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On a Problem Stated by Gyori and Ladas

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In the present paper a problem stated by Gtori and Ladas in [4, p.161] is solved. Sufficient conditions are obtained under which all nonoscillatory solutions of the equation

$$(x(t)+p(t)x(t-\tau))'+q(t)x(t-\sigma)=0 \tag{1}$$

where $\tau>0$, $\sigma\geq 0$ and $p(t)$, $q(t)$ are continuous functions, tend to zero as $t\to\infty$.

1. Introduction

The asymptotic behaviour of solutions of neutral differential equations has been the subject of intensive investigations during the past few years ([1]-[5]). In particular in case when $p(t)$ and $q(t)$ are constant functions the problem of existence of the limit $\lim_{t\to\infty} x(t)$ is almost completely solved. If $p(t)$ and $q(t)$ are nonconstant functions the asymptotic behaviour of nonoscilatory solutions of (1) has been investigated under the following conditions on $q(t)$:

$$Q1.\quad q(t)\in C\left([t_0,\infty),\mathbb{R}_+\right) \text{ and } 0<q_1\leq q(t)\leq q_1$$

$$Q2.\quad q(t)\in C\left([t_0,\infty),\mathbb{R}_+\right) \text{ and } \int_{t_0}^{\infty} q(t)dt=\infty$$

We shall note that no example is known when the limit $\lim_{t\to\infty} x(t)$ (finite of infinite) exists if condition $Q1$ is met, but does not exist if $Q2$ is met. In view of this fact the question asked by Gyori and Ladas in [4, p.161] is quite natural: "For the equation $[x(t)+x(t-\tau)]'+q(t)x(t-\sigma)=0$ let $\int_{t_0}^{\infty} q(t)dt=\infty$. Is it true that all nonoscillating solutions of the equation tend to zero?" In the present paper an example is given answering negatively to the above question. A problem arises of finding sufficient conditions under which each nonoscillating solution of (1) tends to zero.

2. Main Results

By a solution of (1) we mean a continuous function on the interval $[t_0, \infty)$ such that $x(t) + p(t)x(t-\tau)$ is continuously differentiable and x satisfies (1). As is customary, a solution of (1) is said to be oscillatory if it has arbitrary large zeroes, otherwise it is said to be nonoscillatory. In the sequel, for convenience, we will assume that inequalities concerning values of functions are satisfies eventually, that is for all large t.

E x a m p l e 1. Consider the equation

$$[x(t) + x(t-2)]' + q(t)x(t) = 0 \tag{2}$$

where

$$q(t) = \frac{\frac{1}{t^2} + \frac{1}{(t-2)^2}}{\varphi(t) + \frac{1}{t}}$$

and $\varphi(t)$ is a 4-periodic function

$$\varphi(t) = \begin{cases} 0 & , t \in [0,1] \\ t-1 & , t \in (1,2] \\ 1 & , t \in (2,3] \\ 4-t & , t \in (3,4] \end{cases}$$

A straightforward verification yields that $x(t) = \varphi(t) + \frac{1}{t}$ is a nonoscillatory solution of (2) and that the limit $\lim_{t\to\infty} x(t)$ does not exist. On the other hand

$$\int_4^\infty q(t)dt > \sum_{k=1}^\infty \int_{4k}^{4k+1} \frac{\frac{1}{t^2} + \frac{1}{(t-2)^2}}{\varphi(t) + \frac{1}{t}} dt =$$

$$= \sum_{k=1}^\infty \int_{4k}^{4k+1} \frac{\frac{1}{t^2} + \frac{1}{(t-2)^2}}{\frac{1}{t}} dt > \sum_{k=1}^\infty \int_{4k}^{4k+1} \frac{dt}{t}$$

The last series is obviously divergent. Thus $\int_4^\infty q(t)dt = \infty$.

