Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol.9, 1995, Fasc.4

Quantum Stochastic Integral Equations in Fock Space

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We provide an existence and uniqueness theorem for strong solutions of a general class of integral equations involving operators acting on the Finite - Difference Fock space. We also examine the dependence of solutions on initial conditions and coefficients, and provide a necessary and sufficient condition in order for the solutions to be unitary.

1.Introduction

The subject of integral equations in Fock space has been treated most notably by R. L. Hudson and K. R. Parthasaraty in [8] while more general theories, in the context of quantum probability, were constructed by L. Accardi, F. Fagnola and J. Quaegebeur in [1].

The Fock space considered in [8] is related to the Heisenberg - Weyl Lie algebra and it is possible to obtain realizations of the classical Brownian motion and Poisson processes, with the use of the stochastic calculus constructed in [8], as operators acting on that space.

In this paper we consider integral equations involving operators defined on the Finite - Difference (FD) Fock space of [3] which is related to the FD Lie algebra of P. J. Feinsilver defined in [7], and which constitutes a continuous, function analogue of his discrete FD Fock space. In [4] it was shown that the classical exponential process can be realized, in the quantum probability sense, as an essentially self - adjoint operator acting on a dense subspace of the FD Fock space. It should be noted that there is an intrinsic connection between the exponential process and the FD Fock space, so the latter appears as the natural context for its study.

The integral equations considered in [8] had constant coefficients. Here we allow the coefficients to depend on time and provide some estimates for the dependence of solutions on the coefficients as well as on the initial conditions. More precisely, we consider equations of the form

$$X(t) = X_0 + \sum_{(i,j,k)\in\Lambda} \int_0^t A_{i,j,k}(s, X(s))dM_{i,j,k}(s)$$

where Λ is a finite subset of $\mathbb{N}_0^3, t \geq 0, X_0$ is a bounded linear operator acting on the FD Fock space F, for each $(i,j,k) \in \Lambda$ the quantum exponential noise process $(cf: [3],[4],[5])$ $M_{i,j,k}$ is a family of linear operators, mapping a dense subspace E of F into F, indexed by time $t \geq 0$, the integrands $A_{i,j,k} : [0,\infty) \times L(E,F) \to L(E,F)$, where $L(E,F)$ is the space of linear operators $:E \to F$, are $M_{i,j,k}$ integrable in the sense of [5], and the equals sign denotes pointwise equality of the two sides as operators acting on E.

The basic machinery used in our study is the FD stochastic calculus constructed in [5] which was used there to study weak solutions of stochastic differential equations, and which was also used in [6] to prove the existence and uniqueness of quantum diffusions acting on a *-subalgebra of B(F), the Banach algebra of bounded linear operators:$F \to F$.

We remark finally that K. R. Parthasarathy and K. Sinha have recently shown that all known stochastic calculi can be realized in terms of the basic processes of [8]$(cf: [10])$. However the connecting relationships are theoretical rather than practical, therefore a quantum system in the presence of, for example, "exponential noise" is more naturally studied with the use of the FD stochastic calculus which was especially constructed for the study of quantum exponential processes.

2.The FD stochastic calculus : a review

The FD Fock space F is defined as the (separable) Hilbert space completion of the span E of the set of "exponential vectors" $\{y(f)/f \in S\}$, where S is the set of step functions $[0,\infty] \to (-1,1)$, with

$$< y(f), y(g) >= \exp(- \int \log(1 - f(t)g(t))dt).$$

For each $f \in S$ the (closable) linear operators $Q(f), P(f), T(f) : E \to F$ defined by

$$Q(f)y(g) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0}(y(g+\epsilon f) + y(e^{\epsilon f}g)) \qquad \text{(weak derivative)}$$

$$P(f)y(g) = (\int_0^\infty f(t)g(t)dt + Q(fg))y(g)$$

$$T(f)y(g) = (Q(f) + P(f) + \int_0^\infty f(t)dt)y(g)$$

satisfy the commutation relations

$$[P(f), Q(g)] = [P(f), T(g)] = [T(f), Q(g)] = T(fg)$$

and hence constitute a function space realization of the FD Lie algebra of Feinsilver ([4]). The definition of $Q(f), P(f)$ and $T(f)$ can be extended to $f \in L^1_{loc}([0,\infty), R)$ ([3], [4]). The operators $P(f), Q(f)$ are duals of each other while $T(f)$ is symmetric. In particular, the operator $T(t) = T(X_{[0,t]})$ is essentially self - adjoint and has "quantum moment generating function"

$$< e^{sT(t)}y(0), y(0) >= (1-s)^{-t}, \qquad \text{where} \quad s \in (-1,1),$$

thus $\{T(t)/t \geq 0\}$ is a Fock space realization of the classical exponential process.

The basic integrators in the FD stochastic calculus are, for all $(i,j,k) \in \mathbb{N}_0^3$ the processes $\{M_{i,j,k}(t)/t \geq 0\}$ where

$$M_{i,j,k}(t) \stackrel{\text{def}}{=} Q^{(i)}(\chi_{[0,t]})T^j(\chi[0,t])P^{(k)}(\chi_{[0,t]})$$

and superscripts in parentheses denote factorial powers.

