# Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica Balkanica

New Series Vol.10, 1996, Fasc.1

# Preconditioning the Biharmonic Equation by Multilevel Iterations

*R.E. Ewing*†, *S.D. Margenov*††, *P.S. Vassilevski*††

*Presented by P. Kenderov*

A preconditioned conjugate gradient iterative method is studied for the numerical solution of the biharmonic equation with Dirichlet boundary conditions for the solution and its normal derivative in a rectangular domain $\Omega$. We consider two discretizations of the differential equation: by standard finite differences and by quadratic splines from $S_2 \subset C^1(\Omega)$. The proposed preconditioner is constructed by first approximating the biharmonic operator by changing the boundary conditions thus allowing factorizing the new operator into product of two Laplacians. This product is then preconditioned by a recently proposed algebraic multilevel technique based on corresponding discretizations of the Laplace operator. The resulting multilevel preconditioner is used in the preconditioned conjugate gradient method giving rise to a multilevel algorithm with a total cost of $O\left(N^{\frac{5}{4}} \log \frac{1}{\epsilon}\right)$, where $N$ is the number of the unknowns and $\epsilon$ is the accuracy in the conjugate gradient method. Various numerical tests illustrating the properties of the algorithm obtained are presented. In particular, an $O(h^{-1})$ relative condition number of the constructed multilevel preconditioner with respect to the corresponding matrices was observed in the tests confirming the theoretical estimate.

## 1. Introduction

The analysis of bending of elastic plates leads to the solution of the biharmonic equation. The same equation has important applications in the shell theory and in fluid mechanics.

This paper is concerned with a preconditioned conjugate gradient iterative solution of systems of linear algebraic equations arising from finite difference or spline finite element discretization of the biharmonic equation. Some advantages of the iterative methods for solving large-scale fourth-order elliptic boundary value problems are demonstrated by a large set of experiments. In particular, we point out the ill-conditioning of the corresponding matrices $(\text{cond}(A) = O(h^{-4}))$, that makes the construction of efficient preconditioners especially important. We mention a number of papers in this field: O. A x e l s s o n and I. G u s t a f s s o n [1] and I. G u s t a f s s o n [9] consider various incomplete factorization algorithms, domain decomposition methods are considered in T. F. C h a n, E. W e i n a n, and J. S u n [6], and multigrid algorithms are studied by D. B r a e s s and P. P e i s k e r [5] and P. P e i s k e r [10]. Note that the latter two papers deal with mixed finite element formulations of the problem. Some multigrid methods for solving systems of linear algebraic equations that arise from standard nodal basis finite element discretizations of the problem are reported in [4] and [13]. Namely; an optimal multigrid method for the Hsiech-Clough-Tocher triangular finite element discretizations is proposed in [13], and in [4] some very promissing numerical results for a slash-cycle multigrid algorithm with smoothers based on pointwise ILU factorization are reported.

We study an approach based on changing the boundary conditions thus allowing factorization of the new operator into two Laplacians. This results to a nonoptimal approximation of the original operator; namely at this step we lose a factor of $O(h^{-1})$ ($h > 0$ is the mesh size) in the relative condition number of the new (factorizable) operator with respect to the original (nonfactorizable) operator. Then any known (of optimal order if available) preconditioners for the Laplacian can be used to construct a preconditioner for each of the factors and hence for the square of the Laplacian. We used in the presented numerical tests preconditioners constructed on the basis of the algebraic multilevel technique for finite element discretizations of second order eliptic problems (in our case applied for the Laplacian) proposed in O. A x e l s s o n and P. V a s s i l e v s k i [2], [3] as modified in P. V a s s i l e v s k i [12].

The formulation of the problem and the discretizations are given in Section 2. We consider two discretization methods: standard finite differences and quadrilateral finite elements based on quadratic $B$-splines. Section 3 contains the theoretical motivation for the proposed preconditioners and a review of some basic results concerning the algebraic multilevel preconditioners for second order elliptic problems. The numerical tests and some conclusions are presented in the final Section 4.