The example we consider gives a negative answer to the question of Gyori and Ladas and states the problem to find conditions such that $\lim_{t\to\infty} x(t) = 0$ for each nonoscillatory solution. In this direction most general are the results of Lu W u d u [5] and B a i n o v , M y s h k i s , Z a h a r i e v [1].

Theorem 1 ([5]). *Let condition $Q1$ holds and*

$$-1 \le p_1 \le p(t) \le p_2 \tag{3}$$

Then $\lim_{t\to\infty} x(t) = 0$ for each nonoscillatory solution $x(t)$ of (1).

Before formulating the result of Bainov, Myshkis, Zahariev we shall introduce the notation $Q3$. $q(t) \in C([t_0, \infty), \mathbb{R}_+)$ and for each measurable set Q such that $meas(Q \cap [t, t + 2\tau]) \geq \tau$ for each $\geq t_0$, $\int_Q q(t)dt = \infty$.

Theorem 2 ([1]). *Let condition $Q3$ holds and $p(t) \equiv p \geq 0$. Then $\lim_{t\to\infty} x(t) = 0$ for each nonoscillatory solution $x(t)$ of (1).*

We shall say that condition $Q4$ is met if the following condition holds.

$$Q4. \quad q(t) \in C([t_0, \infty), \mathbb{R}_+) \text{ and } \int_{t_0}^{\infty} \tilde{q}(t)dt = \infty$$

where $\tilde{q}(t) = \min\{q(t), q(t + \tau)\}$. Define the function $z(t)$ in the following way:

$$z(t) = x(t) + p(t)x(t - \tau) \tag{4}$$

Theorem 3. *Let conditions $Q4$ and (3) hold. Then, for each nonoscillatory solution of (1) $\lim_{t\to\infty} x(t) = 0$.*

Proof. Without loss of generality let $x(t) > 0$. We shall split up the proof of the theorem into 3 steps: i), ii), iii).

i) $z(t) > 0$

From the inequality $x(t) > 0$ and from the equality

$$z'(t) = -q(t)x(t - \sigma) \tag{5}$$

it follows that $z(t)$ is a nondecreasing function. $Q4$ implies that $z(t)$ is not an eventually constant function. Suppose that $z(t) < 0$. There exists $c > 0$ such that $z(t) < -c$. Then

$$-x(t - \tau) < p(t)x(t - \tau) < x(t) + p(t)x(t - \tau) = z(t) < -c$$

From $Q4$, (5) and from the inequality $x(t) > c$ it follows that $\lim_{t\to\infty} z(t) = -\infty$ and $\lim_{t\to\infty} \sup x(t) = \infty$. On the other hand from (3), (4) and $z(t) < 0$ we obtain the inequalities

$$x(t) < -p(t)x(t - \tau) < x(t - \tau)$$

But the inequality $x(t) < x(t - \tau)$ contradicts the relation

$$\lim_{t\to\infty} \sup x(t) = \infty,$$

hence $z(t) > 0$.

ii) $\lim_{t\to\infty} z(t) = 0$.

From (5) it follows that

(6) $z'(t)-\tilde{q}(t-\tau)p(t-\sigma)x(t-\tau-\sigma) = -q(t)x(t-\sigma)-\tilde{q}(t-\tau)p(t-\sigma)x(t-\sigma-\tau)$

Then making use of the definition of $\tilde{q}(t)$ and (3) we obtain that

$$z'(t) - p_2\tilde{q}(t-\tau)x(t-\tau-\sigma) \leq -\tilde{q}(t-\tau)[x(t-\sigma) + p(t-\sigma)x(t-\sigma-\tau)] =$$
$$= -\tilde{q}(t-\tau)z(t-\sigma)$$