The differentials of the basic integrators can be multiplied using Ito's formula:

$$dM_{i,j,k}(t).dt = dt.dM_{i,j,k}(t) = 0$$

and for $(i,j,k), (I,J,K) \in \mathbb{N}_0^3$

$$dM_{i,j,k}(t).dM_{I,J,K}(t) =$$

$$= \sum_{\lambda=0}^{I}\sum_{\mu=0}^{k}\sum_{v=0}^{\lambda}\sum_{\tau=0}^{\mu}\sum_{\sigma=0}^{\min(i,v)}\sum_{\rho=0}^{\min(K,\tau)} \delta_{k-\mu,I-\lambda}\binom{I}{\lambda}\binom{k}{\mu}$$

$$\binom{\lambda}{v}\binom{\mu}{\tau}\binom{i}{\sigma}\binom{v}{\sigma}\binom{K}{\rho}\binom{\tau}{\rho}$$

$$(k-\mu)!j^{(\lambda-v)}J^{(\mu-\tau)}\sigma!\rho!dM_{i+v-\sigma,j+J+I-\lambda,K+\tau-\rho}(t)$$

$$-M_{i,j,k}(t)dM_{I,J,K}(t)-M_{I,J,K}(t)dM_{i,j,k}(t). \tag{2.1}$$

If $X=\{X(s):E\to F/s\geq 0\}$ ia a process which is adapted in the sense of R. L. Hudson and K. R. Parthasarathy ([9], [8]), the "stochastic integral of X over $[0,t]$ with respect to $M_{i,j,k}$" " is the operator

$$\int_0^t X(s)dM_{i,j,k}(s)\in L(E,F)$$

defined by

$$<\int_0^t X(s)dM_{i,j,k}(s)y(f),y(g)>=$$

$$\sum_{\alpha,\beta,\gamma\in\{0,1\}}\Phi_{i,j,k}^{\alpha,\beta,\gamma}\sum_{I=1}^{i\delta_{\alpha,0}}\sum_{J=1}^{j\delta_{\beta,0}}\sum_{K=1}^{k\delta_{\gamma,0}}\binom{i\delta_{\alpha,0}}{I}\binom{j\delta_{\beta,0}}{J}\binom{k\delta_{\gamma,0}}{K}\int_0^t\sigma_{I,J,K}^{f,g}(s)<X(s)$$

$$M_{\alpha i,\beta j,\gamma k(s)}M_{i\delta_{\alpha,0}-I,j\delta_{\beta,0}-J,k\delta_{\gamma,0}-K}(s)y(f),y(g)>ds \tag{2.2}$$

provided that the integral on the right exists.

Here and in what follows we use the following notation:
$f=\sum_{\lambda=1}^n a_\lambda\chi_{I_\lambda},\quad g=\sum_{\lambda=1}^n b_\lambda\chi_{I_\lambda}\quad\in S,\qquad (i,j,k)\in\mathbb{N}_0^3,$
δ is Kronecker's delta,

$$\sigma_{i,j,k}^{f,g}(s)\stackrel{\text{def}}{=}\begin{bmatrix}\delta_{i,0}\delta_{K,0}(\delta_{j,0}+(1-\delta_{j,0})(j-1)!), & \text{if}\quad s\notin\cup_{\lambda=1}^n I_\lambda\\ (1-\delta_{i+j+k,0})(i+j+k-1)!\frac{f^k(s)g^i(s)(1+f(s))^{i+j}(l+g()^{j+k}}{(1-f(s)g(s))^{i+j+k}}, & \text{if}\quad s\in\cup_{\lambda=1}^n I_\lambda\end{bmatrix}$$

$$\Phi_{i,j,k}^{\alpha,\beta,\gamma}=$$

$$(1-\delta_{\alpha+i,0})(1-\delta_{\beta+j,0})(1-\delta_{\gamma+k,0})[(1-\delta i\delta_{\alpha,0},0)(1-\delta j\delta_{\beta,0},0)(1-\delta k\delta_{\gamma,0},0)+$$
$$(1-\delta_{i\delta_{\alpha,0},0})(1-\delta_{j\delta_{\beta,0},0})\delta_{k\delta_{\gamma,0},0})+(1-\delta_{i\delta_{\alpha,0},0})\delta_{j\delta_{\beta,0},0})(1-\delta_{k\delta_{\gamma,0},0})+$$
$$\delta_{i\delta_{\alpha,0},0})(1-\delta_{j\delta_{\beta,0},0})(1-\delta_{k\delta_{\gamma,0},0})+(1-\delta_{i\delta_{\alpha,0},0})(1-\delta_{j\delta_{\beta,0},0)+k\delta_{\gamma,0},0}+$$
$$(1-\delta_{j\delta_{\beta,0},0})\delta_{i\delta_{\alpha,0},0)+k\delta_{\gamma,0},0}+(1-\delta_{k\delta_{\gamma,0},0})\delta_{i\delta_{\alpha,0},0})+j\delta_{\beta,0},0]$$

