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Controlled Wiener Process by External Intervations Minimizing the Average Time with a Drift [1]

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The paper deals with a problem of control of a Wiener process with positive drifts appearing at random times. External interventions at suitable times are allowed to eliminate these drifts. A model of control of the process is investigated with accordingly defined external interventions so as to minimize the average time when the controlled process runs with a drift. A special case is considered when the problem of optimal control has an explicit solution.

1.Introduction

Several problems of control of Wiener processes were studied during the last years. The models and the results are of interest for both, the theory and the applications to other fields. J.H.R a t h [12] formulates a problem of control of a reflected Wiener process where there is a choice between two sets of drift and diffusion parameters. H. C h e r n o f f and A.J. P e t k a u [3] generalize the Rath's problem. J.M. H a r r i s o n and M.I. T a k s a r [8], M.I. T a k s a r [15], [16], D. P e r r y and S.K. B a r - L e v [10] consider some models of a continuous control of a Wiener process with drift describing also an application in storage systems. The general goal of the proposed control schemes is to minimize suitable cost functions. In a series of papers Dimitrov, Petrov, Barosov, Kolev, Geist, Reynolds, Westal ect. (1976-1990) consider models of control of unreliable processes with implicit or explicit breakdowns (see [5], [6]). In fact, in these papers a mechanism of external interventions is used as well as we do in [2] and in the present paper but for the diffusion type processes.

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Let us note that the Wiener process which acquires a drift after a random time interval may represent the cumulative output of an industrial process, which is under control so long as the average output is zero, but which may go out of control and then must be corrected as soon as possible (see [11]).

Diffusion processes with controlled drift serve as models of systems in which the 'rate' of randomness is exogenous and beyond control, and the varying drift represents the controlled input or output of the system. Examples of such systems can be inventory models, in which the driving diffusion process is equal to the difference between the deterministic production and random demand (see [16]). For queuing models in heavy traffic the driving diffusion process describes the number of the customers in the system, while the variation of drift corresponds to changing the rate of service (see [12], [16]).

2. Statement of the problem

Let $(\Omega, \mathcal{F}, p)$ be a probability space on which a standard one-dimensional Wiener process $W = (W(t), t \in [0, \infty))$ and an independent non-negative random variable (r.v.) ξ are defined.

Definition 1. The random process $Z = (Z(t), t \in [0, \infty))$, defined by the equation:

$$Z(t) = b + \mu.(t - \xi).I\{\xi < t\} + W(t),\ t \in [0, \infty),\ b, \mu \in R,$$

is said to be a Wiener process with initial state b, which acquires a linear drift with parameter μ at time ξ. Here $I\{.\}$ is an indicator of the set $\{.\}$.

Define the random process $X = (X(t), t \in [0, \infty))$:

$$X(t) = W(t) + \mu.(t - \xi)I\{\xi < t\},\ \mu \in [0, \infty).$$

In other words, X is a Wiener process with initial state 0, which acquires a linear drift with parameter $\mu > 0$ at time ξ. The process X is considered up to its first hitting time of a fixed level A. Here A is a positive real number.

Let b be a real non-negative number, and $W^b = (W^b(t), t \in [0, \infty))$ be an one-dimensional Wiener process with an absorbing screen in the state b. We define the following times of first hitting the level b:

$$r(b) = \inf\{t \geq 0 : W(t) = b\};$$

$$r'(b, \xi) = \inf\{t \geq 0 :\ W^b(\xi) + W(t) + \mu t = b\};$$

$$T(b, \xi) = \inf\{t \geq 0 : W(t) + \mu.(t - \xi)I\{\xi < t\} = b\}.$$

Since W is a Markov process and ξ is an independent of W r.v., we have:

$$T(b,\xi) = r(b).I\{\sup_{0\le t\le \xi} W(t) \ge b\} + (\xi + r'(b,\xi)).I\{\sup_{0\le t\le \xi} W(t) < b\};$$

$$T(b,\xi) = r(b).I\{r(b) \le \xi\} + (\xi + r'(b,\xi)).I\{r(b) > \xi\}.$$

If no external interventions in the behavior of the process X are realized, the time interval when we consider the process will be $[0, T(A,\xi)]$. Note that $r'(A,\xi)$ is time period in this interval when the process X runs with non-zero drift.

