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A Jackson Type Theorem for Tchebycheff Systems

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Let $\mathcal{U}_n := \{1, \varphi_1, \dots, \varphi_n\}$ be a T-system on [-1,1]. We give, under certain restrictions on $\mathcal{U}_n$, an estimation for the best uniform approximation of a continuous function f on $[-1, 1]$ by generalized polynomials from span $\mathcal{U}_n$. The estimate is expressed through the modulus of continuity of f.

1. Introduction

Denote by π_n the set of all algebraic polynomials of degree n. Consider the space $C[-1, 1]$ of continuous functions f on $[-1, 1]$ supplied with the uniform norm $\|f\| := \max\{| f(x) |: x \in [-1, 1]\}$. According to the Weierstrass theorem, for every fixed $f \in C[-1, 1]$, the best approximation

$$E_n^0(f) := \inf_{u \in \pi_n} \|f - p\|$$

approaches zero as $n \to \infty$. Jackson proved (see, for example Natanson [4]) that there is a constant $C > 0$ such that

$$E_n^0(f) \leq C\omega(f; \frac{1}{n}),$$

where

$$\omega(f; \delta) := \sup\{| f(x) - f(t) | : x, t \in [-1, 1], | x - t | \leq \delta\}$$

is the *modulus of continuity* of f.

We present here a Jackson type result studying the case of generalized polynomials defined by a given Tchebycheff system. Recall that the functions

$\varphi_0, \varphi_1, \ldots, \varphi_n$ constitute a Tchebycheff system (abbreviated to T-system) on $[-1,1]$ if and only if every generalized polynomial $a_0\varphi_0(x) + \ldots + a_n\varphi_n(x)$ has no more than n distinct zeros in $[-1,1]$, or equivalently,

$$\det\{\varphi_i(x_k)\}_{i=0,k=0}^{n\ \ n} \neq 0$$

for every choice of the points $x_0 < \ldots < x_n$ in $[-1,1]$.

Recently Borwein [3] proved a Weierstrass type theorem for approximation by Tchebycheff systems. The present note uses a different approach, which makes it possible to get an estimation of the error. The starting point of our study is the following known relation between the best uniform approximation by algebraic polynomials of degree n and the divided differences at $n+2$ points.

$$E_n^0(f) = \sup_{-1\le x_0<\cdots<x_{n+1}\le 1} \left| \frac{f[x_0, \ldots, x_{n+1}]}{s[x_0, \ldots, x_{n+1}]} \right| . \tag{1.1}$$

Here $s(x)$ is a function such that $s(x_i) = (-1)^i, \quad i = 0, \ldots, n+1$.

2. Preliminaries. Markov systems

Assume that the sequence $\mathcal{U} := \{\varphi_0, \ldots, \varphi_N\} (N \le \infty)$ of continuous functions is a Markov system on $[-1,1]$. This means that for each $n \le N$ the functions $\varphi_0, \ldots, \varphi_n$ form a T-system on $[-1,1]$. Denote by Π_n the set of all generalized polynomials $\varphi(x) = a_0\varphi_0(x) + \ldots + a_n\varphi_n(x)$ with real coefficients. It follows from the definition of T-systems that the Lagrange interpolation problem

$$\varphi(x_i) = f(x_i), \quad i = 0, \ldots, n,$$

has a unique solution $\varphi(f;x)$ from Π_n for each fixed f and nodes $x_0 < \ldots < x_n$ in $[-1,1]$. We call the leading coefficient a_n of $\varphi(f;x)$ a *divided difference* of f at $x_0, \ldots, x_n$ with respect to $\mathcal{U}$ and denote it by $f[\mathcal{U}; x_0, \ldots, x_n]$. By Kramer's rule

$$f[\mathcal{U}; x_0, \ldots, x_n] = \frac{\begin{bmatrix} \varphi_0, & \ldots, & \varphi_{n-1}, & f \\ x_0, & \ldots, & x_{n-1}, & x_n \end{bmatrix}}{\begin{bmatrix} \varphi_0, & \ldots, & \varphi_n \\ x_0, & \ldots, & x_n \end{bmatrix}}, \tag{2.1}$$

where we have used the notation

$$\begin{bmatrix} g_0, & \ldots, & g_k \\ t_0, & \ldots, & t_k \end{bmatrix} := \det\{g_i(t_j)\}_{i=0,j=0}^{k\ \ k}.$$

