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Embedding Derivatives of $\mathcal{M}$-Harmonic Hardy Spaces into Lebesgue Spaces

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A characterization is given of those measures of μ on B, the open unit ball in C^n, such that differentiation of order m maps the $\mathcal{M}$ -harmonic Hardy space $\mathcal{H}\surd$ boundedly into $L^p(\mu)$, $2 \leq p < \infty$.

1. Introduction

Let B be the open unit ball in C^n with (normalized) volume measure ν and let S denote its boundary. For the most part we will follow the notation and terminology of Rudin [5].

Let $\tilde{\Delta}$ be the invariant Laplacian on B. That is $(\tilde{\Delta}f)(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B taking 0 to z (see [5]). An $\mathcal{M}$-harmonic function on B, $f \in \mathcal{M}$, is a function in $C^2(B)$ which is annihilated by $\tilde{\Delta}$ on B. We shall denote by $\mathcal{H}\surd = \mathcal{H}\surd(\mathcal{B})$ the space of all $\mathcal{M}$-harmonic functions on B that satisfy the growth condition

$$||f||^p_{\mathcal{H}\surd} = \sup_{0<r<1} \int_S |f(r\epsilon)|^p d\sigma(\epsilon) < \infty, \quad 0 < p < \infty.$$

Here σ denotes the rotation invariant probability measure on S.

For $f \in \mathcal{M}$ let $\partial f(z) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_n}\right)$ and for any positive integer m we write $\partial^m f(z) = \left(\partial^\alpha \bar{\partial}^\beta f(z)\right)_{|\alpha|+|\beta|=m}$ and $|\partial^m f(z)|^2 = \sum_{|\alpha|+|\beta|=m} |\partial^\alpha \bar{\partial}^\beta f(z)|^2$, where $\partial^\alpha \bar{\partial}^\beta f(z) = \frac{\partial^{|\alpha|+|\beta|} f(z)}{\partial z_1^{\alpha_1}, \dots, \partial z_n^{\alpha_n}, \partial \bar{z}_1^{\beta_1}, \dots, \partial \bar{z}_n^{\beta_n}}$, α and β are multiindices.

Let μ be a positive measure on B and consider the problem of determining what conditions on μ imply $|\partial^m f| \in L^p(\mu)$ whenever $f \in \mathcal{H}\surd$. A standard application of the closed graph theorem leads to the following equivalent problem.

Characterize the μ for which there exists a constant C satisfying

$$\left(\int_B |\partial^m f|^p d\mu\right)^{1/p} \leq C\|f\|_{\mathcal{H}\surd}.$$

The purpose of this paper is to present a solution of this problem in the case $2 \leq p < \infty$. To state it we need some more notations.

For $z \in B$ and r, $0 < r < 1$, $E_r(z) = \{w \in B \, |\varphi_z(w)| < r\}$. We'll let $|E_r(z)| = \nu(E_r(z))$. Throughout the paper r will be fixed and we will occasionally write $E(z)$ instead of $E_r(z)$. Constants will be denoted by C which may indicate a different constant from one occurrence to the next.

Theorem *Let $2 \leq p < \infty$. For a positive measure μ on B and a positive integer m, necessary and sufficient condition for*

$$\left(\int_B |\partial^m f|^p d\mu\right)^{1/p} \leq C\|f\|_{\mathcal{H}\surd}, \text{ for all } f \in \mathcal{H}\surd \tag{1}$$

is that there exists a constant K for which

$$\mu(E(z)) \leq K(1 - |z|)^{n+mp}, \quad z \in B. \tag{2}$$

For harmonic functions Theorem was proved by Shirokov and Luecking (see [2], [3], [6] and [7]).

2. Proof of Theorem.

The following three preliminary lemmas will beneeded in the proof of the Theorem.

Lemma 2.1 ([4]) *If $f \in C^2(B)$ and $0 < r < 1$, then*

$$f(0) = \int_S f(r\epsilon)d\sigma(\epsilon) - \int_{rB} \tilde{\Delta} f(z)G(|z|, r)(1 - |z|^2)^{-n-1} d\nu(z),$$

where

$$G(t, r) = \frac{1}{2n}\int_t^r \rho^{1-2n}(1 - \rho^2)^{n-1} d\rho, \quad 0 < t < r \leq 1.$$

Lemma 2.2 *If g is a function in $L^p(\sigma)$ for $2 \le p < \infty$ and if*

$$f(z) = P[g](z) = \int_S \frac{(1-|z|^2)^n}{|1- < z, \epsilon > |^{2n}} g(\epsilon) d\sigma(\epsilon),$$

then

$$\int_B |\partial f(z)|^p (1-|z|)^{p-1} d\nu(z) \le C \int_S |g(\epsilon)|^p d\sigma(\epsilon),$$

where C is a constant independent of g.

