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Seperation Properties in Topological Categories

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In this paper, it is shown that there are various ways to charecterize separation axioms in the category of topological spaces in terms of the concepts which make sense in any topological category, e.g. in terms of initial lifts, final lifts, and discreteness. These generalizations include two notions of T_0, one notion of T_1, and four notions of each of T_2, T_3, and T_4.

1. Introduction.

The notion of topological space has been extended in various ways to include convergence spaces, limit spaces, bornological spaces, uniform spaces, nearness spaces, and preordered spaces by D. C. Kent [7], L. D. Nel [10], F. Schwarz [12], O. Wyler [13], among others, to the notion of a topological category. The more general notions of topological functors and topological category were introduced by H. Herrlich [3] and represents a generalization of the ideas of induced and coinduced topologies in terms of initial and final lifts. If one wishes to study the extent to which theorems in general topology can be formulated and proved in the more general setting of a topological category it is necessary to first reformulate certain basic concepts which make sense in any topological category e.g. in terms of initial and final lifts, discreteness, and indiscreteness. Some basic concepts in general topology are the notions of separation properties (T_0, T_1, T_2, T_3, T_4) which appear in many important theorems such as the Urysohn Metrization theorem, the Urysohn lemma, the Tietze extension theorem, among others. In view of this, it is useful to be able to extend these notions to arbitrary topological categories.

Broadly speaking, these separation properties involve 'Separating' certain kinds of sets (points and closed sets) from one another by (disjoint) open sets. In order to generalize these notions to arbitrary topological categories, we need to express, as mentioned above, each of these notions in terms of initial lifts, final lifts, and discreteness. We introduce some notions such as the principal axis map, the skewed axis map, the wedge product, and the fold map which are needed to define the separation properties.

We first define separation properties at a point, p i.e. locally, then we generalize this point to free definitions using the generic element, [5] p.39, method of topos theory for an arbitrary topological category over sets. One reason for doing this is that, in general objects in a topos may not have points, however they always have a generic point. The other reason is the notion of closedness on arbitrary topological categories is defined in terms of T_0 and T_1 at a point, p. 335 [1].

In general topology, the separation properties T_0 and T_1 are hardly used in general. However, we shall see in this paper that they become very important because they are used to define T_1, T_2, and T_4. There is also one other important separation axiom, namely Pre T_2 (which is not used in general topology) that has already appeared in [4] as a generalized Hausdorff condition arising in the study of geometric realization functors that preserve finits limits.

The main purpose of this paper is to charecterize the separation properties in this category of topological spaces in terms of the concepts which make sense in arbitrary topological categories.

Let X be a set and p a point in X. Let $X \vee_p X$ be the wedge product of X with itself, i.e. two distinct copies of X identified at the point p. A point x in $X \vee_p X$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $X \vee_p X$. Let $X^2 = X \times X$ be the cartesian product of X with itself and, $X^2 \vee_\Delta X^2$ be two distinct copies of X^2 identified along the diagonal. A point in $X^2 \vee_\Delta X^2$ will be denoted by $(x, y)_1((x, y)_2)$ if (x, y) is in the first (resp. second) component of $X^2 \vee_\Delta X^2$. Clearly $(x, y)_1 = (x, y)_2$ iff $x = y$.

1.1. Definitions.

The principal p-axis map, $A_p : X \vee_p X \to X^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed p-axis map, $S_p : X V_p X \to X^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, $\nabla_p : X V_p X \to X$ is given by $\nabla_p(x_i) = x$ for $i = 1, 2$.

1.2. Example

If X is the set of real numbers and $p = 0$, then the image of the principal p-axis map is just the union of the $x-$ and $y-$axes, and the image of the skewed

p-axis map is the union of the diagonal, i.e. the line $y = x$, and the y-axis.

This example motivates the terminology in 1.1. Hence, in this way we may view the image of A_p and S_p as 'axes' in X^2 with origin p.

1.3. Definitions

The principal axis map $A : X^2 \vee_\nabla X^2 \to X^3$ is given by $A(x,y)_1 = (x,y,x)$ and $A(x,y)_2 = (x,x,y)$. The skewed axis map $S : X^2 \vee_\nabla X^2 \to X^3$ is given by $S(x,y)_1 = (x,y,y)$ and $S(x,y)_2 = (x,x,y)$, and the fold map, $\nabla : X^2 \vee_\nabla X^2 \to X^2$ is given by $\nabla(x,y)_i = (x,y)$ for $i = 1,2$.