Clearly

$$f[\mathcal{U}; x_0, \ldots, x_n] = \begin{cases} 0 & \text{for } f = \varphi_i, \quad i = 0, \ldots, n-1 \\ 1 & \text{for } f = \varphi_n \end{cases}. \tag{2.2}$$

Denote by $E_n(f)$ the best uniform approximation of f on $[-1, 1]$ by generalized polynomials φ from Π_n. Let $P(f; x)$ be the polynomial of best approximation of f, i.e.,

$$\|f - P(f; \cdot)\| = E_n(f) := \inf_{\varphi \in \Pi_n} \|f - \varphi\|.$$

We shall give an expression of form (1.1) for $E_n(f)$.

By the Tchebycheff criterion for T-systems (see Akhiezer [1], section 46) there exist $n+2$ points $x_0 < \ldots < x_{n+1}$ in $[-1, 1]$ such that

$$f(x_i) - P(f; x_i) = \epsilon(-1)^i E_n(f), \quad i = 0, \ldots, n+1, \tag{2.3}$$

for some $\epsilon = 1$ or $\epsilon = -1$. Operating by the divided difference on the both sides of (2.3) and taking into account (2.2), we get

$$f[\mathcal{U}; x_0, \ldots, x_{n+1}] = s[\mathcal{U}; x_0, \ldots, x_{n+1}] E_n(f), \tag{2.4}$$

where s is any function satisfying the conditions

$$s(x_i) = \epsilon(-1)^i, \quad i = 0, \ldots, n+1.$$

Observe that $s[\mathcal{U}; x_0, \ldots, x_{n+1}] \neq 0$. Indeed, the polynomial $\varphi(s; x)$ from Π_{n+1} which interpolates s at $x_0, \ldots, x_{n+1}$ changes sign at these points and thus, it has at least $n+1$ zeros in $[-1, 1]$. On the other hand, the assumption $s[\mathcal{U}; x_0, \ldots, x_{n+1}] = 0$ would imply that $\varphi(s; x)$ is actually from Π_n and hence $\varphi(s; x)$ may have at most n zeros, a contradiction. Thus $s[\mathcal{U}; x_0, \ldots, x_{n+1}] \neq 0$ and we can determine $E_n(f)$ from (2.4). Therefore

$$E_n(f) = \frac{f[\mathcal{U}; x_0, \ldots, x_{n+1}]}{s[\mathcal{U}; x_0, \ldots, x_{n+1}]}.$$

Now using (2.1), we get

$$E_n(f) = \frac{\begin{bmatrix} \varphi_0, & \ldots, & \varphi_n, & f \\ x_0, & \ldots, & x_n, & x_{n+1} \end{bmatrix}}{\begin{bmatrix} \varphi_0, & \ldots, & \varphi_n, & s \\ x_0, & \ldots, & x_n, & x_{n+1} \end{bmatrix}}, \tag{2.5}$$

which is the expression we shall thoroughly exploit in this paper.

The representation (2.5) of $E_n(f)$ is known (see for example [5], page 28).

3. Main theorem

Let $\mathcal{U} := \{\varphi_0, \ldots, \varphi_N\} (N \leq \infty)$ be a given Markov system on $[-1, 1]$ of differentiable functions. Fix $n < N$. Assume that $\varphi_0, \ldots, \varphi_n$ satisfy the following additional requirement.

Assumption A. *$\varphi_0(x) \equiv 1$ and $\varphi'(x), \ldots, \varphi'(x)$ is a T-system on $[-1, 1]$.*

We shall say that a function f is orthogonal to Π'_n , if

$$\int_{-1}^{1} f(t)\varphi'(t)\, dt = 0 \text{ for } \varphi = \varphi_1, \ldots, \varphi_n.$$

Let $V[f]$ be the *total variation* of the function f in $[-1, 1]$. In other words,

$$V[f] := \sup \left\{ \sum_{i=0}^{m} |f(t_{i+1}) - f(t_i)| \ : \ -1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1, \ \forall m \right\}.$$

Introduce the set $\mathcal{A}_n$ of all integrable functions f on $[-1, 1]$ which are orthogonal to Π'_n and $V[f] \leq 1$. Set

$$\delta_n := \sup \left\{ \int_{-1}^{1} |g(t)|\, dt \ : \ g \in \mathcal{A}_n \right\}.$$

Now we can formulate the main theorem.