Moreover, if $(i,j,k),(I,J,K)\in\mathbb{N}_0^3$ and X,Y are respectively $M_{i,j,k}$, $M_{I,J,K}$ integrable adapted processes then for all $f,g\in S$ and $t\geq 0$:

$$< \int_0^t X(s)dM_{i,j,k}(s)y(f), \int_0^t Y(s)dM_{I,J,K}(s)y(g) >=$$

$$\sum_{\alpha,\beta,\gamma\in\{0,1\}}\sum_{a,b,c\in\{0,1\}} \Phi_{i,j,k}^{\alpha,\beta,\gamma}\Phi_{I,J,K}^{a,b,c}$$

$$\sum_{i'=1}^{i\delta_{\alpha,0}}\sum_{j'=1}^{j\delta_{\beta,0}}\sum_{k'=1}^{k\delta_{\gamma,0}}\sum_{I'=1}^{I\delta_{a,0}}\sum_{J'=1}^{J\delta_{b,0}}\sum_{K'=1}^{K\delta_{c,0}}$$

$$\binom{i\delta_{\alpha,0}}{i'}\binom{j\delta_{\beta,0}}{j'}\binom{k\delta_{\gamma,0}}{k'}\binom{I\delta_{a,0}}{I'}\binom{J\delta_{b,0}}{J'}\binom{K\delta_{c,0}}{K'}$$

$$\Bigg[\sum_{\lambda=0}^{i'}\sum_{\mu=0}^{I'}\sum_{v=0}^{\lambda}\sum_{\tau=0}^{\mu}\sum_{\sigma=0}^{\min(K',v)}\sum_{\rho=0}^{\min(k',\tau)}\delta_{I-\mu,i-}$$

$$-\lambda\binom{i'}{\lambda}\binom{I'}{\mu}\binom{\lambda}{v}\binom{\mu}{\tau}\binom{K'}{\sigma}\binom{v}{\sigma}\binom{k'}{\rho}\binom{\tau}{\rho}$$

$$(I'-\mu)!J'^{(\lambda-v)}j'^{(\mu-\tau)}\sigma!\rho!\int_0^t \sigma^{f,g}_{K'+v-\sigma,J'+j'+i'-\lambda,k'+\tau-\rho}(s) < X(s)$$

$$M_{\alpha i,\beta j,\gamma k}(s)M_{i\delta_{\alpha,0}-i',j\delta_{\beta,0}-j',k\delta_{\gamma,0}-k'}(s)y(f),$$

$$Y(s)M_{aI,bJ,cK(s)}M_{I\delta_{a,0}-I',J\delta_{b,0}-J',K\delta_{c,0}-K'}(s)y(g) > ds+$$

$$+\int_0^t\int_0^t \sigma^{f,g}_{i',j',k'}(s)\sigma^{f,g}_{K',J',I'}(w) < X(s)M_{\alpha i,\beta j,\gamma k}(s)$$

$$M_{i\delta_{\alpha,0}-i',j\delta_{\beta,0}-j',k\delta_{\gamma,0}-k'}(s)y(f)$$

$$Y(w)M_{aI,bJ,cK(w)}M_{I\delta_{a,0}-I',J\delta_{b,0}-J',K\delta_{c,0}-K'}(w)y(g) > dsdw\Bigg] \tag{2.3}$$

3. The existence and uniqueness theorem

Let Λ be a subset of N_0^3 of finite cardinality $|\Lambda|$ and let

$$\mathrm{A}=\{\mathrm{A}_{i,j,k}=\{\mathrm{A}_{i,j,k}(s,w(s))\in L(F,F)/w:[0,\infty)\to L(E,F),s\geq 0\}/(i,j,k)\in\Lambda\}$$

be a family of adapted procesess satisfying:

(a) **Boundedness condition:** For each $\tau\geq 0$ and $f\in S$ there exists $c_{\tau,f}(\mathrm{A})\geq 0$ such that

$\sup_{0\leq s\leq\tau}\|\mathrm{A}_{i,j,k}(s,w(s))\mathrm{N}_{I,J,K}(s)y(f)\|\leq c_{\tau,f}(\mathrm{A}).\sup_{0\leq t\leq\tau}\|w(s)y(f)\|$ for all $(i,j,k)\in\Lambda$, $w:[0,\infty)\to L(E,F)$ which are strongly continuous, $(I,J,K)\in\mathbb{N}_0^3$ and

$$\mathrm{N}_{I,J,K}(s)\stackrel{\mathrm{def}}{=}M_{\alpha i,\beta j,\gamma k}(s)M_{i\delta_{\alpha,0}-i',j\delta_{\beta,0}-j',k\delta_{\gamma,0}-k'}(s)$$

where $\alpha,\beta,\gamma\in\{0,1\}$ and $0\leq i'\leq i\delta_{\alpha,0},\quad 0\leq j'\leq j\delta_{\beta,0},\quad 0\leq k'\leq k\delta_{\gamma,0}$.

R e m a r k: By "strongly continuous" we mean that for each $f\in S$ the map $t\to w(t)y(f)$ is continuous.

