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On the Spectral Properties of Some Classes of Two Parametric Operator Pencils

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1. The role of waveguides in modern technology and physics is well know. From the point of view of physical application, great interest attaches to acoustic, electromagnetic, elastic and other waveguides. Various wagveguiding systems are described by various dynamical equations, but waves have a series of general peculiarities, which admit a unique mathematical description. To this class of wavequiding system corresponds the dynamical equation of the form

$$\mathcal{L}V = V_{tt} - CV_{xx} + iBV_x + AV = f, \tag{1.1}$$

where A, B and C are symmetric, generally speaking, unbounded operators in a Hilbert space H. $V(x, t): R^1 \times R^1 \to H$ is a smooth function which describes the state of system. We consider the solution of equation $\mathcal{L}V = 0$ of the form $V(x, t) = ue^{i(wt-kx)}$, where $u \in H$ is an amplitude, w is frequency and k is a wave number. By substituting $V(x, t)$ in equation

$$V_{tt} - CV_{xx} + iBV_x + AV = 0$$

we obtain

$$\mathcal{L}(k, w)u \equiv (k^2 + kB + A - w^2)u = 0, \tag{1.2}$$

This equation shows the relation between k, w and u. This is two parametric nonlinear spectral problem. The functions $k(w)$ and $w(k)$, $k, w \in R^1$ are called dispersing curves. The solution of equation (1.1), which will be defined below, is understood in a generalized sense. Note that a wide class of regular waveguiding systems is given by the equation (1.1), where coefficients A, B and C, satisfying the following conditions, are operators in H:

1^0) $A = A^*$ is nonnegative operator i.e $(Au, u) \geq 0$, for all $u \in D(A)$ and A has a discrete spectrum

$$(v_n^2)\ , 0 \leq v^0 \leq v^1 \leq \ldots, \text{ so that } (A+I)^{-1} \in S_\infty.$$

2^0) C is bounded and positive definite operator, i.e there exists numbers c_- and c_+,satisfying

$$c_-^2(u, u) \leq (Cu, u) \leq C_+(u, u)$$

3^0) $(A+I)^{-\frac{1}{2}} B (A+I)^{-\frac{1}{2}} \in S_\infty$, where

$(A+I)^{-\frac{1}{2}} = \int_R^1 (\lambda+1)^{-\frac{1}{2}} dE_A(\lambda)$, $E_A(\lambda)$ is a spectral measure of operator A.

4^0) There exists a number $\mu \geq 0$ satisfying

$$(Au, u) + k(Bu, u) + k^2(Cu, u) \geq \mu^2(u, u),$$

for $k \in R^1, u \in D(A+I)^{\frac{1}{2}}$

S_∞ is a set of compact operators.

R e m a r k. Under condition $1^0 - 4^0$ quadratic forms of operators A, B and C are defined on $D(A+I)^{\frac{1}{2}}$- energetic space of operator A. A generalised solution of equation (1.2) is understood in the following sense:

$$(Au, \eta) + (Bu, \eta)k + (Cu, \eta)k^2 - w^2(u, \eta) = 0, \text{for any} u, \eta \in D(A+I)^{\frac{1}{2}}.$$

1.1. E x a m p l e. Let $G = (\overline{x}, \mid x_1 \mid < \infty, x_1 = (x_2, x_3) \in \Omega)$ be a three dimensional cylinder. Consider the equation

$$\frac{d^2V}{dt^2} - \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \frac{d}{dx_\alpha}\left(a_{\alpha\beta} \frac{dV}{dx_\beta}\right) = 0, \tag{1.3}$$

where $a_{\alpha\beta}\ inL_\infty(\Omega)$. Let there exist numbers $\sigma_+ \geq \sigma_- \geq 0$ such that for any $\xi\ inG^3$ in Ω the uniform hyperbolicity condition, i.e

$$q_- \mid \xi_\alpha \mid^2 \leq a_{\alpha\beta} \leq q_+ \mid \xi_\alpha \mid^2, \tag{1.4}$$

is satisfied. We note that the indices α and β imply the sum from 1 to 3. Three forms of boundary condition are considered:

$$I)\ V(t, x_1, x_2, x_3) \mid_s = 0,\ S = d\Omega \times R^1,\ t \in R^1$$

$$II)\, v^{\alpha}.\alpha_{\alpha\beta}\frac{dV}{dx_{\beta}} \mid S = 0,\ t \in R^1$$

$$III)\, (v^{\alpha}a_{\alpha\beta}\frac{dV}{dX_{\beta}} + hV) \mid_s = 0,\ t \in R^1$$

Theorem 1.2. *The equation (1.3) with boundary condition I)-III) can be reduced to equation (1.1) with coefficients A, B and C satisfying the condition* $1^0) - 4^0)$.