Let X be a topological space and p be a point in X.

1.4. Definitions

1. X is said to be $T_\emptyset$ at p iff for any point $q \neq p$, there exists a neighborhood N_p of p not containing q or there exists a neighborhood N_q of q not containing p.

2. X is said to be T_1 at p iff for any point $q \neq p$, there exist neighborhoods N_p and N_q of p and q, respectively such that $p \notin N_q$ and $q \notin N_p$.

3. X is said to be PreT_2 at p iff for each $q \neq p$, if the set {p,q} is not indiscrete, then there exist disjoint neighborhoods N_p and N_q of p and q, respectively.

4. X is said to be T_2 at p iff for each $q \neq p$, there exist disjoint neighborhoods N_p and N_q of p and q, respectively.

5. X is said to be T_3 at p iff X is T_1 at p and X/F is PreT_2 at p for all nonempty closed subsets, F of X missing p, where X/F is the quotient space that is induced from the quotient map $q : X \to X/F$ identifying F to a point, $\star$.

6. X is said to be T_4 at p iff X is T_1 at p and X/F is T_3 at $\star$ for all nonempty closed subsets, F of X containing p, where the point $\star$ defined above.

Let X be a topological space and $p \in X$.

1.5.Theorem. *1. X is $T_\emptyset$ at p iff the induced topology on $X \vee_p X$ via $\{A_p : X \vee_P X \to X^2$ and $\nabla_p : X \vee_p X \to DX\}$ is a discrete topological space, where DX is X equipped with the discrete topology.*

2. X is T_1 at p iff the induced topology on $X \vee_p X$ from X^2 and DX by S_p and ∇_p, respectively, is discrete.

3. X is PreT_2 at p iff the induced topologies on $X \vee_p X$ from X^2 by A_p and S_p agree.

4. X is T_2 at p iff X is T_0 and PreT_2 at p.

5. X is $T_i, i = 0,1,2,3,4$ iff X is T_i at p for all p in $X, i = 0,1,2,3,4$.

Proof 1. Suppose X is T_0 at p. It is sufficient to show that each one-point set is open in $X \vee_p X$. If $q \neq p$, then there exists a neigborhood N_p of p not containing q or there exists a neighborhood N_q of q not containing p. If the first case (resp. the second case) happens, then let $U = \{q\}$ and $W = N_p \times X$ (resp. $W = X \times N_q$) which are open in DX and in X^2, respectively.

Clearly $\nabla_p^{-1}(U) \cap A_p^{-1}(W) = \{q_2\}$ (resp. $\{q_1\}$). If $q = p$, then let $U = \{p\}$ and $W = X^2$ which are open in DX and X^2, respectively. Clearly $\nabla_p^{-1}(U) \cap A_p^{-1}(W) = \{p\}$. Hence the induced topology on $X \vee_p X$ from X^2 and DX via A_p and ∇_p, respectively is discrete.

Conversely, suppose the induced topology is discrete. We show that X is T_0 at p, i.e by 1.4, for each $q \neq p$ and each neighborhood N_q of q that contains p, there exists a neighborhood N_p of p that does not contain q. Since the induced topology is discrete $\{q_1\} = \nabla_p^{-1}(\{q\}) \cap A_p^{-1}(W)$ for some W open in X^2, and consequently there exist open sets N_q and V in X such that $A_p(q_1) = (q,p) \in N_q \times V \subset W$. If q is not in V, then let $N_p = V$. If q is in V, then $(p,q) \in N_q \times V \subset W$ since all neighborhood N_q of q contains p. However, $\{q_1, q_2\} = \nabla_p^{-1}(\{q\}) \cap A_p^{-1}(W)$, a contradiction. Hence $q \notin V$. This completes the proof.