## 2. The discretization methods for the biharmonic equation

Consider the fourth order elliptic equation

$$\begin{aligned} \Delta^2 u &= f, \qquad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial n} = 0, \quad \text{on } \Gamma = \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Delta$ stands for the Laplace operator and $\Omega = [0,1] \times [0,1]$. The corresponding Galerkin variational formulation of (2.1) is to find a function $u \in \overset{\circ}{H}{}^2_0(\Omega)$ satisfying

$$a(u,v) = (f,v), \qquad \forall\, v \in H_0^2(\Omega), \tag{2.2}$$

where

$$a(u,v) = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy})dxdy \tag{2.3}$$

and

$$H_0^2(\Omega) = \left\{ v \in H^2(\Omega) : v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma \right\}.$$

We consider two approaches for numerical solution of the biharmonic equation discretized on a uniform mesh $\omega$ in $\Omega$. As a first method for discretization, we use the well known 13-point finite difference stencil. In this way, we obtain the system of linear algebraic equations

$$K^{(fd)}\mathbf{u} = \mathbf{f}, \tag{2.4}$$

where $\mathbf{u}$ is the vector of the nodal unknowns.

The second method we consider is the finite element method for solving (2.2). As a finite element space, we use the space of quadratic splines $\overset{\circ}{S}_2\,(\Omega)$ [11] with a basis of tensor product of $B_2$-splines, where

$$\begin{aligned} B_{2,k}(x) &= 3\sum_{r=k+1}^{k+3} (x_r - x)_+^2 / w'_{k,2}(x_r), \\ (x_r - x)_+ &= \begin{cases} x_r - x & \text{when } x \le x_r \\ 0 & \text{when } x \ge x_r, \end{cases} \\ w_{k,2}(x) &= (x - x_k)(x - x_{k+1})(x - x_{k+2})(x - x_{k+3}). \end{aligned} \tag{2.5}$$

This results to a system of linear algebraic equations

$$K^{(sp)}\xi = \mathbf{g}, \tag{2.6}$$

where $\xi$ is the coefficient vector of the finite elemnt solution $s_2 \in \overset{\circ}{S}_2\,(\Omega)$.

Both matrices $K^{(fd)}$ and $K^{(sp)}$ can be written in a tensor product form

$$K^{(fd)} = A^{(fd)} \otimes I + 2B^{(fd)} \otimes B^{(fd)} + I \otimes A^{(fd)}, \tag{2.7}$$

(2.8) $$K^{(sp)} = A^{(sp)} \otimes C^{(sp)} + 2B^{(sp)} \otimes B^{(sp)} + C^{(sp)} \otimes A^{(sp)},$$

where $I = \mathrm{diag}(1, 1, \ldots, 1)$ and

$$A^{(fd)} = \frac{1}{h^4} \begin{pmatrix} 7 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ & & & & & 1 & -4 & 7 \end{pmatrix},$$

$$B^{(fd)} = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix},$$

$$A^{(sp)} = \frac{1}{h^5} \begin{pmatrix} 6 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & & - \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ & & & & & 1 & -4 & 6 \end{pmatrix},$$

$$B^{(sp)} = \frac{1}{6h^3}\begin{pmatrix} 6 & -2 & -1 & & & & \\ -2 & 6 & -2 & -1 & & & \\ -1 & -2 & 6 & -2 & -1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & -1 & -2 & 6 & -2 & -1 \\ & & & & -1 & -2 & 6 & -2 \\ & & & & & -1 & -2 & 6 \end{pmatrix},$$

$$C^{(sp)} = \frac{1}{120h}\begin{pmatrix} 66 & 26 & 1 & & & & \\ 26 & 66 & 26 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 26 & 66 & 26 & 1 \\ & & & & 1 & 26 & 66 & 26 \\ & & & & & 1 & 26 & 66 \end{pmatrix}.$$

## 3. Multilevel preconditioning

In this section, we consider an indirect multilevel technique for preconditioning the matrices $K^{(fd)}$ and $K^{(sp)}$.