(b) **Lipschitz condition:** For each $\tau\geq 0$ and $f\in S$ there exists $\rho_{\tau,f}(A)\geq 0$ such that

$$\|[\mathrm{A}_{i,j,k}(s,w(s))-\mathrm{A}_{i,j,k}(s,\tilde{w}(s))]\mathrm{N}_{I,J,K}(s)y(f)\|\leq\rho_{\tau,f}(A).$$

$\|[w(s)-\tilde{w}(s)]y(f)\|$ for all $(i,j,k)\in\Lambda,\quad w,\tilde{w}:[0,\infty)\to L(E,F),s\in[0,\tau]$ and all $N_{I,J,K}$ defined as in condition (a) above.

R e m a r k: If $A_{i,j,k}(s,0)=0$, for all $s\geq 0$ and $(i,j,k)\in\Lambda$, then $(b)\to(a)$.

For each $\tau\geq 0$ and $f\in S$, we define

$$\xi_{\tau,f}\stackrel{\mathrm{def}}{=}\|\Lambda\|\sum_{(i,j,k)\in\Lambda}\sum_{\alpha,\beta,\gamma\in\{0,1\}}$$

$$\Phi_{i,j,k}^{\alpha,\beta,\gamma}\sum_{i'=1}^{i\delta_{\alpha,0}}\sum_{j'=1}^{j\delta_{\beta,0}}\sum_{k'=1}^{k\delta_{\gamma,0}}\sum_{I'=1}^{i\delta_{\alpha,0}}\sum_{J'=1}^{j\delta_{\beta,0}}\sum_{K'=1}^{k\delta_{\gamma,0}}\binom{i\delta_{\alpha,0}}{i'}$$

$$\binom{j\delta_{\beta,0}}{j'}\binom{k\delta_{\gamma,0}}{k'}\binom{i\delta_{\alpha,0}}{I'}\binom{j\delta_{\beta,0}}{J'}\binom{k\delta_{\gamma,0}}{K'}\Bigg[\sum_{\lambda=0}^{i'}\sum_{\mu=0}^{I'}\sum_{v=0}^{\lambda}\sum_{\tau=0}^{\mu}\sum_{\sigma=0}^{\min(K',v)}\sum_{\rho=0}^{\min(k',\tau)}$$

$$\delta_{I'-\mu,i'-\lambda}\binom{i'}{\lambda}\binom{I'}{\mu}\binom{\lambda}{v}\binom{\mu}{\tau}\binom{K'}{\sigma}\binom{v}{\sigma}\binom{k'}{\rho}\binom{\tau}{\rho}(I'-\mu)!J'^{(\lambda-v)}j'^{(\mu-\tau)}\sigma!\rho$$

$$\sup_{0\leq s\leq\tau}|\sigma^{f,f}_{K'+v-\sigma,J'+j'+i'-\lambda,k'+\tau-\rho}(s)|+\tau\sup_{0\leq s,w\leq\tau}|\sigma^{f,f}_{i',j',k'}(s)\sigma^{f,f}_{K',J',I'}(w)|\Big] \tag{3.1}$$

Then, we can prove the following:

Theorem3.1 *Let $X_0 \in B(F)$. There exists a unique, strongly continuous, adapted process $X = \{X(t) \in L(E,F)/t \geq 0\}$ such that $X(0) = X_0$ and for $t \geq 0$*

$$X(t) = X_0 + \sum_{(i,j,k)\in\Lambda}\int_0^t A_{i,j,k}(s, X(s))dM_{i,j,k}(s) \tag{3.2}$$

Proof. Let $\tau \geq 0$. For $t \in [0,\tau]$ define a sequence of adapted processes $\{X^\tau_n = \{X^\tau_n(t) \in L(E,F)/t \geq 0\}\}_{n=0}^\infty$ by
$X^\tau_0(t) = X_0$ and for $n \geq 1$,

$$X^\tau_n(t) = X_0 + \sum_{(i,j,k)\in\Lambda}\int_0^t A_{i,j,k}(s, X^\tau_{n-1}(s))dM_{i,j,k}(\tag{3.3}$$

(for the adaptedness of this sequence see, e.g., [8]).
If $0 \leq t_1 \leq t_2 \leq \tau$ then for $n \geq 1$ and $f \in S$,

$$\|[X^\tau_n(t_2) - X^\tau_n(t_1)]y(f)\|^2 \leq \xi_{\tau,f}(t_2-t_1)c^2_{\tau,f}(A)\sup_{0\leq s\leq\tau}\|X^\tau_{n-1}(s)y(f)\|^2$$

and so, by induction, $\{X^\tau_n\}_{n=0}^\infty$ is a sequence of strongly continuous adapted processes.