2. Suppose X is T_1 at p i.e. by 1.4 for each $q \neq p$, each has a neighborhood not containing the other. Note that $\{q_1\} = \nabla_p^{-1}(\{q\}) \cap S_p^{-1}(N_q \times N_q)$ and $\{q_2\} = \nabla_p^{-1}(\{q\}) \cap S_p^{-1}(N_p \times N_q)$, where N_p and N_q are open in X with q is not in N_p and p is not in N_q. Note also $\{p\} = \nabla_p^{-1}\{p\} \cap S_p^{-1}(X^2)$. Hence the induced topology is discrete.

Conversely, suppose the condition holds, and let $q \neq p$. $\{q_1\} = \nabla_p^{-1}(\{q\}) \cap S_p^{-1}(W)$ for some W open in X^2. In particular, $S_p(q_1) = (q,q) \in W$ and consequently there exists a neighborhood N_q of q such that $(q,q) \in N_q \times N_q \subset W$. Clearly, p is not in N_q otherwise $(p,q) \in W$ and consequently $\{q_1, q_2\} = \nabla_p^{-1}(\{q\}) \cap S_p^{-1}(W)$, a contradiction. Similiarly $\{q_2\} = \nabla_p^{-1}(\{q\}) \cap S_p^{-1}(W)$ for some open W in X^2 and consequently there exists a neighborhood N_q and N_p of q and p, respectively such that $(p,q) \in N_p \times N_q \subset W$. Clearly, q is not in N_p. Hence X is T_1 at p.

3. To prove the part (3) we need the following lemma ([9], p.80):

1.6. Lemma. *Let B and B' be bases for the topologies ξ and ξ', respectively, on X. Then the following are equivalent. a) $\xi \subset \xi'$. b) For each x in X and each basis element B in B containing x, there is a basis element B' in B' such that $x \in B' \subset B$.*

We now proceed to prove part (3) of 1.5.

Suppose X is PreT_2 at p. We show that $A^{-1}(\xi) = S_p^{-1}(\xi)$ where ξ is a product topology on X^2. To show $A^{-1}(\xi) \subset S_p^{-1}(\xi)$, we use Lemma 1.6. Let U be any basis element of $A_p^{-1}(\xi)$ and $q \in U$. It follows that $U = A_p^{-1}(W)$ for some basis element W of ξ. $A_p(q) = (q,p)$ or (p,q). Suppose $A_p(q) = (p,q) \in W$ and consequently there exist neighborhoods N_p and N_q of p and q, respectively, and $W = N_p \times N_q$. If $\{p,q\}$ is indiscrete, then let $W_1 = W$ and clearly $q = q_2 \in S_p^{-1}(W) \subset A_p^{-1}(W)$. If $\{p,q\}$ is not indiscrete, then there exist disjoint neighborhoods N'_p and N'_q of p and q, respectively. Let $W_1 = (N_p \cap N'_p) \times (N_q \cap N'_q)$ and clearly $q = q_2 \in S_p^{-1}(W_1) \subset A_p^{-1}(W)$. Suppose $A_p(q) = (q,p) \in W = N_q \times N_p$. If $\{p,q\}$ is indiscrete, then clearly $q = q_1 \in S_p^{-1}(W_1) \subset A_p^{-1}(W)$. If $\{p,q\}$ is not indiscrete, then there exist disjoint neigborhoods N'_p and N'_q of p and q, respectively (since X is PreT_2 at p). Let $W_1 = (N_q \cap N'_q) \times (N_q \cap N'_q)$ and clearly $q = q_1 \in S_p^{-1}(W_1)$. We now show that $S_p^{-1}(W_1) \subset A_p^{-1}(W)$. Let $r \in S_p^{-1}(W_1)$, then $S_p(r)$ must be (r,r) (since $p \notin N'_q$) and consequently $(r,p) \in W$. Hence $r \in A_p^{-1}(W)$. Therefore $A_p^{-1}(\xi) \subset S_p^{-1}(\xi)$.

We next show that $S_p^{-1}(\xi) \subset A_p^{-1}(\xi)$. Again we use Lemma 1.6. Let $U = S_p^{-1}(W)$ be a basis element of $S_p^{-1}(\xi)$, where W is a basis element of the product topology ξ and let q be in U. $S_p(q) = (q,q)$ or (p,q).