If we introduce an external intervention before the process X hits level A, e.g., if we eliminate the drift appeared, then the process X and the first hitting time of the level A will change.

In [2] a model of control of the process X is investigated; external interventions of type (A) are defined and the average time until the controlled process hits level A is maximized. Now we consider another model of control of the process X. Our purpose is to find a rule to control the process by correcting it so as to minimize the average time when the controlled process runs with a drift.

We need some definition before giving a precise formulation of the problem.

Definition 2. Let b be a real number, $b \in [0, A]$. The control of the process X by an external intervention of type (B) is realized according to the following three steps:

1. When the process X hits level b we check if X has a non-zero drift. If 'no', we go to step 2. If 'yes', we eliminate the drift and go to step 3.

2. The observation on the process is started until either the process hits level b or acquires a drift. Then the observation is stopped and we return the process to state b and go to step 3. If during the observation no drift has appeared, we go to step 1.

3. The process behaves as a Wiener process with initial state b and it can acquire a linear drift with parameter μ after a random time interval ξ'. Here ξ' is a r.v. which is independent of the process and of the r.v. ξ.

The level b is said to be a level of external intervention of type (B) or briefly an intervention level.

R e m a r k 1. The intervention of type (B) includes a check for a non-zero drift of the process as well as a detecting an appearance of a drift during the observation. These are well known problems for detecting a change in the drift of Wiener process. Among the papers on this topic let us mention those of E.S. P a g e [9], A.N. S h i r y a e v [14], S.V. R o b e r t s [13], M. P o l l a k and D. S i e g m u n d [11]. We do not dwell on this subject because the function to be optimized does not depend on the time for intervention of type (B). Therefore, the cost function does not depend on the time for a check and detecting a non-zero drift of the controlled process.

R e m a r k 2. The control of the process X by one external intervention of type (B) is determined by choosing a particular value of intervention level $b \in [0, A]$. In this model the last intervention level is admissible to coincide with level A which makes the difference between the model presented in [2] and the present one.

R e m a r k 3. The controlled process is considered until it hits the level A after the last intervention.

Let us define the r.v.

$$T'(b,\xi) = \begin{cases} \inf\{t \geq 0 : W(\xi) + W(t) + \mu t = b\}, & \text{if} \quad W(\xi) < b; \\ 0, & \text{if} \quad W(\xi) \geq b \end{cases}$$

and notice that $T'(b,\xi) + \tau'(A - b, \xi')$ is the time when the controlled process runs with a non-zero drift, or briefly, the time with a drift.

Consider the case when $n-1$ interventions of type (B) may be realized. Let b_i, $i = 1, \ldots, n-1$, be the intervention levels. Clearly, they form a nondecreasing sequence:

$$0 \leq b_1 \leq b_2 \leq \ldots \leq b_{n-1} \leq A.$$

Denote by ξ_1 the random time when the process X acquires a drift if no interventions are realized. Denote by ξ_{i+1} the length of the random time interval expiring when the process acquires a drift after i-th intervention, $i = 1, 2, \ldots, n-1$. At the first hitting time of the level b_1 the first intervention of type (B) is realized. Then the controlled process is represented by a Wiener process with initial state b_1 which acquires a linear drift with parameter μ after the random time ξ_2. At the time of hitting the level b_2 the second intervention of type (B) is realized. We continue in this manner until the considered process hits level A after the last intervention.

Introduce the notations:

$$a_1 = b_1, a_2 = b_2 - b_1, \ldots a_k = b_k - b_{k-1}, \ldots, a_n = A - b_{n-1}.$$

Note that the time with a drift of the controlled process is

$$\sum_{i=1}^{n-1} T'(a_i, \xi_i) + r'(a_n, \xi_n).$$

In the next three definitions we clarify the meaning of the terms 'control' and 'optimal control' within the framework of our task.

Definition 3. Every choice of the intervention levels $b_1, b_2, \ldots, b_{n-1}$ satisfying the condition $0 \leq b_1 \leq b_2 \leq \ldots \leq b_{n-1} \leq A$, is said to be a control of the process X by $n-1$ external interventions of type (B).

This definition is equivalent to the following one.

Definition 4. Every choice of values of $a_1, a_2, \ldots, a_n$ satisfying the conditions $a_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} a = A$, is said to be a control of the process X by $n-1$ external interventions of type (B).