**Lemma 3.1.** *The matrices $K^{(fd)}$ and $K^{(sp)}$ are spectrally equivalent; i.e.,*

$$\text{cond}\left(K^{(sp)-1}K^{(fd)}\right) = O(1). \tag{3.1}$$

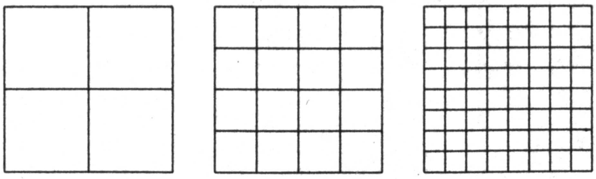
P r o o f. The estimate (3.1) follows from the relations:

- $C^{(sp)}$ is a positive definite matrix with cond $\left(C^{(sp)}\right) = O(1)$.
- The pairs of matrices $\left(A^{(fd)}, A^{(sp)}\right)$ and $\left(B^{(fd)}, B^{(sp)}\right)$ are spectrally equivalent with
cond $\left(B^{(sp)-1}B^{(fd)}\right) = O(1)$ and cond $\left(A^{(sp)-1}A^{(fd)}\right) = O(1)$.

A 2D local analysis can be used (see [7]) for a precise computation of the constant in (3.1). ∎

Assume that, in order to obtain a sufficiently accurate solution of the problem considered, a uniform refinement procedure is used. In this way, we have constructed a sequence of meshes $\omega_1 \subset \omega_2 \subset \ldots \subset \omega_\ell$ (see Figure 1) and corresponding matrices $K_1, K_2, \ldots, K_\ell$.

Figure 1: Uniform mesh refinement of square domain



Then the system (2.4) (respectively (2.6)) corresponds to the finest level of discretization $\omega = \omega_\ell$.

We use the recently proposed AMLI (algebraic multilevel preconditioning) technique (for more details see [2,3]). Let $M_{\Delta_5}$ be the optimal-order algebraic multilevel preconditioner for the discrete 5-point stencil Laplace matrix $K_{\Delta_5}$. Then consider the matrix

$$M = (M_{\Delta_5})^2, \qquad \text{where} \qquad \text{cond}\left(M_{\Delta_5}^{-1} K_{\Delta_5}\right) = O(1), \tag{3.2}$$

as a preconditioner of $K^{(fd)}$ and $K^{(sp)}$. Assuming $H^2$-regularity of the Dirichlet problem for the Laplacian (this is the case for our model rectangular domain $\Omega$) one can show that the AMLI methods from Axelsson and Vassilevski [2,3] as modified in Vassilevski [12] are actually of optimal order in the $L^2$-norm as well. Hence $M = (M_{\Delta_5})^2$ is an optional order preconditioner for $K_{\Delta_5}^2$. But since the purpose of this paper is to demonstrate the performance of the method we shall not go into more details here. We performed several numerical tests confirming that $M$ is an optimal-order preconditioner for $K_{\Delta_5}^2$. We now consider the structure of $K_{\Delta_5}^2$, which corresponds to a 13-point finite difference approximation of the biharmonic equation with boundary conditions $u = u_{nn} = 0$ on $\Gamma$. We have

$$K_{\Delta_5}^2 = \tilde{A}^{(fd)} \otimes I + 2B^{(fd)} \otimes B^{(fd)} + I \otimes \tilde{A}^{(fd)} \tag{3.3}$$

where

$$\tilde{A}^{(fd)} = \frac{1}{h^4} \begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix}$$

We note that our original finite difference matrix can not be factored into product of matrices representing any known discretization of the Laplace operator (with appropriate boundary conditions). Based on the analysis of Braess and Peisker [5] the following result follows.