Since $z(t)$ is a positive nonincreasing function then there exist the non-negative limit $c = \lim_{t\to\infty} z(t)$. Suppose that $c > 0$. Then $z(t) \geq c$ and (6) takes the form

$$z'(t) - p_2\tilde{q}(t-\tau)x(t-\tau-\sigma) \leq -c\tilde{q}(t-\tau)$$

Integrate the last inequality from t_1 to t and obtain

$$z(t) - z(t_1) - p_2 \int_{t_1}^{t} \tilde{q}(s-\tau)x(s-\tau-\sigma)ds \leq -c \int_{t_1}^{t} \tilde{q}(s-\tau)ds.$$

Since $z(t)$ is a bounded function $Q4$ implies that $\int_{t_1}^{\infty} \tilde{q}(t)x(t-\sigma)dt = \infty$. From the definition of $\tilde{q}(t)$ it follows that

$$\int_{t_1}^{\infty} q(t)x(t-\sigma)dt = \infty \tag{7}$$

Integrating (5) from t_2 to t we obtain the equality

$$z(t) - z(t_2) = -\int_{t_2}^{t} q(s)x(s-\sigma)ds$$

Then (7) implies the relation $\lim_{t\to\infty} z(t) = -\infty$. The contradiction obtained shows that $c = 0$, i.e. $\lim_{t\to\infty} z(t) = 0$.

iii) $\lim x(t) = 0$.

Let us assume that $x(t)$ is an unbounded function. There exist a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} x(t_n) = \infty$ and $\max_{[t_1,t_n]} x(s) = x(t_n)$. Then

$$z(t_n) = x(t_n) + p(t_n)x(t_n-\tau) \geq x(t_n) + p_1x(t_n-\tau) \geq (1+p_1)x(t_n)$$

Since $1+p_1 > 0$ then $\lim_{n\to\infty} z(t_n) = \infty$ which contradicts the relation $\lim_{t\to\infty} z(t) = 0$ proved in ii). Hence $x(t)$ is bounded. Let $d = \lim_{t\to\infty} \sup x(t)$. Choose a sequence $\{\alpha_n\}$ such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} x(\alpha_n) = d$.

Since $\{p(\alpha_n)\}$ and $\{x(\alpha_n-\tau)\}$ are bounded we can choose a subsequence $\{n_k\} \subseteq \{n\}$ such that $\{p(\alpha_{n_k})\}$ and $\{x(\alpha_{n_k}-\tau)\}$ converge. Then

$$0 = \lim_{k\to\infty} z(\alpha_{n_k}) = \lim_{t\to\infty} \sup x(t) + \lim_{k\to\infty} p(\alpha_{n_k}) \lim_{k\to\infty} x(\alpha_{n_k}-\tau).$$

If $\lim_{k\to\infty} p(\alpha_{n_k}) \geq 0$ then $0 \geq \lim_{t\to\infty} \sup x(t) \geq 0$.
If $\lim_{k\to\infty} p(\alpha_{n_k}) < 0$ then

$$0 = \lim_{t\to\infty} \sup x(t) + \lim_{k\to\infty} p(\alpha_{n_k}) \lim_{k\to\infty} x(\alpha_{n_k}-\tau) \geq$$

$$\leq \left(1 + \lim_{k\to\infty} p(\alpha_{n_k})\right) \lim_{t\to\infty} \sup x(t) \geq 0.$$

Hence $\lim_{t\to\infty} x(t) = 0$.

It is clear that the assertion of Theorem 3 is stronger than that of Theorem 1. The relation between $Q3$ and the condition $Q4$ which we introduced is not so obvious. Let us denote with $F3$ ($F4$) the set of functions satisfying condition $Q3$ ($Q4$).

Proposition 1. $F3 \subseteq F4$.