Definition 5. A control minimizing the average time when the controlled process runs with a drift until its hitting the level A after the last intervention is said to be an optimal control by fixed number external interventions of type (B).

3. General form of the optimal control problem of a Wiener process by external interventions of type (B)

Let $n \geq 1$ be a fixed integer number. The problem of optimal control of a Wiener process by $n-1$ external interventions of type (B) can be defined in the following way.

Problem (B). Determine the numbers $a_1, a_2, \ldots, a_n$ so that

$$\min \mathbf{E}\Big(\sum_{i=1}^{n-1} T'(a_i, \xi_i) + r'(a_n, \xi_n)\Big) \tag{1}$$

is attained under the restrictions:

$$a_i \geq 0,\ i = 1, \ldots, n, \quad \sum_{i=1}^{n} a_i = A. \tag{2}$$

Obviously, $b_i = a_1 + \ldots + a_i, \quad i = 1, \ldots, n-1$.

Further on we find the form of the object function and prove that under some conditions Problem (B) is a convex optimization problem having a unique solution.

Theorem 1. *The conditional expectation* $\mathbf{E}(T'(a,\xi)|\xi)$ *of the r.v.* $T'(a,\xi)$ *a.s. has the form:*

(3)
$$\mathbf{E}(T'(a,\xi)|\xi) =$$
$$= (a/\mu)I\{\xi = 0\} + (a/(2\mu) + a\,.\ \mathrm{erf}(a/\sqrt{2\xi})/(2\mu) +$$
$$+ \sqrt{\xi}\exp(-a^2/(2\xi))/(\mu\sqrt{2\pi})I\{\xi > 0\}.$$

(Recall that $\mathrm{erf}(x) = 1 - \mathrm{erfc}(x) = (2/\sqrt{\pi})\exp(-t^2)dt, \quad x \in R$*).*

P r o o f. Denote by P_ξ the generated by the r.v. ξ probability measure in $(R, \mathbb{B}, (\mathbb{R}))$, where $\mathbb{B}(R)$ is the Borel δ-algebra in $\mathbb{R}$.

The conditional probability density of the r.v. $W(\xi)$ given ξ is

$$f_{W(\xi)}(u|\xi = y) = (1/\sqrt{2\pi y})\exp(-u^2/(2y)), \quad y > 0.$$

The probability density of the first hitting time of the level a by a Wiener process with an initial state $u < a$ and a drift with parameter $\mu > 0$ is (see [17])

$$f(s) = ((a-u)/\sqrt{2\pi s^3})\exp(-(a-u-\mu s)^2/(2s)), \quad s > 0.$$

The distribution function (d.f.) of the r.v. $T'(a,\xi)$ has an atom at zero, $P(T'(a,\xi) = 0) = P(W(\xi) \geq a)$ and we have

(4)
$$\begin{aligned} P(T'(a,\xi) &= 0|\xi = y) = P(W(\xi) \geq a|\xi = y) = \\ &= (1/\sqrt{2\pi y})\int_a^\infty \exp(-u^2/(2y))du = \\ &= (1/\sqrt{\pi y})\int_{a/\sqrt{2y}}^\infty \exp(-x^2)dx = \\ &= (1/2)\mathrm{erfc}(a/\sqrt{2y}), \quad y > 0, \quad P_\xi - \text{a.s.} \end{aligned}$$

Denote by $p'(s)$, $s > 0$, the conditional probability density of the r.v. $T'(a,\xi)$ when $W(\xi) < a$. Then we have

$$\begin{aligned} p'(s|\xi = y) &= \int_{-\infty}^a ((a-u)/(2\pi\sqrt{ys^3}))\exp(-(a-u-\mu s)^2/(2s))\exp(-u^2/(2y))\mathrm{d}u \\ &= (1/(2\pi s\sqrt{sy}))\int_0^\infty \exp(-(x-\mu s)^2/(2s) - (x-a)^2/(2y))\mathrm{d}x, \quad y > 0. \end{aligned}$$

$$\mathbf{P}(T'(a,\xi) > 0|\xi = y) = \int_0^\infty p'(s|\xi = y)ds = \exp(-a^2/(2y))/(2\pi\sqrt{y})\times$$

$$\times \int_0^\infty \int_0^\infty x/\sqrt{s^3}.\exp(ax/y - x^2/(2y) - x^2/(2s) + \mu x - \mu^2 s/2) dxds =$$