Now, for each $f \in S$,

$$\|[X^\tau_n(t) - X^\tau_{n-1}(t)]y(f)\|^2 =$$

$$\|\textstyle\sum_{(i,j,k)\in\Lambda}\int_0^t[A_i,j,k(t_1, X^\tau_{n-1}(t_1)) - A_i,j,k(t_1, X^\tau_{n-2}(t_1))]dM_{i,j,k}(t_1)y(f)\|^2 \leq$$

$|\Lambda| \sum_{(i,j,k)\in\Lambda} \| \int_0^t [A_{i,j,k}(t_1, X_{n-1}^\tau(t_1)) - A_{i,j,k}(t_1, X_{n-2}^\tau(t_1))] dM_{i,j,k}(t_1) y(f)\|^2 \leq$
$\xi_{\tau,f}\rho_{\tau,f}^2(A) \int_0^t \|[X_{n-1}^\tau(t_1) - X_{n-2}^\tau(t_1)] y(f)\|^2 dt_1$ (by (2.3) and condition (b))
$\leq [\xi_{\tau,f}\rho_{\tau,f}^2(A)]^2 \int_0^t \int_0^{t_1} \|[X_{n-2}^\tau(t_2) - X_{n-3}^\tau(t_2)] y(f)\|^2 dt_2 dt_1$
$\leq [\xi_{\tau,f}\rho_{\tau,f}^2(A)]^{n-1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \|[X_1^\tau(t_{n-1}) - X_0^\tau(t_{n-1})] y(f)\|^2 dt_{n-1} \ldots dt_2 dt_1$
$\leq [\xi_{\tau,f}\rho_{\tau,f}^2(A)]^{n-1} \xi_{\tau,f} c_{\tau,f}^2(A) \|X_0\|^2 \|y(f)\|^2$

$$\cdot \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} dt_{n-1} \ldots dt_2 dt_1$$

$\leq \frac{[\xi_{\tau,f}\rho_{\tau,f}^2(A)\tau]^{n-1}}{(n-1)!} \xi_{\tau,f} c_{\tau,f}^2(A) \|X_0\|^2 \|y(f)\|^2$
Thus $\sum_{n=1}^{\infty} \|[X_n^\tau(t) - X_{n-1}^\tau(t)] y(f)\| < \infty$
and by the completeness of F we conclude as in the usual O.D.E case that for each $f \in S$ and $t \in [0,\tau]$ we may define

$$X^\tau(t) y(f) = \lim_{n\to\infty} X_n^\tau(t) y(f) \tag{3.4}$$

where the convergence is uniform on $[0,\tau]$

Being a strong limit of adapted procesess, the process $X^\tau = \{X^\tau(t)/0 \leq t \leq \tau\}$ is also adapted. The uniformity of the convergence in the definition of X^τ implies that it is a strongly continuous process.

By the triangle inequality, (2.3) and condition (b) we have,

$$\|[X^\tau(t) - X_0 - \sum_{(i,j,k)\in\Lambda} \int_0^t A_{i,j,k}(s, X^\tau(s)) dM_{i,j,k}(s)] y(f)\|^2$$

$$\leq 2\|[X^\tau(t) - X_{n+1}^\tau(t)] y(f)\|^2 + 2\xi_{\tau,f}\rho_{\tau,f}^2(A) \int_0^\tau \|[X_n^\tau(s) - X^\tau(s)] y(f)\|^2 ds$$

$$\leq 2\|[X^\tau(t) - X_{n+1}^\tau(t)] y(f)\|^2 + 2\xi_{\tau,f}\rho_{\tau,f}^2(A)\tau \cdot \sup_{0\leq s\leq\tau} \|[X_n^\tau(s) - X^\tau(s)] y(f)\|^2$$

which goes to zero as $n \to \infty$

Thus, for each $t \in [0,\tau]$,

$$X^\tau(t) = X_0 + \sum_{(i,j,k)\in\Lambda} \int_0^t A_{i,j,k}(s, X^\tau(s)) dM_{i,j,k}(s)$$

pointwise on E.

If $Y^\tau = \{Y^\tau(t)/0 \le t \le \tau\}$ is another adapted process with the same property then, for each $f \in S$,

$$\|[X^\tau(t) - Y^\tau(t)]y(f)\|^2 \le \xi_{\tau,f}\rho^2_{\tau,f}(A)\int_0^\tau \|[X^\tau(s) - Y^\tau(s)]y(f)\|^2 ds$$

and by Gronwall's inequality,

$$\|[X^\tau(t) - Y^\tau(t)]y(f)\| = 0$$

i.e. X^τ agrees with Y^τ on E for all $t \in [0, \tau]$.

Thus, the process $X = \{X(t)/t \ge 0\}$ defined by

$$X(t) = X^\tau(t), \tag{3.5}$$

where τ is any real number such that $0 \le t \le \tau$, is the process in the statement of the theorem. ■

Remarks:

(i) Since, by the definition of $Q(t), P(t)$ and $T(t)$, $dt = dM_{0,0,1}(t) - dM_{0,1,0}(t) - dM_{1,0,0}(t)$,the above theorem also covers "time integrals".

(ii) If $A_{i,j,k}(s, w(s)) = a_{i,j,k}(s)w(s)b_{i,j,k}(s)$ then $c_{\tau,f}(A)$ is denoted by $c_{\tau,f}(a; b)$. Moreover, if $a_{i,j,k}$ or $b_{i,j,k} = 0$ then $c_{\tau,f}(a; b) = 0$ (see section 5).