Proof. Let $q \in F_3$. We shall prove that $q \in F4$. Consider the intervals

$$\{[t_0 + 2k\tau, t_0 + (2k+2)\tau]\} \quad , k = 0, 1, 2, ..$$

$$\{[t_0 + (2k+1)\tau, t_0 + (2k+3)\tau]\} \quad , k = 0, 1, 2, ..$$

Fix $n \in \mathbb{N} \cup \{0\}$ and introduce the sets

$$E_{n1} = \{s \in [t_0 + 2n\tau, t_0 + (2n+1)\tau]/\tilde{q}(s) = q(s)\}$$

$$E_{n2} = \{s \in [t_0 + (2n+1)\tau, t_0 + (2n+2)\tau]/\tilde{q}(s-\tau) = q(s)\}$$

$$F_{n1} = \{s \in [t_0 + (2n+1)\tau, t_0 + (2n+2)\tau]/\tilde{q}(s) = q(s)\}$$

$$F_{n2} = \{s \in [t_0 + (2n+2)\tau, t_0 + (2n+3)\tau]/\tilde{q}(s-\tau) = q(s)\}$$

$$E_n = E_{n1} \cup E_{n2}, \quad F_n = F_{n1} \cup F_{n2}, \quad E = \cup_0^\infty E_n, \quad F = \cup_0^\infty F_n$$

We shall prove that $meas\{(E \cup F) \cap [\alpha, \alpha + 2\tau]\} \geq \tau$ for each $\alpha \geq t_0$.

Fix $\alpha \in [t_0, \infty)$. Let $\alpha \in [t_0 + 2n\tau, t_0 + (2n+1)\tau]$ (the case when $\alpha \in [t_0 + (2n+1)\tau, t_0 + (2n+2)\tau]$ is considered analogously). Then

$$(E \cup F) \cap [\alpha, \alpha + 2\tau] \supseteq (E_n \cup F_n) \cap [\alpha, \alpha + 2\tau] =$$

$$= (E_n \cap [\alpha, \alpha + 2\tau]) \cup (F_n \cap [\alpha, \alpha + 2\tau]) =$$

$$= (E_{n_1} \cap [\alpha, \alpha+2\tau]) \cup (E_{n_2} \cap [\alpha, \alpha+2\tau]) \cup (F_{n_1} \cap [\alpha, \alpha+2\tau]) \cup (F_{n_2} \cap [\alpha, \alpha+2\tau])$$

Let

$$\begin{aligned} G_{n1} &= E_{n1} \cap [\alpha, t_0 + (2n+1)\tau] \\ G_{n2} &= E_{n2} \cap [\alpha + \tau, t_0 + (2n+2)\tau] \\ H_{n1} &= F_{n1} \cap [t_0 + (2n+1)\tau, \alpha + \tau] \\ H_{n2} &= F_{n2} \cap [t_0 + (2n+2)\tau, \alpha + 2\tau] \end{aligned}$$

Obviously

$$G_{ni} \subseteq E_{ni} \cap [\alpha, \alpha + 2\tau] \quad i = 1, 2.$$

$$H_{ni} \subseteq F_{ni} \cap [\alpha, \alpha + 2\tau] \quad i = 1, 2.$$

Let $\tilde{G}_{n2} = \{s - \tau / s \in G_{n2}\}$, $\tilde{H}_{n2} = \{s - \tau / s \in H_{n2}\}$. Then

$$[\alpha, t_0 + (2n+1)\tau] = G_{n1} \cup \tilde{G}_{n2}$$

$$[t_0 + (2n+1)\tau, \alpha + \tau] = H_{n1} \cup \tilde{H}_{n2}$$

Hence

$$\begin{aligned} meas(G_{n1} \cup G_{n2}) &\geq t_0 + (2n+1)\tau - \alpha \\ meas(H_{n1} \cup H_{n2}) &\geq \alpha + \tau - (t_0 + (2n+1)\tau) \end{aligned}$$

Since $meas(G_{n1} \cup G_{n2}) \cap (H_{n1} \cup H_{n2}) = 0$ then $meas(G_{n1} \cup G_{n2} \cup H_{n1} \cup H_{n2}) \geq \tau$