**Lemma 3.2.** *The matrices $K^2_{\Delta_5}$ and $K^{(fd)}$ satisfy the following spectral equivalence relation*

$$\text{cond}\left(\left(K^2_{\Delta_5}\right)^{-1} K^{(fd)}\right) = O(h^{-1}). \tag{3.5}$$

Therefore, we have from Lemma 3.1 and Lemma 3.2 that

$$\text{cond}\left(\left(K^2_{\Delta_5}\right)^{-1} K^{(sp)}\right) = O(h^{-1}). \tag{3.6}$$

Following (3.5) and (3.6), we can expect, for the preconditioner $M$ from (3.2), the estimates

$$\text{cond}\left(M^{-1} K^{(fd)}\right) = O(h^{-1}), \tag{3.7}$$

and

$$\text{cond}\left(M^{-1} K^{(sp)}\right) = O(h^{-1}), \tag{3.8}$$

to hold. The last two relations lead to an $O(N^{5/4})$ arithmetic cost of the proposed multilevel preconditioning algorithm. Here $N$ is the number of the unknowns. This has been confirmed in our numerical tests.

## 4. Numerical tests

In this section, we present some numerical tests that demonstrate the properties of the proposed multilevel technique for iterative solution of the biharmonic equation.

The model problem (2.1) is discretized by finite differences leading to the system (2.4) and by the quadratic spline finite elements method yielding the system (2.6). Tables 1–3 show the behavior of the preconditioned conjugate gradient method with preconditioner defined by (3.2) related to the finite difference discrete system. The numerical tests corrsesponding to the solution of the spline finite element systems are similarly presented in Tables 4–6. The meshsize $h = 2^{-k}$ is varied as a function of the number of the refinement levels $k = 3, 4, 5, 6, 7$. The measures used to study the proposed preconditioner are the number of iterations and the corresponding CPU time in seconds. The tests presented are performed on the Ardent computer.

The preconditioner $M$ is defined by (3.2), where $M_{\Delta_5}$ is the hybrid algebraic multilevel preconditioner proposed in [12]. We denote by $\nu^i = (j_1, j_2, \ldots, j_i)$ the degrees of the accelerating polynomials used in the algebraic multilevel algorithm. For more details we refer to [12].

In the second columns of the tables, we show the relative condition numbers $\kappa = \text{cond}\left(M_{\Delta_5}^{-1} K_{\Delta_5}\right)$ and the reduction factor $\rho$ of the corresponding algebraic multilevel algorithm for solving systems with a matrix $K_{\Delta_5}$. One can see that $\kappa$ and $\rho$ are uniformly bounded with respect to $k$.

The average reduction factor of the iterative methods studied for solving the biharmonic equation and the relative iteration factors are presented in the last two columns of the tables.

The numerical tests show an attractive (but not optimal, as must be expected) convergence rate of the proposed preconditioning iterative technique (i.e., the increase of the number of iterations per level). The iteration factor is near $\sqrt{2}$, which is in full agreement with (3.7) and (3.8), respectively.

We finally remark that the proposed approach of preconditioning the (nonfactorizable) biharmonic operator discretized by finite differences was successfully applied in Ewing et al. [8] for solving 2D nonstationary Navier-Stokes equations in the stream-function ($\psi$) formulation based on a fourth order equation for $\psi$.

**TABLE 1. Biharmonic equation; finite differences**

$\nu^7 = (1,2,2,2,2,2,1)$, $\nu^4 = (1,2,2,1)$
$\nu^6 = (1,2,2,2,2,1)$, $\nu^3 = (1,2,1)$
$\nu^5 = (1,2,2,2,1)$

| levels | $\kappa/\rho$ | iterations | cpu/sec | average red. factor | iteration factor |
|---|---|---|---|---|---|
| 3 | 1.93<br>0.16 | 8 | 0.68 | 0.26 | – |
| 4 | 2.17<br>0.19 | 10 | 3.42 | 0.37 | 1.25 |
| 5 | 2.28<br>0.20 | 13 | 17.58 | 0.46 | 1.3 |
| 6 | 2.34<br>0.21 | 18 | 95.26 | 0.56 | 1.37 |
| 7 | 2.37<br>0.21 | 19 | 402.56 | 0.60 | 1.06 |