$$= \exp(-a^2/(2y))/(2\pi\sqrt{y}). \int_0^\infty x.\exp(-x^2/(2y) + ax/y + \mu x) \times$$

$$\times \int_0^\infty \exp(-x^2/(2s) - \mu^2 s/2)/\sqrt{s^3} dsdx =$$

$= \exp(-a^2/(2y))/(\sqrt{2\pi y}) \int_0^\infty \exp(-x^2/(2y) + ax/y)dx =$
$= \ \mathrm{erfc}(-a/\sqrt{2y})/2 = (1/2)(1 + \mathrm{erf}(a/\sqrt{2y})),\ y > 0, \quad \mathbf{P}_\xi - a.s.$

From (4) and the equality $\mathrm{erf}(x) + \mathrm{erfc}(x) = 1$, $x \in R$, we obtain

$$\mathbf{P}(T'(a,\xi) = 0) + \mathbf{P}(T'(a,\xi) > 0) = 1.$$

Further we get $\mathbf{P}_\xi - a.s.$

$\mathbf{E}(T'(a,\xi)| = y) = \int_0^\infty sp'(s|\xi = y)ds = \exp(-a^2/(2y))/(2\pi\sqrt{y}).$
$. \int_0^\infty x.\exp(-x^2/(2y) + ax/y + \mu x) \int_0^\infty \exp(-x^2/(2s) - \mu^2 s/2)/\sqrt{s} dsdx =$

$= \exp(-a^2/(2y))/(\mu\sqrt{2\pi y}) \int_0^\infty \exp[-x^2/(2y) + ax/y]dx =$
$a\,.\,\mathrm{erfc}(-a/\sqrt{2y})/(2\mu) + \sqrt{y}\exp(-a^2/(2y))/(\mu\sqrt{2\pi}) =$
$= a/(2\mu) + a\,.\,\mathrm{erf}(a/\sqrt{2y})/(2\mu)) + \exp(-a^2/(2y))/(\mu\sqrt{2\pi}), \quad y > 0;$
$\mathbf{E}(T'(a,\xi)|\xi = 0) = \mathbf{E}(r'(a,\xi)|\xi = 0) = a/\mu.$

The proof is completed. ■

Corollary 1. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be non-negative, independent r.v.'s with* $\mathbf{E}\ \sqrt{\xi_i} < \infty$, $i = 1, \ldots, n$. *Then Problem (B) is a convex optimization problem with a unique solution. The object function has the form:*

$$(5) \qquad \mathbf{E}(\sum_{i=1}^{n-1} T'(a_i,\xi_i) + r'(a_n,\xi_n)) = \left[\sum_{i=1}^{n-1} \int_0^\infty G_i'(x)dF_i(x)\right] + a_n/\mu,$$

where $G_i'(x)$, $x \geq 0$, is given as follows:

$$\begin{aligned} G_i'(x) &= \mathbf{E}(T'(a_i,\xi_i)|\xi_i = x) = \\ &= (a_i/\mu)I\{x = 0\} + (a_i/(2\mu) + a_i \mathrm{erf}(a_i/\sqrt{2x})/(2\mu) + \\ &+ \sqrt{x}\exp(-a_i^2/(2x))/(\mu\sqrt{2\pi}))I\{x > 0\}, \quad P_{\xi_i} - a.s. \end{aligned}$$

and $F_i(x)$, $x \geq 0$, *is the d.f. of the r.v.* ξ_i, $i = 1, \ldots, n$

Proof. The representation (5) is obvious. The form of $G_i'(x)$, $i = 1, \ldots, n-1$ follows from (3). Moreover, for every $i = 1, \ldots, n-1$, we get

$$\int_0^\infty G_i'(x) dF_i(x) < \infty$$

as well as the following inequalities, valid for $x \in (0, \infty)$:

$$\int_0^\infty \sqrt{y} dF_i(y) < \infty;$$

$$\operatorname{erf}(a_i/\sqrt{2x}) \leq 1;$$

$$\exp(-a_i^2/(2x)) \leq 1.$$

Since $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ and $d \operatorname{erf}(x)/dx = (2/\sqrt{\pi}) \exp(-x^2)$, $x \in R$ (see [1], Ch.7 and Ch.22), we obtain:

$$dG_i'(x)/da_i = 1/(2\mu) + (1/(2\mu))\operatorname{erf}(a_i/\sqrt{2x});$$

$$d^2 G_i'(x)/da^2 = (1/\mu\sqrt{2\pi x}) \exp(-a_i^2/(2x)) > 0$$

for every $x \in (0, \infty)$ and for every $a_i \in [0, A]$, $i = 1, \ldots, n-1$.