Theorem 1 *Let $\mathcal{U} := \{\varphi_0, \ldots, \varphi_N\}$ be a Markov system of differentiable functions on $[-1, 1]$ and $n < N$. Suppose that $\varphi_0, \ldots, \varphi_n$ satisfy Assumption A. Then, for every function $f \in C[-1, 1]$,*

$$E_n(f) \leq \frac{3}{2}\omega(f; \delta_n).$$

P r o o f. Let $x_0 < \ldots < x_{n+1}$ be the alternation points for f. Then $E_n(f)$ can be presented as in (2.5). Now we shall make some manipulations on this expression of $E_n(f)$. Consider the determinant A in the numerator. Let us number its rows by $0, 1, \ldots, n+1$. Subtracting from the row $k+1$ the previous one and doing this for $k = n, n-1, \ldots, 0$, we get

$$A \ := \ \begin{bmatrix} \varphi_0, & \ldots, & \varphi_n, & f \\ x_0, & \ldots, & x_n, & x_{n+1} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & \varphi_1(x_0) & \cdots & \varphi_n(x_0) & f(x_0) \\ 0 & \varphi_1(x_1)-\varphi_1(x_0) & \cdots & \varphi_n(x_1)-\varphi_n(x_0) & f(x_1)-f(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \varphi_1(x_{n+1})-\varphi_1(x_n) & \cdots & \varphi_n(x_{n+1})-\varphi_n(x_n) & f(x_{n+1})-f(x_n) \end{bmatrix}$$

$$= \det \begin{bmatrix} \varphi_1(x_1)-\varphi_1(x_0) & \cdots & \varphi_n(x_1)-\varphi_n(x_0) & f(x_1)-f(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_1(x_{n+1})-\varphi_1(x_n) & \cdots & \varphi_n(x_{n+1})-\varphi_n(x_n) & f(x_{n+1})-f(x_n) \end{bmatrix}.$$

Next we expand the latter determinant along the elements of the last column and obtain the expression

$$A = (-1)^n \sum_{k=0}^{n} (-1)^k [f(x_{k+1}) - f(x_k)] \det \mathbf{D}_k,$$

where $\mathbf{D}_k$ is obtained from the matrix

$$\mathbf{D} := \{[\varphi_i(x_{j+1}) - \varphi_i(x_j)]\}_{i=1}^{n}{}_{j=0}^{n},$$

deleting the row k. In the same fashion we get the expression

$$B = \epsilon(-1)^{n+1} \sum_{k=0}^{n} 2 \det \mathbf{D}_k,$$

for the denominator B of (2.5). Therefore

$$E_n(f) = \sum_{k=0}^{n} (-1)^k c_k [f(x_{k+1}) - f(x_k)], \tag{3.1}$$

where

$$c_k := -\frac{\epsilon \det \mathbf{D}_k}{2\sum_{j=0}^{n} \det \mathbf{D}_j}, \quad k = 0, \dots, n. \tag{3.2}$$

We have mentioned already that $B \neq 0$. Next we prove that all determinants $\{\det \mathbf{D}_k\}_{k=0}^{n}$ are distinct from zero and have the same sign. Indeed, consider the functional

$$D[f] := \sum_{k=0}^{n} (-1)^k c_k [f(x_{k+1}) - f(x_k)].$$

It is seen that $D[f]$ equals the best uniform approximation of f on the finite set $x_0, \dots, x_{n+1}$ by generalized polynomials from Π_n. Therefore

$$D[f] = 0 \text{ for all } f \in \Pi_n.$$

On the other hand $D[f]$ may be rewritten in the form

$$D[f] = \sum_{k=0}^{n}(-1)^k c_k \int_{x_k}^{x_{k+1}} f'(t)\,dt$$

$$= \int_{-1}^{1} \psi(t) f'(t)\,dt,$$

where we have introduced the piece-wise constant function

$$\psi(t) := \begin{cases} (-1)^k c_k & \text{for } t \in (x_k, x_{k+1}), \quad k = 0, \ldots, n, \\ 0 & \text{on } [-1, x_0) \cup (x_{n+1}, 1]. \end{cases}$$

