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Two Simple Explicit Forms of the Taylor Expansion of the Bounded Koebe Function

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In this paper we give two simple explicit forms of the coefficients of the bounded Koebe function.

Let $k(z)$ denote the Koebe function

$$w = k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n, \ |z| < 1, \tag{1}$$

which maps analytically and univalently the disc $|z| < 1$ onto the domain D that consists of the entire complex plane except for a slit along the negative real axis from $w = -\infty$ to $w = -1/4$ (see, for example , in [1, pp.15-16]). Let

$$z = K(w) = \frac{1 + 2w - \sqrt{1+4w}}{2w}, \ \sqrt{1} = 1, \ w \in D, \tag{2}$$

be the inverse function for the Koebe function (1). The function (2) has the Taylor expansion

$$z = K(w) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} \binom{2\nu}{\nu-1} w^{\nu}, \ |w| < \frac{1}{4}. \tag{3}$$

From (1) and (2) it follows that, for $0 < t < 1$, the function

$$w = f(z) \equiv \frac{1}{t} K(tk(z)) = \tag{4}$$

$$= \frac{(1-z)^2 + 2tz - (1-z)\sqrt{(1-z)^2 + 4tz}}{2t^2 z}, \ \sqrt{1} = 1,$$

maps analytically and univalently the disc $|z| < 1$ onto the disc $|w| < 1/t$ with a slit from $-1/t$ to

$$\frac{1}{t}K(-\frac{t}{4}) = -\frac{1}{t} + \frac{2}{t^2}(\sqrt{1-t} - 1 + t) < 0$$

where the last radical is positive. From (1) and (4) it is clear that the function $w = f(z)$ can also be determined by the equation

$$\frac{z}{(1-z)^2} = \frac{w}{(1-tw)^2}, \ |z| < 1, \ 0 < t < 1. \tag{5}$$

The function (4) is known as the bounded Koebe function (see, for example, in [1, pp.37-40]). This function can also be found in [2]-[7]. The authors [4] gave a cumbersome explicit form of the Taylor expansion of the function (4). Now we shall give two simple explicit forms of this expansion.

Theorem 1. *The bounded Koebe function $f(z)$, determined by (4) (or (5)), has the following Taylor expansion*

$$f(z) = \sum_{n=1}^{\infty} nF(1-n, 1+n, 3, t)z^n, \ |z| < 1, \ 0 < t < 1, \tag{6}$$

where $F(1-n, 1+n, 3, t)$ is the hypergeometric polynomial

$$F(1-n, 1+n, 3, t) = \sum_{\nu=0}^{n-1} \frac{(1-n)_\nu (1+n)_\nu}{(3)_\nu \nu!} t^\nu, \ n \geq 1, \tag{7}$$

where $(a)_\nu$ for an arbitrary number a denotes

$$(a)_\nu = a(a+1)...(a+\nu-1), \ \nu = 1, 2, ...; (a)_0 = 1. \tag{8}$$

Proof. From (3) and (4) we find the formula

$$f(z) = \sum_{n=1}^{\infty} z^n \sum_{\nu=0}^{n-1} \mu_\nu(n) t^\nu, \ |z| < 1, \ 0 < t < 1, \tag{9}$$

where

$$\mu_\nu(n) = \frac{(-1)^\nu}{\nu+1} \binom{2\nu+2}{\nu} \binom{\nu+n}{2\nu+1}, \ 0 \leq \nu \leq n-1, \ n \geq 1. \tag{10}$$

From (10) it follows that

$$\frac{\mu_\nu(n)}{\mu_{\nu-1}(n)} = \frac{(\nu-n)(\nu+n)}{(\nu+2)\nu}, \; 1 \le \nu \le n-1, \; n \ge 2, \tag{11}$$

with

$$\mu_0(n) = n, \; n \ge 1. \tag{12}$$

From (11) and (12) we deduce that

$$\mu_\nu(n) = n\frac{(1-n)_\nu(1+n)_\nu}{(3)_\nu \nu!}, \; 0 \le \nu \le n-1, \; n \ge 1, \tag{13}$$

having in mind the notation (8). Thus from (9) and (13) we obtain the formulas (6) and (7) which completes the proof of Theorem 1. ■

For $n \ge 2$, the hypergeometric polynomial (7) can be written as

$$F(1-n, 1+n, 3, t) = 1+ \tag{14}$$

$$+2\sum_{\nu=1}^{n-1}(-1)^\nu \frac{(n^2-1^2)(n^2-2^2)...(n^2-\nu^2)}{(\nu+2)!\nu!}t^\nu.$$

