, w), d(z, Sz), d(w, Tw), \tfrac{1}{2}[d(z, Tw) + d(w, Sz)], b\sqrt{d(z, Tw)d(w, Sz)}\}.$$

This implies the right hand side of the equation is zero and so $z = w$ is a common fixed point of S and T.

It follows easily that the common fixed point is unique if $b < 1$.

Corollary. *Let T be a continuous self-mapping of a compact metric space (X, d) satisfying*

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), \tfrac{1}{2}[d(x, Ty) + d(y, Tx)], b\sqrt{d(x, Ty)d(y, Sx)}\}$$

for all x, y in X for which the right hand side of the inequality is positive, where $b > 0$. Then T has a fixed point. Further, if $b < 1$, then the fixed point is unique.

References

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