Moreover, for every $x \in (0, \infty)$ and for every $i = 1, \ldots, n-1$, we have

$$\int_0^\infty dG_i'(x)/da_i) dF_i(x) < \infty, \quad \int_0^\infty d^2 G_i'(x)/da_i^2) dF_i(x) < \infty,$$

because

$(1/\sqrt{x}) \exp(-a_i^2/(2x)) = 1/(\sqrt{x}. \exp(a_i^2/(2x)) =$

$= 1/(\sqrt{x}. \sum_{n=0}^{\infty} (a_i^2/(2x))^n/n!) =$

$= \sqrt{x}/(x + a_i^2/2 + \sum_{n=2}^{\infty} (a_i^2/(2x))^n.x/n!) <$

$< \sqrt{x}(a_i^2/2) = 2\sqrt{x}/a_i^2.$

Since a_i belongs to a finite interval, $a_i \in [0, A]$, $i = 1, \ldots, n$, the following transformations are valid

$$\frac{d}{da_i}\left(\int_0^\infty G_i'(x)dF_i(x)\right) = \int_0^\infty \frac{dG_i'(x)}{da_i}dF_i(x);$$

$$\frac{d^2}{da_i^2}\left(\int_0^\infty G_i'(x)dF_i(x)\right) = \int_0^\infty \frac{d^2G_i'(x)}{da_i^2}dF_i(x).$$

$$\frac{\partial^2}{\partial a_i \partial a_j}\left(\int_0^\infty G_i'(x)dF_i(x)\right) =$$

$$\int_0^\infty \frac{\partial^2 G_i'(x)}{\partial a_i \partial a_j}dF_i(x) = 0 \quad \text{for} \quad i \neq j, \quad i,j = 1,\ldots,n.$$

Denote by C' the matrix

$$\partial^2 \left\{ \frac{\mathbf{E}(\sum_{k=1}^{n-1} T'(a_k,\xi_k) + r'(a_n,\xi_n))}{\partial a_i \partial a_j} \right\}, \qquad i,j = 1,\ldots,n.$$

Then we have

$$\text{(6)} \qquad \frac{\partial^2 \left[\mathbf{E}\left(\sum_{k=1}^{n-1} T'(a_k,\xi_k) + r'(a_n,\xi_n)\right)\right]}{\partial^2 a_i^2} = \int_0^\infty \frac{d^2 G_i'(x)}{da_i^2}dF_i(x) > 0$$

for every $a_i \in [0, A]$, $i = 1,\ldots,n-1$;

$$\frac{\partial^2 \left[\mathbf{E}\left(\sum_{k=1}^{n-1} T'(a_k,\xi_k) + r'(a_n,\xi_n)\right)\right]}{\partial^2 a_n^2} = 0;$$

$$\text{(7)} \qquad \frac{\partial^2 \left[\mathbf{E}\left(\sum_{k=1}^{n-1} T'(a_k,\xi_k) + r'(a_n,\xi_n)\right)\right]}{\partial a_i \partial a_j} = 0$$

for $i \neq j$, where $i, j = 1, \ldots, n$.

According to the Silvester criterion, the quadratic form $y'C'y$, $y \in R^n$, is a positive semi-definite one for every vector $(a_1, a_2, \ldots, a_n)$ with components $a_i \in [0, A]$, i 1, $\ldots, n$, Thus, the object function $\mathbf{E}(\sum_{i=1}^{n-1} T'(a_i, \xi_i) + r'(a_n, \xi_n))$ is a convex function on the domain defined by the conditions (2) (see [4], 6.2). As the domain (2) is also convex, then Problem (B) is a convex optimization problem.

In order to prove that Problem (B) has a unique solution we consider the following optimization problem.