**TABLE 2. Biharmonic equation; finite differences**

$\nu^7 = (1,3,1,3,1,3,1)$, $\nu^4 = (1,1,3,1)$
$\nu^6 = (1,1,3,1,3,1)$, $\nu^3 = (1,3,1)$
$\nu^5 = (1,3,1,3,1)$

| levels | $\kappa/\rho$ | iterations | cpu/sec | average red. factor | iteration factor |
|---|---|---|---|---|---|
| 3 | 1.85<br>0.15 | 7 | 0.15 | 0.25 | – |
| 4 | 1.95<br>0.16 | 9 | 3.35 | 0.31 | 1.29 |
| 5 | 1.98<br>0.17 | 12 | 17.70 | 0.41 | 1.33 |
| 6 | 1.98<br>0.17 | 16 | 90.77 | 0.51 | 1.33 |
| 7 | 1.99<br>0.17 | 18 | 406.35 | 0.56 | 1.13 |

**TABLE 3. Biharmonic equation; finite differences**

$\nu^7 = (1,3,1,1,3,1,1)$, $\nu^4 = (1,3,1,1)$
$\nu^6 = (1,3,1,3,1,1)$, $\nu^3 = (1,1,1)$
$\nu^5 = (1,1,3,1,1)$

| levels | $\kappa/\rho$ | iterations | cpu/sec | average red. factor | iteration factor |
|---|---|---|---|---|---|
| 3 | 2.12<br>0.18 | 8 | 0.57 | 0.31 | – |
| 4 | 2.71<br>0.24 | 14 | 3.95 | 0.50 | 1.75 |
| 5 | 2.89<br>0.25 | 16 | 11.69 | 0.54 | 1.14 |
| 6 | 2.96<br>0.26 | 20 | 87.87 | 0.61 | 1.25 |
| 7 | 2.97<br>0.26 | 28 | 476.14 | 0.70 | 1.40 |

**TABLE 4. Biharmonic equation; splines**
$\nu^7 = (1,2,2,2,2,2,1)$, $\nu^4 = (1,2,2,1)$
$\nu^6 = (1,2,2,2,2,1)$, $\nu^3 = (1,2,1)$
$\nu^5 = (1,2,2,2,1)$

| levels | $\kappa/\rho$ | iterations | cpu/sec | average red. factor | iteration factor |
|---|---|---|---|---|---|
| 3 | 1.93<br>0.16 | 12 | 0.97 | 0.31 | – |
| 4 | 2.17<br>0.19 | 19 | 6.15 | 0.47 | 1.58 |
| 5 | 2.28<br>0.20 | 22 | 28.70 | 0.52 | 1.16 |
| 6 | 2.34<br>0.21 | 29 | 150.05 | 0.60 | 1.32 |
| 7 | 2.37<br>0.21 | 36 | 740.08 | 0.67 | 1.24 |

**TABLE 5. Biharmonic equation; splines**
$\nu^7 = (1,3,1,3,1,3,1)$, $\nu^4 = (1,1,3,1)$
$\nu^6 = (1,1,3,1,3,1)$, $\nu^3 = (1,3,1)$
$\nu^5 = (1,3,1,3,1)$

| levels | $\kappa/\rho$ | iterations | cpu/sec | average red. factor | iteration factor |
|---|---|---|---|---|---|
| 3 | 1.85<br>0.15 | 12 | 1.12 | 0.30 | – |
| 4 | 1.95<br>0.16 | 18 | 6.28 | 0.44 | 1.50 |
| 5 | 1.98<br>0.17 | 21 | 29.67 | 0.50 | 1.17 |
| 6 | 1.98<br>0.17 | 24 | 133.16 | 0.55 | 1.14 |
| 7 | 1.99<br>0.17 | 30 | 659.90 | 0.62 | 1.27 |