Corollary 3.1: *The matrix elements of the solution of (3.2) are given for each $f, g \in S$ and $t \ge 0$ by:*

$$< X(t)y(f), y(g) >= \lim_n < [X_0 + \sum_{(i,j,k)\in\Lambda} \int_0^t A_{i,j,k}(s, X_n(s))dM_{i,j,k}(s)]y(f), y(g) >$$

where X_n is defined as in (3.3).

Proof. The proof follows directly from (3.5), (3.4), (3.3). ■

4. Dependence of solutions on initial conditions

Proposition 4.1 : *Let $X_0, Y_0 \in B(F)$ and let $X = \{X(t)/t \ge 0\}$ and $Y = \{Y(t)/t \ge 0\}$ be the unique solutions of*

$$X(t) = X_0 + \sum_{(i,j,k)\in\Lambda} \int_0^t A_{i,j,k}(s, X(s))dM_{i,j,k}(s) \tag{4.1}$$

and

$$Y(t) = Y_0 + \sum_{(i,j,k)\in\Lambda} \int_0^t A_{i,j,k}(s, Y(s))dM_{i,j,k}(s) \tag{4.2}$$

respectively. Then, for each $f \in S$ and $\tau \geq 0$,

$$\sup_{0\leq t\leq\tau} \|[X(t) - Y(t)]y(f)\|^2 \leq 2\|X_0 - Y_0\|^2\|y(f)\|^2 e^{2.\xi_{\tau,f}(A).\rho^2_{\tau,f}(A).\tau}$$

where $\rho_{\tau,f}(A)$ and $\xi_{\tau,f}(A)$ are defined in condition (b) of section 3 and (3.1) respectively.

Proof. For each $t \in [0,\tau]$,

$$\|[X(t) - Y(t)]y(f)\|^2 \leq 2\|[X_0 - Y_0]y(f)\|^2 + 2.\xi_{\tau,f}.\rho^2_{\tau,f}(A)$$

$$\cdot \int_0^t \|[X(s) - Y(s)]y(f)\|^2 ds$$

and by Gronwall's inequality,

$$\sup_{0\leq t\leq\tau} \|[X(t) - Y(t)]y(f)\|^2 \leq 2\|[X_0 - Y_0]y(f)\|^2 e^{2.\xi_{\tau,f}.\rho^2_{\tau,f}(A).\tau}$$

■

Remark:
If $A(s,0) = 0$ for all s then, Prop. 4.1 implies that

$$\sup_{0\leq t\leq\tau} \|X(t)y(f)\|^2 \leq 2\|X_0\|^2\|y(f)\|^2 e^{2.\xi_{\tau,f}.\rho^2_{\tau,f}(A).\tau}$$

5. Dependence of solutions on coefficients

Suppose that for each $(i,j,k) \in \Lambda$ the adapted processes
$a_{i,j,k}, b_{i,j,k}, c_{i,j,k}, d_{i,j,k} : [0,\infty) \to B(F)$
have the following properties (similar to those assumed in [1]):

(a) **Invariance property:** If $N_{I,J,K}$ is defined as in condition (a) of section 3 then, for each $s \geq 0$, the linear operators
$a_{i,j,k}(s)N_{I,J,K}(s),\quad b_{i,j,k}(s)N_{I,J,K}(s)$,
$c_{i,j,k}(s)N_{I,J,K}(s)$, and $d_{i,j,k}(s)N_{I,J,K}(s)$, map E into itself.

(b) **Boundedness and Lipschitz property:** The families of adapted processes

$$A = \{A_{i,j,k} = \{A_{i,j,k}(s, w(s)) \stackrel{\text{def}}{=} a_{i,j,k}(s)w(s)b_{i,j,k}(s)/s \geq 0,$$

$$w : [0,\infty) \to L(E,F)\}/(i,j,k) \in \Lambda\}$$

and

$$B = \{B_{i,j,k} = \{B_{i,j,k}(s, w(s)) \stackrel{\text{def}}{=} c_{i,j,k}(s)w(s)d_{i,j,k}(s)/s \geq 0,$$

$$w : [0,\infty) \to L(E,F)\}/(i,j,k) \in \Lambda\}$$

satisfy conditions (a) and (b) of section 3.

Then, by the theorem of section 3, there exist strongly continuous adapted processes $X = \{X(t)/t \geq 0\}$ and $Y = \{Y(t)/t \geq 0\}$ uniquely solving the integral equations

$$X(t) = X_0 + \sum_{(i,j,k)\in\Lambda} \int_0^t a_{i,j,k}(s)X(s)b_{i,j,k}(s)dM_{i,j,k}(s) \tag{5.1}$$

and

$$Y(t) = X_0 + \sum_{(i,j,k)\in\Lambda} \int_0^t c_{i,j,k}(s)Y(s)d_{i,j,k}(s)dM_{i,j,k}(s) \tag{5.2}$$

where $X(0) = Y(0) = X_0 \in B(F)$ and we can prove the following:

Proposition 5.1.: *Let $X = \{X(t)/t \geq 0\}$ and $Y = \{Y(t)/t \geq 0\}$ be the (unique) solutions of (5.1) and (5.2) respectively, with $X(0) = Y(0) = X_0 \in B(F)$. Then, for each $f \in S$ and $\tau \geq 0$,*

$$\sup_{0\leq t\leq\tau} \|[X(t) - Y(t)]y(f)\|^2 \leq 6\tau.\xi_{\tau,f}.[c^2_{\tau,f}(a; b - d) + c^2_{\tau,f}(a - c; d)]$$

$$.\|X_0\|^2\|y(f)\|^2. \exp(3.\tau.\xi_{\tau,f}.[\rho^2_{\tau,f}(A) + \rho^2_{\tau,f}(B)])$$

where $\xi_{\tau,f}$ is defined by (3.1) and $c_{\tau,f}$, $\rho_{\tau,f}$ are defined in conditions (a) and (b) of section 3 respectively.

Proof.

By (2.3) and (3.1) for each $f \in S$ and $t \in [0, \tau]$,

$$\|[X(t) - Y(t)]y(f)\|^2 \leq \xi_{\tau,f}. \max_{\alpha,\beta,\gamma\in\{0,1\}} \max_{(i,j,k)\in\Lambda} \max_{\substack{0\leq i' \leq i\delta_{\alpha,0}\\ 0\leq j' \leq j\delta_{\beta,0}\\ 0\leq k' \leq k\delta_{\gamma,0}}}$$

$$\int_0^\tau \|[a_{i,j,k}(s)X(s)b_{i,j,k}(s) - c_{i,j,k}(s)Y(s)d_{i,j,k}(s)]N_{I,J,K}(s)y(f)\|^2 ds$$

where $N_{I,J,K}$ is as in condition (a) of section 3.

Since $\|[a_{i,j,k}(s)X(s)b_{i,j,k}(s) - c_{i,j,k}(s)Y(s)d_{i,j,k}(s)]N_{I,J,K}(s)y(f)\|^2$
$\leq 3\|a_{i,j,k}(s)X(s)[b_{i,j,k}(s) - d_{i,j,k}(s)]N_{I,J,K}(s)y(f)\|^2+$
$3\|[a_{i,j,k}(s) - c_{i,j,k}(s)]X(s)d_{i,j,k}(s)N_{I,J,K}(s)y(f)\|^2+$
$3\|c_{i,j,k}(s)[X(s) - Y(s)]d_{i,j,k}(s)N_{I,J,K}(s)y(f)\|^2$
$\leq 6[c^2_{\tau,f}(a; b - d) + c^2_{\tau,f}(a - c; d)]\|X_0\|^2\|y(f)\|^2 e^{2.\xi_{\tau,f}.\rho^2_{\tau,f}.(A).\tau}$
$+3\rho^2_{\tau,f}(B)\|[X(s) - Y(s)]y(f)\|^2$ we have
$\|[X(t) - Y(t)]y(f)\|^2$
$\leq 6.\tau.\xi_{\tau,f}[c^2_{\tau,f}(a; b - d) + c^2_{\tau,f}(a - c; d)]\|X_0\|^2\|y(f)\|^2 e^{2.\xi_{\tau,f}.\rho^2_{\tau,f}.(A).\tau}$

$$+3.\xi_{\tau,f}.\rho^2_{\tau,f}(B) \int_0^t \|[X(s) - Y(s)]y(f)\|^2 ds$$

and by Gronwall's inequality

$\sup_{0\leq t\leq\tau} \|[X(t)-Y(t)]y(f)\|^2 \leq 6\tau.\xi_{\tau,f}.[c^2_{\tau,f}(a; b-d)+c^2_{\tau,f}(a-c; d)].\|X_0\|^2\|y(f)\|^2$
$.e^{2.\xi_{\tau,f}.\rho_{\tau,f}(A).\tau}. \exp(3.\tau.\xi_{\tau,f}.\rho^2_{\tau,f}(B))$. ■

6. Unitarity of solutions

In [8] and [1] the subject of unitarity of solutions of equations of the type considered in section 5 was examined with the use of the linear independence (in the sense of [1]) of the basic integrators. Due to the complexity of the Ito's formula in the FD case, that method does not always work nicely and so we derive a necessary and sufficient condition on the coefficients of

$$X(t) = X_0 + \sum_{(i,j,k)\in\Lambda} \int_0^t a_{i,j,k}(s)X(s)b_{i,j,k}(s)dM_{i,j,k}(s) \tag{6.1}$$

in order for $X = \{X(t)/t \geq 0\}$ to be unitary, by using Corollary 3.1 (here $a_{i,j,k}$ and $b_{i,j,k}$ are as in section 5). We have:

Proposition 6.1.: Let $\alpha = (i, j, k)$ and let $X = \{X(t)/t \geq 0\}$ be the unique solution of

$$X(t) = X_0 + \sum_{\alpha\in\Lambda} \int_0^t a_\alpha(s)X(s)b_\alpha(s)dM_\alpha(s) \tag{6.2}$$

Then X is unitary iff for all $t \geq 0$ and $f, g \in S$:

$< [X_0 + \sum_{\lambda=0}^{\infty} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{\lambda+1}\in\Lambda} \int_0^{t_0}\int_0^{t_1}\cdots\int_0^{t_\lambda} a_{\alpha_1}(t_1)\ldots$
$a_{\alpha_{\lambda+1}}(t_{\lambda+1})X_0 b_{\alpha_{\lambda+1}}(t_{\lambda+1})\ldots b_{\alpha_1}(t_1)dM_{\alpha_{\lambda+1}}(t_{\lambda+1})\ldots dM_{\alpha_1}(t_1]y(f),$
$[X_0 + \sum_{\mu=0}^{\infty} \sum_{\beta_1,\beta_2,\ldots,\beta_{\mu+1}\in\Lambda} \int_0^{t_0}\int_0^{t_1}\cdots\int_0^{t_\mu} a_{\beta_1}(t_1)\ldots a_{\beta_{\mu+1}}(t_{\mu+1}).$

$$X_0 b_{\beta_{\mu+1}}(t_{\mu+1})\ldots b_{\beta_1}(t_1)dM_{\beta_{\mu+1}}(t_{\mu+1})\ldots dM_{\beta_1}(t_1)]y(g) =< y(f), y(g) > \tag{6.3}$$

where $t_0 = t$.

Proof.

The process $X = \{X(t) \in L(E, F)/t \geq 0\}$ is unitary iff for each $t \geq 0$ the operator $X(t)$ is unitary i. e. iff for all $f, g \in S$:

$$< X(t)y(f), X(t)y(g) =< y(f), y(g) > \iff \quad (\textit{by Corollary } 3.1)$$

$$\lim_{n,m} < [X_0 + \sum_{\alpha_1\in\Lambda} \int_0^t a_{\alpha_1}(s)X_n(s)b_{\alpha_1}(s)dM_{\alpha_1}(s)]y(f),$$

$$[X_0+\sum_{\beta_1\in\Lambda}\int_0^t a_{\beta_1}(s)X_m(s)b_{\beta_1}(s)dM_{\beta_1}(s)]y(g)>=<y(f),y(g)> \iff \quad (iteratively)$$

to (6.3). ∎

E x a m p l e : Let $\lambda \in \mathbb{R}$ and $i=\sqrt{-1}$. By thm. 3.1 and cor. 3.1, the integral equation

$$X(t)=I+i\lambda\int_0^t X(s)dT(s) \tag{6.4}$$

has a unique solution $X=\{X(t)/t\geq 0\}$ whose matrix elements are given for each $f,g\in S$ and $t\geq 0$ by:
$<X(t)y(f),y(g)>=\lim_n <X_n(t)y(f),y(g)>$
$=<[I+\sum_{n=1}^{\infty}\int_0^{t_0}\dots\int_0^{t_{n-1}}(i\lambda)^n dT(t_n)\dots dT(t_1)]y(f),y(g)>$
(where $t_0=t$)
$=<y(f),y(g)>+\sum_{n=1}^{\infty}(i\lambda)^n<\int_0^{t_0}\dots\int_0^{t_{n-1}}dT(t_n)\dots dT(t_1)y(f),y(g)>$
$=[1+\sum_{n=1}^{\infty}(i\lambda)^n\int_0^{t_0}\dots\int_0^{t_{n-1}}\phi(t_n)\dots\phi(t_1)dt_n\dots dt_1]<y(f),y(g)>$
(by (2.2), with $\phi(t)\overset{\text{def}}{=}\frac{(1+f)(1+g)}{1-fg}(t)$)
$=[1+\sum_{n=1}^{\infty}(i\lambda)^n N_n(t)]<y(f),y(g)>$
$=\sum_{n=0}^{\infty}(i\lambda)^n N_n(t)<y(f),y(g)>$
where $N_0(t)=1,\quad N_1(t)=\int_0^t\phi(s)ds$ and for $n=2,3,\dots$

$$N_n(t)=\int_0^t N_{n-1}(s)\phi(s)ds.$$

Thus $N_n(t)=\frac{[N_1(t)]^n}{n!},\quad n=0,1,2,\dots$ and so

$$<X(t)y(f),y(g)>\exp(i\lambda\int_0^t\Phi(s)ds)<y(f),y(g)>.$$

One might be tempted to think that X is unitary for all $\lambda\in\mathbb{R}$ To see that is not the case, it is easier to use the Hudson - Parthasarathy method for unitarity (cf: [8]). So suppose that X is unitary.

Since,

$$dX=i\lambda X\,dT$$

and

$$dX^* = -i\lambda X^* dT$$

we have

$$XX^* = I \iff d(XX^*) = 0$$

$$\iff X\dot{d}X^* + dX\dot{X}^* + dX\dot{d}X^* = 0 \quad (by\ lemma\ 3.3\ of\ [5])$$

$$\iff -i\lambda(XX^*)dT + i\lambda(XX^*)dT + \lambda^2(XX^*)(dT)^2 = 0$$

$$\iff \lambda^2(dT)^2 = 0$$

$$\iff \lambda = 0.$$

Thus X is unitary iff $\lambda = 0$.

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Received 16.07.1993