Proof. For definiteness we consider the I) - boundary condition. The remaining cases are considered analogously (see [3], [4]). Rewrite equation (1.3) in the following form:

$$\frac{d^2V}{dt^2} - a_{11}\frac{d^2V}{dx_1^2} - \sum_{m=2}^{3}[a_{1m}\frac{d^2V}{dx_1 dx_m} + \frac{d}{dx_m}(a_{m1}\frac{dV}{dx_1})] -$$

$$- \sum_{m,n=2}^{3} \frac{d}{dx_m}(a_{mn}\frac{dV}{dx_n}) = 0 \tag{1.5}$$

Consider the function $V(t, x_1, x_2, x_3) : R^1 \times R^1 \to L_2(\Omega)$ where t and x_1 are fixed. By comparing (1.5) with equation (1.1) we obtain ,

$$Au = -\frac{d}{dx_m}(a_{mn}\frac{du}{dx_n});\ Bu = i[a_{1m}\frac{du}{dx_m} + \frac{d}{dx_m}(a_{m1})];\ Cu = a_{11}u \text{ where}$$

$$u = u(x_2, x_3) \in C_0^2(\Omega) = (u,\ u \in C^2(\Omega) \frown C(\overline{\Omega}), u \mid_{\Gamma} = 0)$$

The generalized approach gives

$$\frac{d^2}{dt^2}\int_{\Omega} V\overline{\eta}dx_2dx_3 - \frac{d^2}{dx_1^2}\int_{\Omega} a_{11}V\overline{\eta}dx_2dx_3 -$$

$$-\frac{d}{dx_1}\int_{\Omega}[a_{1m}\frac{dV}{dx_m}\overline{\eta} - (a_{m1}V)\frac{\overline{d\eta}}{dx_m}]dx_2dx_3 +$$

$$+\int_{\Omega} a_m n\frac{dV}{dx_m}.\frac{d\overline{\eta}}{dx_n}dx_2dx_3 - \int_{\Gamma} a_{m\alpha}\frac{dV}{dx_{\alpha}}v^{\alpha}\overline{\eta}ds = 0$$

Now define the operators A, B and C by the use of bilinear forms: where v is vector of normal to Γ and

$$V(x, t) \in W_2^1, x_1,\ t \in R^1, \eta \in W_2^1(\Omega).$$

$$(Au,\eta)=\int_\Omega a_{mn}\frac{du}{dx_m}\cdot\frac{\overline{d\eta}}{dx_n}dx_2dx_3,(Bu,\eta)=\int_\omega a_{1m}[i\frac{du}{dx_m}\overline{\eta}+u(\frac{\overline{id\eta}}{dx_m})]dx_2dx_3,$$

$$(Cu,u)=\int_\Omega a_{11}u\overline{\eta}dx_2dx_2\cdot(Au,u)\approx\|u\|_{w_2^1},u\in W_2^1(\Omega).$$

Therefore the form (Au,u) is closed. Then it is known (see [2]) that to the form (Au,u) corresponds a selfadjoint operator A and $H_A = D(A^{\frac{1}{2}})$. Thus,in the given case $D(A^{\frac{1}{2}}) = D((A+I)^{\frac{1}{2}})$ and the condition $A = A^* \geq 0$ is fulfilled. The discreteness of spectrum of A follows from Rellich's theorem, i.e $W_2^1(\Omega) \in L_2(\Omega)$ is a compact imbedding. Hence $(A+I)^{-1} \in S_\infty$. Condition 2^0) is the consequence of (1.4)- the uniform hyperbolicity condition.