**TABLE 6. Biharmonic equation; splines**
$\nu^7 = (1,3,1,1,3,1,1)$, $\nu^4 = (1,3,1,1)$
$\nu^6 = (1,3,1,3,1,1)$, $\nu^3 = (1,1,1)$
$\nu^5 = (1,1,3,1,1)$

| levels | $\kappa/\rho$ | iterations | cpu/sec | average red. factor | iteration factor |
|---|---|---|---|---|---|
| 3 | 2.12<br>0.18 | 14 | 0.95 | 0.37 | – |
| 4 | 2.71<br>0.24 | 21 | 5.80 | 0.50 | 1.50 |
| 5 | 2.89<br>0.25 | 24 | 26.02 | 0.54 | 1.15 |
| 6 | 2.96<br>0.26 | 30 | 129.80 | 0.62 | 1.25 |
| 7 | 2.97<br>0.26 | 42 | 707.47 | 0.71 | 1.4 |

## Acknowledgment

This paper was partially supported by the Bulgarian Ministry of Education and Science under grants MM/415-94 and MM/417-94.

## References

1. O. Axelsson, I. Gustafsson. An iterative solver for a mixed variable variational formulation of the first biharmonic problem. *Comput. Meth. Appl. Mech. Engng.* **20** (1979), 9–16.
2. O. Axelsson, P. S. Vassilevski. Algebraic multilevel preconditioning methods. I., *Numer. Math.* **56** (1989), 157–177.
3. O. Axelsson, P. S. Vassilevski. Algebraic multilevel preconditioning methods. II., *SIAM J. Numer. Anal.* **27** (1990), 1569–1590.
4. R. Banerjee, M. Benson. An approximate inverse based multigrid approach to the biharmonic problem. *Int. J. Comp. Math.* **40** (1991), 201-210.
5. D. Braess, P. Peisker. On the numerical solution of the biharmonic equation and the role of squaring matrices for preconditioning. *IMA J. Numer. Anal.* **6** (1986), 393–404.
6. T.F. Chan, E. Weinan, J. Sun. Domain decomposition interface preconditioners for fourth order elliptic problems. *CAM Report 87-11*, UCLA, 1990.
7. V. Eijkhout, S. Margenov. Preconditioned iterative solution of spline finite element systems *Mathematika Balkanica* **4** (1990), 350–367.

8. R. E. E w i n g, O. P. I l i e v, S. D. M a r g e n o v, P. S. V a s s i l e v s k i. A multilevel technique for solving 2D unsteady Navier–Stokes equations. *in Adaptive, Multilevel, and Hierarchical Computational Strategies*, Ahmed K. Noor, ed., AMD–Vol.**157** (1992), 235–247.
9. I. G u s t a f s s o n. A preconditioned iterative method for the solution of the biharmonic problem. *IMA J. of Numer. Anal.* **4** (1984), 55–67.
10. P. P e i s k e r. A multilevel algorithm for the biharmonic problem. *Numer. Math.* **46** (1985), 623–634.
11. S. D. M a r g e n o v. Application of spline macro-elements in solving the biharmonic equation. - In: *Numerical Methods '84, Proceedings of the Int. Conf. on Numer. Meth.*, Sofia, Bulgaria, 1985 (Russian), 408–413.
12. P. S. V a s s i l e v s k i. Hybrid $V$-cycle algebraic multilevel preconditioners. *Math. Comp.*, **58** (1992), 489-512.
13. S. Z h a n g. An optimal order multigrid method for biharmonic $C^1$ finite element equations. *Numer. Math.* **56** (1989), 613-624.

†*Institute for Scientific Computation,*
*TA&MU*
*College Station, TX 77840,*
*U.S.A.*

††*CICT,*
*Bulgarian Academy of Sciences*
*Acad. G.Bontchev str. Bl.25-A,*
*1113 Sofia,*
*BULGARIA*

*Received 22.09.1994*