The formulas (6)-(7) and (14) appear to be basic formulas in the theory of the bounded Koebe function (4). Using the Gauss hypergeometric function theory (see, for example, in [8, Vol.II]) for our formula (6)-(7), we can obtain other formulas that suit our purposes better. For instance, we have the following

Theorem 2. *If $n \ge 2$, the hypergeometric polynomial (7) vanish for $t = 1$, i.e.*

$$F(1-n, 1+n, 3, 1) = 0, \; n = 2, 3, ... \tag{15}$$

Proof. According to the well known Gauss formula for the limit of the hypergeometric function as its argument tends to unit (see, for example, in [8, Vol. II, p.143, formula (381.7)]), applied to the hypergeometric polynomial (7) for $n \ge 2$, we have the equation

$$F(1-n, 1+n, 3, 1) = \frac{\Gamma(1)\Gamma(3)}{\Gamma(2-n)\Gamma(2+n)}, \; n \ge 2, \tag{16}$$

where Γ is the gamma-function. Since $\Gamma(1) = 1$, $\Gamma(3) = 2$, $\Gamma(2-n) = \infty$ and $\Gamma(2+n) = (n+1)!$ for $n = 2, 3, ...$ (see, for example, in [8, Vol. I, p.287]), the equation (16) is reduced to (15), which completes the proof of Theorem 2. ■

From (14) and (15) we obtain the identities

$$(17)\qquad 1+2\sum_{\nu=1}^{n-1}(-1)^{\nu}\frac{(n^2-1^2)(n^2-2^2)...(n^2-\nu^2)}{(\nu+2)!\nu!}=0$$

for $n=2,3,\ldots$.

Further we have

Theorem 3. *The bounded Koebe function $f(z)$, determined by (4) (or (5)), has the following Taylor expansion*

$$(18)\qquad f(z)=z+(1-t)\sum_{n=2}^{\infty}nF(2-n,2+n,3,t)z^n$$

for $|z|<1$ and $0<t<1$, where $F(2-n,2+n,3,t)$ is the hypergeometric polynomial

$$(19)\qquad F(2-n,2+n,3,t)=\sum_{\nu=0}^{n-2}\frac{(2-n)_\nu(2+n)_\nu}{(3)_\nu\nu!}t^\nu,\ n\geq 2,$$

having in mind the notation (8).

Proof. (First) According to the basic linear transformations of the Gauss hypergeometric function (see, for example, in [8, Vol. II, p.146, formulas (382.9)]), applied to the hypergeometric polynomial (7) for $n\geq 2$, we have the factorization

$$(20)\qquad F(1-n,1+n,3,t)=(1-t)F(2-n,2+n,3,t),\ n\geq 2,$$

with the help of the hypergeometric polynomial (19). Therefore, the identities (20) transform the expansion (6)-(7) into the form (18)-(19). ∎

Proof. (Second) The factorization (20) can be directly proved as follows. From (7) we obtain the Taylor expansion

$$(21)\qquad \frac{F(1-n,1+n,3,t)}{1-t}=\sum_{j=0}^{\infty}a_j(n)t^j,\ 0<t<1,\ n\geq 2,$$

where

$$(22)\qquad a_j(n)=\sum_{\nu=0}^{j}\frac{(1-n)_\nu(1+n)_\nu}{(3)_\nu\nu!},\ j\geq 0,\ n\geq 2.$$

Now (22), by induction on j, yields

$$a_j(n) = \frac{(2-n)_j(2+n)_j}{(3)_j j!}, \; j \geq 0, \; n \geq 2. \tag{23}$$

From (23), having in mind (8), it follows that $a_j(n) = 0$ for $j \geq n-1$, $n \geq 2$. Hence the series (21) is reduced to the hypergeometric polynomial (19), i.e. the representation (20) holds.

Let us note that the identities (22)-(23), valid for arbitrary integers $j \geq 0$ and $n \geq 2$, generalize the identities (17) which again follow from (22)-(23) for $j = n-1$ and $n \geq 2$.

From (19) we obtain the table

$$\begin{aligned} 2F(0,4,3,t) &= 2, \\ 3F(-1,5,3,t) &= 3-5t, \\ 4F(-2,6,3,t) &= 2(2-8t+7t^2), \\ 5F(-3,7,3,t) &= 5-35t+70t^2-42t^3, \\ 6F(-4,8,3,t) &= 2(3-32t+108t^2-144t^3+66t^4), \\ 7F(-5,9,3,t) &= 7-105t+525t^2-1155t^3+1155t^4-429t^5, \end{aligned} \tag{24}$$

....The table (24) determines the coefficients of (18) (or the same ones of (6)) for $n = 2,3,4,5,6,7,\ldots.$

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