Suppose $S_p(q) = (q,q)$. We can take $W = N_q \times N_q$, where N_q is a neighborhood of q. If $\{p,q\}$ is indiscrete, then clearly $A_p^{-1}(W) \subset S_p^{-1}(W)$. If $\{p,q\}$ is not indiscrete , then there exist disjoint neighborhoods N_p and N'_q of p and q, respectively. Let $W_1 = (N_q \cap N'_q) \times N_p$ and clearly $q \in A_p^{-1}(W)$. If $r \in A_p^{-1}(W_1)$, then $A_p(r)$ must be (r,p) since $p \notin Nq'$ and consequently $A_p^{-1}(W_1) \subset S_p^{-1}(W)$.

Suppose $S_p(q) = (p,q)$. We can take $W = N_p \times N_q$,where N_q is a neighborhood of q, If $\{p,q\}$ is indiscrete, then clearly $A_p^{-1}(W) \subset S_p^{-1}(W)$. If (p,q) is not indiscrete, then there exist disjoint neighborhoods N'_p and N'_q of p and q, respectively.

Let $W_1 = (N_p \cap N'_p) \times (N_q \cap N'_q)$.It is easy to see $A_p^{-1}(W_1) \subset S_p^{-1}(W)$ and consequently $S_p^{-1}(\xi) \subset A_p^{-1}(\xi)$. Therefore they are equal.

Conversely, suppose $S_p^{-1}(\xi) = A_p^{-1}(\xi)$. Will show that X is PreT_2 at p. Suppose $\forall q \neq p$, the set $\{p,q\}$ is not indiscrete i.e. there exists either a neighborhood N_p of p with $N_p \cap \{p,q\} = \{p\}$ or a neighborhood N_q of q with $N_q \cap \{p,q\} = \{q\}$. Suppose $N_q \cap \{p,q\} = \{q\}$. Note that $q_2 \in S_p^{-1}(X \times N_q)$ and by assumption there exists a basis element $A_p^{-1}(W)$ of $A_p^{-1}(\xi)$ containing q_2 and $A_p^{-1}(W) \subset S_p^{-1}(X \times N_q)$, where $W = N_p \times (N_q \cap N'_q)$. Again by assumption, there exists a basis element $S_p^{-1}(W_1)$ of $S_p^{-1}(\xi)$ containing q_2 and $S_p^{-1}(W_1) \subset A_p^{-1}(W)$ where $W_1 = (N_p \cap N'_p) \times (N_q \cap N'_q \cap N''_q) = N \times M$. We claim that $N \cap M = 0$. Otherwise, suppose $r \in N \cap M$, and consequently $r_1 \in S_p^{-1}(W)$. But r_1 is not in $A_p^{-1}(W)$ since $A_p(r_1) = (r,p) \in W$. This implies $p \in N_q$, a contradiction. Hence $N \cap M = \phi$. Suppose $N_p \cap \{p,q\} = \{p\}$. Then $q_1 \in A_p^{-1}(X \times N_p = U)$. By assumption there exists a basis element $S_p^{-1}(W)$ of $S_p^{-1}(\xi)$ such that $q_1 \in S_p^{-1}(W) \subset A_p^{-1}(U)$, where $W = N_q \times N_q \cdot p$ is not in N_q. Otherwise $q_2 \in S_p^{-1}(W)$ but q_2 is not in $A_p^{-1}(U)$(since $q \notin N_p$). Hence p is not in N_q i.e $N_q \cap \{p,q\} = \{q\}$ and by the first case in above we get the result. Therefore X is PreT_2 at p.

The proof of the parts (4) and (5) follows easily.

1.7 Lemma. *The induced topology on $X \vee_p X$ from X^2 via A_p and the wedge topology ξ' of $X \vee_p X$ i.e the coinduced topology on $X \vee_p X$ from X via injections i_1 and i_2 are the same.*