In view of the previous observation ψ is orthogonal to f' if $f \in \Pi_n$. Then ψ is orthogonal to $\varphi_1'(x), \ldots, \varphi_n'(x)$. But according to Assumption A these functions form a T-system. Then ψ must have at least n sign changes in (x_0, x_{n+1}). Therefore $\psi(x)$ changes sign at $x_1, \ldots, x_n$. This implies that $c_k \neq 0$ for all $k = 0, \ldots, n+1$ and $\{c_k\}$ have the same sign. Hence all determinants $\{\det \mathbf{D}_k\}$ have a constant sign. Our claim is proved. It follows then from (3.2) that

$$\sum_{k=0}^{n} | c_k | = \frac{1}{2}\,.$$

Now let us return to (3.1). Clearly

$$E_n(f) \leq \sum_{k=0}^{n} | c_k | \, |f(x_{k+1}) - f(x_k)|$$

$$\leq \sum_{k=0}^{n} | c_k | \, \omega(f; | x_{k+1} - x_k |).$$

Since $\omega(f; \lambda\delta) \leq (\lambda + 1)\omega(f; \delta)$ for each positive λ,

$$E_n(f) \leq \sum_{k=0}^{n} | c_k | \left(\frac{| x_{k+1} - x_k |}{\delta} + 1 \right) \omega(f; \delta)$$

$$= \left\{ \frac{1}{\delta} \sum_{k=0}^{n} | c_k | | x_{k+1} - x_k | + \frac{1}{2} \right\} \omega(f; \delta)$$

for each $\delta > 0$. Note that

$$\sum_{k=0}^{n} | c_k | | x_{k+1} - x_k | = \int_{-1}^{1} | \psi(t) | \, dt =: \|\psi\|_1.$$

Therefore, choosing $\delta = \|\psi\|_1$, we obtain the estimation

$$E_n(f) \leq \frac{3}{2}\omega(f; \|\psi\|_1). \tag{3.3}$$

Now observe that ψ has a bounded variation on $[-1,1]$. Precisely,

$$V[\psi] \leq 2 \mid c_0 + \ldots + c_n \mid = 1.$$

Hence $\psi \in \mathcal{A}_n$. This yields $\|\psi\|_1 \leq \delta_n$ and the proof is completed.

Next we give some immediate consequences from the main theorem.

Let $x_0(f) < \cdots < x_{n+1}(f)$ be a set of alternating points for f. Set

$$H_n(f) := \max\{\mid x_{k+1}(f) - x_k(f) \mid : 0 \leq k \leq n\}.$$

Corollary 1 *Suppose that $f \in C[-1,1]$. Then*

$$E_n(f) \leq \frac{1}{2}\omega(f; H_n(f)).$$

P r o o f. In view of (3.1),

$$E_n(f) \leq \sum_{k=0}^{n} \mid c_k \mid\mid f(x_{k+1}) - f(x_k) \mid$$

$$\leq \sum_{k=0}^{n} \mid c_k \mid \omega(f; H_n(f)) = \frac{1}{2}\omega(f; H_n(f)).$$

Corollary 1 asserts that a particular continuous function f may be approximated by polynomials from span $\mathcal{U}$ with any preassigned accuracy, provided $\liminf H_n(f) = 0$. This is related to the result of Borwein [3] which showed that any function from $C[-1,1]$ can be approximated if $\liminf H_n(T_n) = 0$ for the Tchebycheff generalized polynomial T_n .

Corollary 2 *Let $\xi_k = -1 + kh$, $k = 0, \ldots, n+1$, $h = 2/(n+1)$. Then the best uniform approximation $E_n(f; \bar{\xi})$ of $f \in C[-1,1]$ by polynomials from Π_n on the finite set $\bar{\xi} := \{\xi_0, \ldots, \xi_{n+1}\}$ satisfies the inequality*

$$E_n(f; \bar{\xi}) \leq \omega(f : \frac{1}{n+1}).$$

P r o o f. From (3.1),

$$E_n(f; \bar{\xi}) \leq \sum_{k=0}^{n} \mid c_k \mid\mid f(\xi_{k+1}) - f(\xi_k) \mid \leq \frac{1}{2}\omega(f; \frac{2}{n+1}) \leq \omega(f : \frac{1}{n+1}).$$

Corollary 3 *If $f \in C[-1,1]$ and the modulus of continuity of f is convex, then* $E_n(f) \le \omega(f;\delta_n)$.