Problem (B'). Determine the numbers $a_1, a_2, \ldots, a_{n-1}$ so that

$$\min \mathbf{E}((\sum_{i=1}^{n-1} T'(a_i, \xi_i) + (A - \sum_{i=1}^{n-1} a_i)/\mu)$$

is attained under the restrictions

$$a_i \geq 0,\ i = 1, \ldots, n-1,\ \sum_{i=1}^{n-1} a_i \leq A. \tag{8}$$

Problem (B') is equivalent to Problem (B), because $\mathbf{E}r'(a_n, \xi_n) = a_n/\mu = (A - \sum_{i=1}^{n-1} a_i)/\mu$ (see [2], 2). The matrix of the second partial derivatives of the object function in Problem (B') has elements

$$\left\{ \frac{\partial^2 \left[\mathbf{E} \left(\sum_{k=1}^{n-1} T'(a_k, \xi_k) + (A - \sum_{k=1}^{n-1} a_k \right) / \mu \right]}{\partial a_i \partial a_j} \right\}; \quad i, j = 1, \ldots, n-1.$$

It follows from (6) and (7) that this matrix is positively definite for every vector $(a_1, a_2, \ldots, a_{n-1})$ with components $a_i \in [0, A]$, $i = 1, \ldots, n-1$. This means that the object function of Problem (B') is a strictly convex function on the domain defined by (8) (see [4], 6.2). Since the domain (8) is bounded and convex, then Problem (B') is a convex optimization problem with a unique solution.

This completes the proof. ∎

Corollary 2. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be a non-negative, independent r.v.'s with $\mathbf{E}\sqrt{\xi_i} < \infty$, $i = 1, \ldots, n$, $n > 1$. Under the optimal control by $n-1$ external interventions of type (B), the last intervention level coincides with level A, i.e. $a_n^{(n)} = 0$. The time when the controlled process is without drift is given by* *the r.v.* $\sum_{i=1}^{n} \xi_i$, *and the time with drift - by the r.v.* $\sum_{i=1}^{n-1} T'(a_i^{(n)}, \xi_i)$. *Here* $a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}$ *are the optimal values of* $a_1, a_2, \ldots, a_n$.

Proof. Since Problem (B) is a convex optimization problem, its solution satisfies the Kuhn-Tucker theorem's conditions (see [4], 7,8) with a Lagrange function

$$\Phi(a_1, \ldots, a_n, u) = \mathbf{E}(\sum_{i=1}^{n-1} T'(a_i, \xi_i)) + a_n/\mu + u\left(\sum_{i=1}^{n} a_i - A\right).$$

In this case the local Kuhn-Tucker conditions concerning $a_n^{(n)}$, are

$$1/\mu + u \geq 0,\ a_n^{(n)}(1/\mu + u) = 0,\ u \geq 0$$

Therefore, $a_n^{(n)} = 0$. Since $r'(0, \xi_n) = 0$, then the time with drift is $\sum_{i=1}^{n-1} T'(a_i^{(n)}, \xi_i)$. It follows from $b_{n-1} = A - a_n^{(n)} = A$ and Definition 2 that the time without drift is $\sum_{i=1}^{n} \xi_i$. ■

Corollary 3. *Let* $\xi_1, \xi_2, \ldots$ *be non-negative, independent r.v.'s with* $\mathbf{E}\sqrt{\xi_i} < \infty$, $i \geq 1$. *Under the optimal control by external interventions of type* (B) *the minimum of the average time for the controlled process with drift increases with the number of interventions.*

Proof. Denote by $H(n)$ the minimum of the average time for the controlled process with draft.

$$H(n) = \min_{\{a_i\}_1^\infty \in M(n,A)} \left\{\mathbf{E}\left(\Sigma_{i=1}^{n-1} T'(a_i, \xi_i) + r'(a_n, \xi_n)\right)\right\},$$

where

$$M(n, A) = \{\{a_i\}, i \geq 1 : a_i \geq 0 \text{ for } 1 \leq i \leq n; a_i = 0 \text{ for } i > n, \Sigma_{i=1}^{n} a_i = A\}.$$

Let $a \geq 0$ and ξ be a non-negative r.v. with $\mathbf{E}\sqrt{\xi} < \infty$. Let us show first that

$$\mathbf{E}T'(a, \xi) > \mathbf{E}r'(a, \xi) \quad \text{if} \quad \mathbf{P}(\xi = 0) < 1; \tag{9}$$

$$\mathbf{E}T'(a, \xi) = \mathbf{E}r'(a, \xi) = a/\mu \quad \text{if} \quad \mathbf{P}(\xi = 0) = 1. \tag{10}$$

To prove (9) we establish that $\mathbf{P}\xi$ -a.s.