For fulfillment of 3^0 it is necessary and sufficient that ([5],[8]) the following inequality must be satisfied

$$|(Bu,u)|\leq C(|u|_{H_A}\cdot\|\eta\|_H+|\eta|_{H_A}\cdot\|u\|_H),\ \|u\|_{\overline{H}}$$

is an ordinary norm in H and $|u|_{H_A} = (Au,u)^{\frac{1}{2}}$. Now consider the condition 4^0. In physics it refers to the condition of non-negativity of energy. But mathematically, in given case, it is the condition of inform hyperbolicity. Indeed, from (1.4) it follows that

$\int_\Omega a_{\alpha\beta}\frac{dV}{dx_\alpha}\cdot\frac{d\overline{V}}{dx_\beta}dx_2dx_3 \geq q_- \int_\Omega |\frac{dV}{dx_\alpha}|^2 dx_2dx_3.$

Take $V = e^{ikx_1}$. Then

$$\int_\Omega a_{\alpha\beta}\frac{d}{dx_\alpha}(ue^{ikx_1}).\frac{d}{dx_\beta}(\overline{u}.e^{-ikx_1})dx_2dx_3=(A(k)u,u)\geq$$

$$\geq q_-\int_\Omega|\frac{d^2u}{dx_2^2}|^2$$

$$+|\frac{d^2u}{dx_3^2}|^2)dx_2dx_3+k^2\int_\Omega|u|^2dx_2dx_3\geq(q_-\lambda_1+k^2)\int_\Omega|u|^2dx_2dx_3$$

The least constant among $q_-\lambda_1 + k^2$, for all $k \in R^1$ will be $\mu = q_-\lambda_1$, where λ_1 is the first eigenvalue of operator

$$\Delta u=\frac{d^2u}{dx_2^2}+\frac{d^2u}{dx_3^2}.$$

Thus, condition 4^0) is fulfilled with $\mu = q_-\lambda^1 > 0$. ∎

1.3. A problem on oscillation of countable number of noninteracting strings.

The following equation of this system can be reduced to the form (1.1).

$$\frac{d^2\Phi_n}{dt^2} - \gamma_n^2 \frac{d^2\Phi_n}{dx^2} + 2i\beta_n \frac{d\Phi_n}{dx} + v_n^2 \Phi_n = 0$$

We shall study this problem in the frame of abstract mathematical model. Now consider a problem of spectral structure of two parameter pencils (1.2). For this purpose we introduce the following definition: Pair w and k is called a spectral if there exists a vector $u \neq 0$ for which $\mathcal{L}(w, k)\, u = 0$ i.e $Au + Bu + k^2 Cu - w^2 u = 0$. A set of such pairs is denoted by M. Let $M_1(w) = \{k : (k, w) \in M\}$ be a set of wave numbers and $M_2 = \{w : (k, w) \in M\}$ a set of eigenfrequencies.

1.4. Theorem A spectral set M is defined by the following inequalities

$$c_+^2 (Imk)^2 + (Rew)^2 - (Imw)^2 - \mu^2 \geq 0, \tag{1.5}$$

$$c_+^2 (Imk)^2 [c_+^2 (Imk)^2 + (Rew)^2 - (Imw)^2 - \mu^2] \geq (Rew)^2 . (Imw)^2 \tag{1.6}$$

Proof. Condition 4^0) is equivalent to the following inequality $I(Bu, u)I \leq 2(A_\mu u, u)^{\frac{1}{2}} . (Cu, u)^{\frac{1}{2}} where A_\mu = A - \mu^2 I, u \in D(A + I)^{\frac{1}{2}} = H_A$. If u is replaced by $\varphi + \psi$, where $\varphi \in H_A$, $\psi \in H_A$, then we obtain

$$\mid (B(\varphi + \psi), \varphi + \psi) \mid \leq (A_\mu(\varphi + \psi), \varphi + \psi)^{\frac{1}{2}} . (C(\varphi + \psi), \varphi + \psi)^{\frac{1}{2}} \tag{1.7}$$

$$\mid (B(\varphi + \psi), \varphi + \psi) \mid = (B\varphi, \psi) + 2Re(B\varphi, \psi) + (B\psi, \psi)$$

$$\mid (B\varphi, \varphi) + 2Re(B\varphi, \psi) + (B\psi, \psi) \mid \leq [(A_\mu\varphi, \varphi)^{\frac{1}{2}} + (A_\mu\psi, \psi)^{\frac{1}{2}}].$$

$$.[(C\varphi, \varphi)^{\frac{1}{2}} + (C\psi, \psi)^{\frac{1}{2}}] \tag{1.8}$$

Now from (1.8) and $\mid (Bu, u) \mid \leq (A_\mu u, u)^{\frac{1}{2}} . (Cu, u)^{\frac{1}{2}}$
it follows that

$$\mid Re(B\varphi, \varphi) \mid \leq 2[(A_\mu\varphi, \varphi)^{\frac{1}{2}} . (C\varphi, \varphi)^{\frac{1}{2}} + (A_\mu\psi, \psi)^{\frac{1}{2}} . (C\psi, \psi)^{\frac{1}{2}}] +$$

$$+(A_\mu\varphi,\varphi)(A_\mu\psi,\psi)^{\frac{1}{2}}.(C\varphi,\varphi)^{\frac{1}{2}}. \tag{1.9}$$

Replacing in (1.8) ψ by $i\psi$ we arrive to the inequality (1.9) for $|\ Im(B\varphi,\varphi)\ |$. Similarly, replacing φ and ψ by $\overline{(B\varphi,\ \psi)}^{\frac{1}{2}}$ and ψ respectively, we obtain (1.9) for $|\ (B\varphi,\ \psi)\ |$.