Proof. To show $\xi' \subset A_p^{-1}(\xi)$, where ξ is the product topology on X^2, let $U \in \xi'$. Hence $i_1^{-1}(U)$ and $i_2^{-1}(U)$ are open in X. If p is in U, then let $V = i_1^{-1}(U) \times i_2^{-1}(U)$. Note that V is open in X^2 and $U = A_p^{-1}(V)$. Hence $U \in A_p^{-1}(\xi)$. If $p \notin U$, then let $V = (i_1^{-1}(U) \times B) \cup (B \times i_2^{-1}(U))$ which is open in X^2. It is easy to see that $U = A_p^{-1}(V)$. Hence $U \in A_p^{-1}(\xi)$. Therefore $\xi' \subset A_p^{-1}(\xi)$. We now show that $A_p^{-1}(\xi) \subset \xi'$. Let $U \in A_p^{-1}(\xi)$ i.e $U = A_p^{-1}(W)$ for some open $W = \bigcup_{i\in I}(N_i \times M_i)$ in X^2 and N_i M_i are open in X for each i. It is easy to see that $i_k^{-1}(A_p^{-1}(N_i \times M_i)) = N_i, M_i$ or Φ, the empty set, for $k = 1,2$. Consequently $i_k^{-1}(U)$ is open in X for $k = 1,2$. Hence $U \in \xi'$. Therefore $A_p^{-1}(\xi) \subset \xi$. This completes the proof.

1.8 Remarks

By Lemma 1.7 we have:

1. X is $T_\emptyset$ at p iff the induced topology on $X \vee_p X$ from the wedge space $(X \vee_p X, \xi')$ and the discrete space, DX, via the identity map and the fold map at p, ∇_p, respectively, is discrete.

2. X is PreT_2 at p iff $\xi' = S_p^{-1}(\xi)$, where ξ' is the wedge topology defined above and ξ is the product topology on X^2.

In order to give a theorem related to a point free definition of the separation properties(usual ones) for a topological category over sets, which works for a topos also, we turn to the generic point method of topos theory [5] p.39. Since in general, objects in a topos may not have points, but they always have the generic point.

Let *Top* and *Sets* denote the category of topological spaces and the category of sets, respectively. Recall [8] p.279 that the forgetful topological functor $U : Top \to Sets$ has a left adjoint, called the discrete functor.

Let X be a set and $Sets/X$ be the localization of X,[6] p.46. It is well known that the functor $(\ )^* : Sets \to Sets/X$ given by $A^* = (\pi_1 : X \times A \to X)$ has a left adjoint $\sum : Sets/X \to Sets$ defined by $\sum(f : A \to X) = A$, [5] p.35. Recall that under $\sum$ the generic element $\delta : 1 \to X^*X$,[5]p.39, of X corresponds to the diagonal $\Delta : X \to X^2$, where 1 is the teminal object of $Sets/X$. It is not hard to see that after applying $\sum$ to the principal δ axis map, A_δ, the skewed δ axis map, S_δ, and the fold map, ∇_δ at δ, we get the principal axis map, A, the skewed axis map, S, and the fold map ∇, respectively.

Let X be a topological space and Top/X be the localization of X. Theorem 1.5 at the generic point, δ corresponds to the following theorem which is a point free version of 1.5.

2.1 Theorem *1. X is $T_\emptyset$ iff the induced topology on $X^2 \vee_\Delta X^2$ from X^3 and DX^2 via A and ∇, respectively is discrete, where DX^2 is X^2 with discrete topology.*

2. X is T_1 iff the induced topology on $X^2 \vee_\Delta X^2$ from X^3 and DX^2 via S and ∇, respectively is discrete.

3. X is $PreT_2$ i.e. for each distinct pair x and y, if the set $\{x, y\}$ is not indiscrete, then there exist disjoint neighborhoods of x and y iff the induced topologies on $X^2 \vee_\Delta X^2$ from X^3 by A and S agree.

4. X is T_2 iff X is $T_\emptyset$ and $PreT_2$.

5. X is T_3 iff X is T_1 and X/F is $PreT_2$ for all nonempty closed subsets, F of X, where X/F is defined in 1.5.

6. X is T_4 iff X is T_1 and X/F is T_3 for all nonempty closed subsets, F of X.

Proof. 1. Suppose the induced topology is discrete. We show that for any distinct points x and y if each neighborhood N_x of x contains y, then there exists a neighborhood N_y of y not containing x. Note that $\{(x,y)_1\} = \nabla^{-1}(U) \cap A^{-1}(W)$ for some open sets U and W in DX^2 and X^3, respectively. It follows that $U = \{(x,y)\}$ and $W \supset N_x \times N_y \times N_x$ for some neighborhoods N_x and N_y of x and y, respectively. Note that $x \notin N_y$ since, otherwise $(x,x,y) = A((x,y)_2) \in W$ and consequently $\{(x,y)_2, (x,y)_1\} = \nabla^{-1}(U) \cap A^{-1}(W)$, a contradiction.