$\mathbf{E}T'(a, \xi)|\xi = x) > a/\mu$ for $x > 0$.

We use the following representation of function $\operatorname{erf}(a/\sqrt{2x})$ (see [7]):

$$\operatorname{erf}(a/\sqrt{2x}) = 1 - \sqrt{2x/\pi}\exp(-a^2/(2x)) \int_0^\infty \exp\left(-t^2 x/2 - ta\right) dt.$$

Since $\mathbf{P}\xi$ - a.s.

$\mathbf{E}r'(a,\xi)|\xi = x) = a/\mu$ for $x \geq 0$ (see [2], §2), we have $\mathbf{P}\xi$ - a.s. for $x > 0$

$\mathbf{E}T'(a,\xi)|\xi = x) - a/\mu =$

$= (a/(2\mu))\,\mathrm{erf}(a/\sqrt{2x})/(2\mu) + \sqrt{x}\exp(-a^2/(2x))/(\mu\sqrt{2\pi} - a/(2\mu) =$

$= (a/(2\mu))(\mathrm{erf}(a/\sqrt{2x}) - 1 + \sqrt{2x/\pi}\exp(-a^2/(2x)))/a =$

$= (a/(2\mu))\sqrt{2x/\pi}\exp(-a^2/(2x))(1/a - \int_0^\infty \exp(-t^2x/2 - ta)dt) >$

$> (a/(2\mu))\sqrt{2x/\pi}\exp(-a^2/(2x))(1/a - \int_0^\infty \exp(-ta)dt) =$

$= (a/(2\mu))\sqrt{2x/\pi}\exp(-a^2/(2x))(1/a - 1/a) = 0.$

Thus the proof of (9) is completed. ■

Another proof of (9) can also be given. Notice that the r.v. $T'(a,\xi)$ has the following representation:

$$T'(a,\xi) = \begin{cases} r'(a,\xi), & \text{if } W^a(\xi) < a; \\ \eta, & \text{if } W^a(\xi) = a \text{ and } W(\xi) < a; \\ 0, & \text{if } W(\xi) \geq a, \end{cases}$$

where η is a non-negative r.v. Then we get

$$\begin{aligned} \mathbf{E}T'(a,\xi) &= \mathbf{E}(r'(a,\xi).I\{W^a(\xi) < a\}) \\ &+ \mathbf{E}(\eta.I\{\{W^a(\xi) = a\} \cap \{W(\xi) < a\}\}); \end{aligned}$$

$\mathbf{E}(\eta.I\{\{W^a(\xi) = a\} \cap \{W(\xi) < a\}\}) > 0$, if $P(\xi = 0) < 1$;

$\mathbf{E}(r'(a,\xi).I\{W^a(\xi) < a\}) = \mathbf{E}r'(a,\xi) = a/\mu$.

Hence $\mathbf{E}T'(a,\xi) > \mathbf{E}r'(a,\xi)$.

If $P(\xi = 0) = 1$ we have

$$\mathbf{E}(T'(a,\xi)|\xi = 0) = \mathbf{E}(r'(a,\xi)|\xi = 0) = a/\mu$$

and (10) is true.

It follows from Corollary 2, (9) and (10) that

$$H(n) = \mathbf{E}(\sum_{i=1}^{n-1} T'(a_i^{(n)},\xi_i)) \geq \mathbf{E}(\sum_{i=1}^{n-2} T'(a_i^{(n)},\xi_i)) + r'(a_{n-1}^{(n)},\xi_{n-1})) \geq$$

$$\geq \min_{\{a_i\}_1^\infty \in M(n-1,A)} \mathbf{E}(\sum_{i=1}^{n-2} T'(a_i,\xi_i)) + r'(a_{n-1},\xi_{n-1})) = H(n-1),$$

i.e. $H(n) \geq H(n-1)$ for $n \geq 2$. The inequality is strict if and only if $P(\xi_{n-1} = 0) < 1$. In general, the sequence $\{H(n)\}$, $n \geq 1$, is non-decreasing. It is strictly increasing if for every $n \geq 1$ we have $P(\xi_n = 0) < 1$.