P r o o f. In this case

$$E_n(f) \le \sum_{k=0}^{n} | c_k | \, \omega(f; | x_{k+1} - x_k |)$$

$$\le \frac{1}{2}\omega(f; 2\sum_{k=0}^{n} | c_k || x_{k+1} - x_k |) = \frac{1}{2}\omega(f; 2\delta_n) \le \omega(f;\delta_n).$$

An estimate of form (3.3) was found recently by Babenko and Schalaev [2] in the case of approximation of periodic functions by trigonometric polynomials. In their case $\|\psi\|_1 \le \frac{\pi}{2n}$, by the classic Bohr inequality, and this implies an estimation of Jackson type with the best constant.

It is important to find sharp estimations for the quantity δ_n for some Tchebycheff systems. This may be not so easy problem. We give below un upper bound of δ_n which could be of some interest.

For any fixed $t \in [-1,1]$, denote by $\theta_n(t;x)$ the polynomial from Π_n of best uniform approximation to the truncated function $\sigma_t(x)$,

$$\sigma_t(x) = (x-t)_+ := \begin{cases} x-t & \text{for } x > t \\ 0 & \text{for } x \le t, \end{cases}$$

which satisfies the additional constraint to interpolate σ_t at the end points -1 and 1. In other words, $\theta_n(t;x)$ is the extremal polynomial from Π_n to the problem:

$$\|\sigma_t - \varphi\| \to \inf$$

over the set $\{\varphi \in \Pi_n \,:\, \varphi(-1) = \sigma_t(-1), \quad \varphi(1) = \sigma_t(1)\}$. Denote

$$\Delta_n := \max_{t\in[-1,1]} \|\sigma_t - \theta_n(t;\cdot)\|.$$

Take now an arbitrary function $g \in \mathcal{A}_n$ with integrable first derivative on $[-1,1]$ and such that $g(-1) = 0$. By Taylor's formula,

$$g(x) = \int_{-1}^{1} (x-t)_+^0 g'(t)\,dt$$

and hence

$$\int_{-1}^{1} g^2(x)\,dx = \int_{-1}^{1} \left[\int_{-1}^{1} (x-t)_+^0 g(x)\,dx\right] g'(t)\,dt.$$

But g is orthogonal to Π'_n. Thus

$$\int_{-1}^{1}(x-t)_+^0 g(x)\,dx = \int_{-1}^{1}\left[(x-t)_+^0 - \theta'_n(t;x)\right] g(x)\,dx$$

$$= \int_{-1}^{1} g(x) d\left[\sigma_t(x) - \theta_n(t;x)\right]$$

$$= \int_{-1}^{1}\left[\sigma_t(x) - \theta_n(t;x)\right] g'(x)\,dx$$

$$\leq \Delta_n \int_{-1}^{1} \mid g'(x) \mid dx = \Delta_n V[g] = \Delta_n.$$

Therefore

$$\int_{-1}^{1} g^2(x)\,dx \leq \Delta_n \int_{-1}^{1} \mid g'(t) \mid \, dt = \Delta_n. \tag{3.4}$$

Since every function g from $\mathcal{A}_n$ can be approximated by a function from $C^1[-1,1]$, the estimation (3.4) holds for every $g \in \mathcal{A}_n$ and particularly for $g = \psi$. Therefore, by Hölder's inequality,

$$\|\psi\|_1 \leq \sqrt{2}\left\{\int_{-1}^{1} \psi^2(t)\,dt\right\}^{\frac{1}{2}} \leq \sqrt{2\Delta_n}.$$

This inequality implies the estimate

$$E_n(f) \leq \frac{3}{2}\omega(f; \sqrt{2\Delta_n}). \tag{3.5}$$

It will be interesting to find those conditions on the Markov system $\mathcal{U}$ which imply the relation $\delta_n = \mathcal{O}(\Delta_n)$.

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