Conversely, suppose X is $T_\emptyset$ i.e. for any distinct points x and y, there exists a neighborhood N_x of x not containing y or there exists a neighborhood N_y of y not containing x. It is easy to see that if the first case holds, then $\{(x,y)_2\} = \nabla_{-1}(\{(x,y)\}) \cap A^{-1}(N_x \times N_x \times X)$. If the second case holds, then $\{(x,y)_1\} = \nabla - 1(\{(x,y)\}) \cap A^{-1}(X \times N_y \times X)$. If $x = y$, then $\{(x,y)_1 = (x,y)_2\} = \nabla^{-1}(\{(x,y)\}) \cap A^{-1}(X^3)$. Hence each singleton in $X^2 \vee_\Delta X^2$ is open and consequently the induced topology is discrete.

The proof of the part (2) is similar to the part (1) and the part (2) of Theorem 1.5.

3. Suppose $A^{-1}(\xi) = S^{-1}(\xi)$ where ξ is the product topology on X^3. We show that X is PreT_2. Suppose for any distinct points x and y of X, the set $\{x, y\}$ is not indiscrete i.e. there exists a neighborhood N_y of y not containing x or there exists a neighborhood N_x of x not containing y. We consider the first case. Note that, by assumption and consequently by Lemma 1.6 $(x,y)_2 \in S^{-1}(W_2) \subset A^{-1}(W_1) \subset S^{-1}(W)$, where $W = X^2 \times N_y, W_1 = N_x \times N_x \times (N_y \cap N'_y)$, and $W_2 = (N_x \cap N'_x) \times (N_x \cap N'_x) \times (N_y \cap N'_y \cap N''_y) = N \times N \times M$. We claim that $N \cap M = \Phi$. Suppose there exists $r \in N \cap M$. Note that $(x, r, r) \in W_2$. Hence $(x,r)_1 \in S^{-1}(W_2)$ and consequently $(x,r)_1 \in A^{-1}(W_1)$, a contradiction since $x \notin N_y$. For the remaining case just change the role of x and y. Hence X is PreT_2.

Conversely, suppose X is PreT_2. To show $S^{-1}(\xi) \subset A^{-1}(\xi)$ we use Lemma 1.6. Let $U = S^{-1}(W)$ be any basis element of $S^{-1}(\xi)$, where W is a basis element of the product topology, ξ on X^3. Let $(x,y) \in U$. Then $S(x,y) = (x,y,y)$ or (x,x,y). Suppose $S(x,y) = (x,y,y) \in W$,where $W = N_x \times N_y \times N_y$ and N_x and N_y are neighborhoods of x and y, respectively. If the set $\{x,y\}$ is indiscrete, then clearly $(x,y)_1 \in A^{-1}(W) \subset S^{-1}(W)$. If $\{x,y\}$ is not indiscrete, then there exist disjoint neighborhoods N'_x and N'_y of x and y, respectively. Let $W_1 = (N_x \cap N'_x) \times (N_y \cap N'_y) \times (N_x \cap N'_x)$ and clearly $(x,y)_1 \in A^{-1}(W_1)$. To show $A^{-1}(W_1) \subset S^{-1}(W)$, let $(c,d) \in A^{-1}(W)$. Then $A(c,d) = (c,d,c)$ or (c,c,d). The second case can not happen since $N'_x \cap N'_y$ is empty. For the first case, we have $(c,d,d) \in W$ since $d \in N_y$, and consequently $(c,d) \in S^{-1}(W)$. Now suppose $S(x,y) = (x,x,y) \in W = N_x \times N_x \times N_y$. If $\{x,y\}$ is indiscrete, then let $W_1 = N_x \times (N_x \cap N_y) \times N_y \subset W$. It is easy to see that $(x,y) \in A^{-1}(W_1)$ and $A^{-1}(W_1) \subset S^{-1}(W)$. If the set $\{x,y\}$ is not indiscrete, then by assumption there exist disjoint neighborhoods N'_x and N'_y of x and y, respectively. Let $W_1 = (N_x \cap N'_x) \times (N_x \cap N'_x) \times (N_y \cap N'_y)$ and clearly $(x,y)_2 \in A^{-1}(W_1)$. To show $A^{-1}(W_1) \subset S^{-1}(W)$, let $(c,d) \in A^{-1}(W_1)$. Then $A(c,d) = (c,d,d)$ or (c,c,d). The first case not happen since $N'_x \cap N'_y$ is empty. For the second case, we have $(c,d) \in S^{-1}(W)$. Therefore $S^{-1}\xi \subset A^{-1}(\xi)$.