4. Solution of Problem (B) in the case of exponentially distributed times of acquiring a drift.

Let the r.v.'s ξ_i be exponentially distributed with parameters $\lambda_i > 0$, $i = 1, 2, \ldots, n$. In this case the explicit form of $\mathbf{E}T'(a,\xi)$, $a \geq 0$, is given by the following statement:

Theorem 2. *If the r.v. ξ is exponentially distributed with parameter λ, $\lambda > 0$, then the conditional expectation $\mathbf{E}T'(a,\xi)$ has the form*

$$\mathbf{E}T'(a,\xi) = a/\mu + \exp(-a\sqrt{2\lambda})/(2\mu\sqrt{2\lambda}). \tag{11}$$

Proof. The representation (11) is obtained from (3) and uses some calculations and the formula for the total expectation. ■

Theorem 3. *Let $\xi_1, \ldots, \xi_n$ be independent r.v.'s exponentially distributed with parameters $\lambda_1 > 0, \ldots, \lambda_n > 0$. Then the optimal control by $n-1$ external interventions of type (B) is realized by a choice of intervention levels $b_1, \ldots, b_{n-1}$ in the form $b_i = a_1 + \ldots + a_i$, $i = 1, 2, \ldots, n-1$, where*

$$a_j = A/(\sqrt{\lambda_j} \textstyle\sum_{k=1}^{n-1} (1/\sqrt{\lambda_k}), \ j = 1, \ldots, n-1, \qquad a_n = 0.$$

Proof. We have shown that Problem (B) is a convex optimization problem for arbitrary distributions of the r.v.'s ξ_i with $\mathbf{E}\sqrt{\xi_i} < \infty$, $i = 1, 2, \ldots, n$. If ξ_i is an exponentially distributed r.v. with parameter λ_i, $i = 1, 2, \ldots, n$, then the condition (1) has the form:

$$\min \left(\sum_{i=1}^{n-1} (a_i/\mu + \exp(-a_i\sqrt{2\lambda_i})/(2\mu\sqrt{2\lambda_i})) + a_n/\mu \right). \tag{12}$$

Problem (12) under the restrictions (2) can be solved by using the Kuhn-Tucker theorem (see [4], 7). In this case the Lagrange function and the Kuhn-Tucker local conditions have the form:

$$\Phi = \sum_{i=1}^{n-1}(a_i/\mu + \exp(-a_i\sqrt{2\lambda_i})/(2\mu\sqrt{2\lambda_i})) + a_n/\mu + v(\sum_{i=1}^{n} a_i - A).$$

$$\exp(-a_i\sqrt{2\lambda_i}) \le u, \quad i = 1, 2, \ldots, n-1; \quad u = 2\mu(v = 1/\mu); \tag{13}$$

$$\sum_{i=1}^{n} a_i = A \tag{14}$$

$$u > 0; \qquad a_i \ge 0, \qquad i = 1, 2, \ldots, n; \tag{15}$$

$$a_i(u - \exp(-a_i\sqrt{2\lambda_i}) = 0, \quad i = 1, 2, \ldots, n-1; \tag{16}$$

$$a_n = 0. \tag{17}$$

It follows from (16) that if for some $i = 1, 2, \ldots, n$ we have $a_i > 0$, then for this i, $\exp(-a_i\sqrt{2\lambda_i}) = u$. it follows from (14) the existance of k, $1 \le k \le n-1$, such that $a_k > 0$. Therefore $u = \exp(-a_k\sqrt{2\lambda_k}) < 1$. Let us assume that there exists j, $1 \le j \le n-1$, such that $a_j = 0$. Hence we get from (13) that $u \ge \exp(-a_j\sqrt{2\lambda_j}) = 1$ contrary to $u < 1$. Therefore, for every i, $1 \le i \le n-1$, we have $a_i > 0$. Further we get

$$\ln u = -a_i\sqrt{2\lambda_i};$$

$$a_i = -\ln u/\sqrt{2\lambda_i}, \quad i = 1, \ldots, n-1. \tag{18}$$

The condition (14) implies that

$$-\sum_{i=1}^{n-1}((\ln u)/\sqrt{2\lambda_i}) = A; \ln u = -A/(\sum_{i=1}^{n-1}(1/\sqrt{2\lambda_i})).$$

Taking into account (18) we obtain

$$a_i = A/(\sqrt{\lambda_i}\sum_{j=1}^{n-1}(1/\sqrt{\lambda_j})), \quad i = 1, \ldots, n-1.$$

The proof of Theorem 3 is completed. ■

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