Therefore

$$meas\,\{(E_{n1} \cap [\alpha, \alpha + 2\tau]) \cup (E_{n2} \cap [\alpha, \alpha + 2\tau]) \cup$$

$$\cup (F_{n1} \cap [\alpha, \alpha + 2\tau]) \cup (F_{n2} \cap [\alpha, \alpha + 2\tau])\} \geq \tau$$

and $meas\,\{(E \cup F) \cap [\alpha, \alpha + 2\tau]\} \geq \tau$.

$$\int_{E \cup F} q(t)dt \leq \int_E q(t)dt + \int_F q(t)dt = \sum_{i=0}^{\infty} \int_{E_i} q(t)dt + \sum_{i=0}^{\infty} \int_{F_i} q(t)dt$$

Consider the integral $\int_{E_k} q(t)dt$.

$$\int_{E_k} q(t)dt = \int_{E_{k1}} q(t)dt + \int_{E_{k2}} q(t)dt =$$

$$= \int_{E_{k1}} \tilde{q}(t)dt + \int_{E_{k2}} \tilde{q}(t - \tau)dt = \int_{E_{k1}} \tilde{q}(t)dt + \int_{\tilde{E}_{k2}} \tilde{q}(t)dt$$

$(\tilde{E}_{k2} = \{s - \tau / s \in E_{k2}\})$

Since $E_{k1} \cup \tilde{E}_{k2} = [t_0 + 2k\tau, t_0 + (2k+1)\tau]$ then

$$\int_{E_{k1}} \tilde{q}(t)dt + \int_{\tilde{E}_{k2}} \tilde{q}(t)dt \leq 2 \int_{t_0+2k\tau}^{t_0+(2k+1)\tau} \tilde{q}(t)dt.$$

Hence

$$\int_{E_k} q(t)dt \leq 2 \int_{t_0+2k\tau}^{t_0+(2k+1)\tau} \tilde{q}(t)dt.$$

and

$$\int_E q(t)dt \leq 2 \sum_0^{\infty} \int_{t_0+2k\tau}^{t_0+(2k+1)\tau} \tilde{q}(t)dt.$$

Analogously

$$\int_F q(t)dt \leq 2 \sum_0^{\infty} \int_{t_0+(2k+1)\tau}^{t_0+(2k+2)\tau} \tilde{q}(t)dt.$$

Thus

$$\int_{E\cup F} q(t)dt \leq 2 \int_{t_0}^{\infty} \tilde{q}(t)dt.$$

$Q3$ implies that $\int_{E\cup F} q(t)dt = \infty$ and therefore $2\int_{t_0}^{\infty} \tilde{q}(t)dt = \infty$.
The following example shows that the inclusion in proposition 1 is strict.

E x a m p l e 2. Consider the function

$$f(x) = \begin{cases} -\frac{2^k}{\tau}x + 1 + k2^k & , x \in [k\tau, (k + \frac{1}{2^k})\tau] \\ 0 & , x \in [(k + \frac{1}{2^k})\tau, (k + \frac{1}{2} - \frac{1}{2^k})\tau] \\ \frac{2^k}{\tau}x + 1 - (k + \frac{1}{2})2^k & , x \in [(k + \frac{1}{2} - \frac{1}{2^k})\tau, (k + \frac{1}{2})\tau] \\ 1 & , x \in [(k + \frac{1}{2})\tau, (k+1)\tau] \end{cases}$$

where $k = 1, 2, 3, \ldots$. It is immediately verified that $f(x) \in C([\tau, \infty), \mathbb{R}_+)$ and $\int_{\tau}^{\infty} \tilde{f}(x)dx = \infty$. On the other hand if we consider the set $Q = \cup_{k=1}^{\infty}[k\tau, (k+\frac{1}{2})\tau]$ then $meas\{Q \cap [t_0, t_0 + 2\tau]\} = \tau$ for each $t_0 \geq \tau$ but $\int_Q f(x)dx < \infty$. Hence $f \in F_4$ but $f \notin F_3$.

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