To show the converse, we use Lemma 1.6, again. Let $U = A^{-1}(W)$ be any basis element of $A^{-1}(\xi)$ and $(x,y) \in U$. Then $A(x,y) = (x,x,y)$ or (x,y,x). Suppose $A(x,y) = (x,x,y) \in W = N_x \times N_x \times N_y$. If $\{x,y\}$ is indiscrete, then let $W_1 = (N_x \cap N_y) \times N_x \times N_y \subset W$, and it follows easily that $S^{-1}(W_1) \subset A^{-1}(W)$. If $\{x,y\}$ is not indiscrete, then there exist disjoint neighborhoods N'_x and N'_y of x and y, respectively. Let $W_1 = (N_x \cap N'_x \times N_x) \cap (N'_x) \times (N_y \cap N'_y)$. It is clear $(x,y) \in S^{-1}(W_1) \subset A^{-1}(W)$. Suppose $A(x,y) = (x,y,x) \in W = N_x \times N_y \times N_x$. If the set $\{x,y\}$ is indiscrete, then $S^{-1}(W) \subset A^{-1}(W)$. If $\{x,y\}$ is not indiscrete, then let $W_1 = (N_x \cap N'_x) \times (N_y \cap N'_y) \times (N_y \cap \mathrm{N}'_y)$, where $N'_x \cap N'_y = \Phi$ (since X is PreT_2). It follows easily that $(x,y) \in S^{-1}(W_1) \subset A^{-1}(W)$. Hence $A^{-1}(\xi) \subset$

$S^{-1}(\xi)$. Therefore $A^{-1}(\xi) = S^{-1}(\xi)$.

The parts (4),(5) and (6) follows easily from 1.4 and 1.5.

2.2 Remarks

1. It follows from 1.5 and 1.8 that $A^{-1}(\xi) = \xi'$, where ξ' is the coinduced topology on $X^2 \vee_\Delta X^2$ from X^2 via injections i_1 and i_2.

2. We can also have :X is $T_\emptyset$ iff the induced topology on $X^2 \vee_\Delta X^2$ from the wedge topological space $X^2 \vee_\Delta X^2$ end discrete topological space DX^2 via the identity map and the fold map, respectively is discrete.

3. X is PreT_2 iff $S^{-1}(\xi) = \xi'$ where ξ is the product topology on X^3 and ξ' is the wedge topology on $X^2 \vee_\Delta X^2$.

4. By parts (2) and (3) and Theorem 2.1. we can have two ways of characterizing each of T_0, PreT_2, T_3, and T_4, and four ways of characterizing T_2. Furthermore in [1] the notion of closedness was introduced in terms of T_1 at p and T_0 at p. In view of this and the above results, the various generalizations of the separation properties, namely, two notions of each of T_0 and PreT_2, and four notions of each of T_2, T_3, and T_4 are defined in [1] for an arbitrary topological category over sets.

5. There are several well-known generalizations of the usual $T_\emptyset$ axiom of topology (see[12]). F. Schwarz in [12], has showed that these notions lead to two different concepts: $T_\emptyset$ and separatedness. In [2], it is shown that our $T_\emptyset$'s and known ones, in general, are different.

6. General results involving relationships among our various generalized separation properties as well as interrelationships among their various forms will be established in a subsequent paper. Furthermore, in the subsequent paper, four various generalizations of each of T_2 (Hausdorff) and T_5 (completely normal) will be given.

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