Proof. Let λ be the measure $d\lambda(z) = (1-|z|^2)^{-1} d\nu(z)$ on B and let T be the operator $Tg(z) = (1-|z|^2)|\partial P[g](z)|$ on $g \in L^2(\sigma)$. We show that T, as a mapping from the measure space $(S, d\sigma)$ to the measure space $(B, d\lambda)$, is of types (2.2) and (∞, ∞).

Let $f = P[g]$, $g \in L^2(\sigma)$. As a simple consequence of Lemma 2.1 we have

$$\int_S |g(\epsilon) - f(0)|^2 d\sigma(\epsilon) = \int_B G(|z|, 1) \tilde{\Delta} |f|^2(z) (1-|z|^2)^{-n-1} d\nu(z).$$

In terms of ordinary differential operators the invariant Laplacian $\tilde{\Delta}$ is as follows:

$$\tilde{\Delta} = 4(1-|z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \overline{z_k}) \frac{\partial^2}{z_j \partial \overline{z_k}},$$

where δ_{jk} denotes the Kronecker delta (see [5], section 4.1, for details). Using this from $\tilde{\Delta}$ and the fact that $\tilde{\Delta} f = \tilde{\Delta} \bar{f} = 0$ and $\overline{\frac{\partial f}{\partial z_j}} = \frac{\partial \bar{f}}{\partial \overline{z_j}}$, $1 \le j \le n$, we find that

$$\tilde{\Delta} |f|^2 = 4(1-|z|^2)(|\partial f(z)|^2 - |Rf(z)|^2 - |R\bar{f}(z)|^2),$$

where, as usual, R denotes the radial derivative $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$.

Since $|Rf(z)|^2 + |R\bar{f}(z)|^2 \le |z|^2 |\partial f(z)|^2$, we have

$$\tilde{\Delta} |f|^2(z) \ge 4(1-|z|^2)^2 |\partial f(z)|^2.$$

Thus

$$\int_S |g(\epsilon)|^2 d\sigma(\epsilon) \ge C \int_B |\partial f(z)|^2 (1-|z|^2) d\nu(z) = C \int_B (Tf(z))^2 d\lambda(z).$$

To show that T is of type (∞, ∞) it is suffices to show that

$$\frac{\partial f}{\partial z_j}(z) = \mathrm{O}\left(\frac{1}{1-|z|}\right) \text{ and } \frac{\partial f}{\partial \overline{z_j}}(z) = \mathrm{O}\left(\frac{1}{1-|z|}\right), \quad j = 1, 2, ..., n.$$

Since $\left|\frac{\partial f}{\partial z_j}(z)\right|$ and $\left|\frac{\partial f}{\partial \overline{z_j}}(z)\right|$ are at most $C\|g\|_\infty \int_S \frac{d\sigma(\epsilon)}{|1-<z,\epsilon>|^{n+1}}$, the desired conclusion follows by Proposition 1.4.10 ([5], p.17). ■

By the Marcinkiewitz interpolation theorem Lemma 2.2. follows

Lemma 2.3 ([1]) *Let $k \geq m$ be non-negative integers, $0 < p < \infty$ and $0 < r < 1$. There exists a constant $C = C(k, m, p, n, r)$ such that if $f \in \mathcal{M}$ then*

$$|\partial^k f(w)|^p \leq C(1-|w|)^{(m-k)p} \int_{E_r(w)} |\partial^m f(z)|^p (1-|z|)^{-n-1} d\nu(z), \text{ for all } w \in B.$$

Proof. The necessity of the condition (1.2) follows from theShirokov-Luecking theorem mentioned above. The sufficiency can be gotten by the following argument.

Let $f \in \mathcal{H}\checkmark$, $2 \leq p < \infty$. There exists $g \in L^p(\sigma)$ such that $f = P[g]$ (see [5]). Using Lemma 2.3, Fubini's theorem and Lemma 2.2. we conclude that

$$\int_B |\partial^m f(z)|^p d\mu(z) \leq C \int_B |\partial f(z)|^p (1-|z|)^{p-n-1-mp} \mu(E(z)) d\nu(z)$$
$$\leq C \int_B |\partial f(z)|^p (1-|z|)^{p-1} d\nu(z) \leq C \int_S |g(\epsilon)|^p d\sigma(\epsilon)$$
$$= C\|f\|^p_{\mathcal{H}\checkmark}.$$

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