Now in condition 4^0 we substitute k to $k+k^1$. Then

$$(Au,\ u)+(k+k^1)(Bu,\ u)+(k+k^1)^2.(Cu,\ u)\geq\mu^2(u,\ u)$$

Hence after regrouping

$(A_\mu(k)u,\ u)+k^1(A(k)u,\ u)+k^{1^2}(Cu,\ u)\geq 0, u\in R^1$, where

$A_\mu(k)=A+kB+k^2C-\mu^2I$, $A'(k)=2kC+B$. This inequality in its turn is equivalent to

$$|\ (A'(k)u,\ u)\ |\leq 2(A_\mu(k)u,\ u)^{\frac{1}{2}}.(Cu,\ u)^{\frac{1}{2}}.$$

Consequently, we again obtain the condition of type 4^0) and the inequality equivalent to it. But in given case $B=A'(k)$ and $A_\mu=A_\mu(k)$. Then, by repeating the same arguments we obtain:

$$|\ Re(A'(k)\varphi,\ \varphi)\ |,|\ Im(A'(k)\varphi,\ \psi)\ |,|\ (A'(k)\varphi,\ \psi)\leq 2[(A_\mu(k)\varphi,\ \varphi)^{\frac{1}{2}}+$$

$$(A_\mu(k)\psi,\ \psi)^{\frac{1}{2}}]+(A_\mu(k)\varphi,\ \varphi)^{\frac{1}{2}}.(C\psi,\ \psi)^{\frac{1}{2}}+(A_\mu(k)\psi,\ \psi)^{\frac{1}{2}}.(C\varphi,\ \varphi)^{\frac{1}{2}}.$$

For the proof of (1.6) we need the inequality (1.10). Scalar multiplying (1.2) by u we obtain

$$(Au,\ u)+k(Bu,\ u)+k^2(Cu,\ u)=w^2(u,\ u).$$

Expansion to real and imaginary parts gives the following equalities:

$$(Au,\ u)+(Rek)(Bu,\ u)+[(Rek)^2-(Imk)^2].(Cu,\ u)-[(Rew)^2-(Imw)^2](u,\ u)=0,$$

$$(Imk)[(Bu,\ u)+2(Rek)(Cu,\ u)]-2(Rew)(Imw)(u,\ u)=0,$$

or the same

$$(A(Rek)u,\ u)-(Imk)^2(Cu,\ u)-[(Rew)^2-(Imw)^2](u,\ u)=0, \tag{1.11}$$

where

$$A(k)=A+kB+k^2C,$$

(1.12). $(Imk)(A'(Rek)u, u) - 2(Rew)(Imw)(u, u) = 0.$

Now (1.11)and condition 4^0 give (1.5). And (1.12) whit application of (1.10) and (1.11) directly give (1.6). Thus the theorem is proved. ■

1.5. **Corollary.** *If $Imk \neq 0$, then $Imw \neq 0$ besides $Imk \geq h^2(w)0.$ If $Imw = 0$ then $(Imk)^2 \geq \frac{\mu^2 - w^2}{c_+^2}$*

R e m a r k. From inequality $(Imk)^2 \geq \frac{\mu^2 - w^2}{C_2}$ it follows that if $w^2 < \mu^2$ then $Imk > 0$ i.e k is pure complex. Indeed, it is the corollary of condition 4^0). If $\mu^2 \leq w^2$ then $Imk \geq 0$. This means that a real wave number be only in given case.

1.6. **Corollary.** *From inequality (1.5) and (1.6) it follows that if $Imk = 0$, then $Imw = 0$. Besides $(Rew)^2 \geq \mu^2$.*

The following result is derived from the operator theory ([3], [8], [2]).

1.7. **Theorem.** *For any $k, w \in G$ the sets $M_1(w)$ and $M_2(k)$ are discrete spectrum i.e the sets $M_1(w)$ and $M_2(k)$ are infinite sequences of eigenvalues with a unique limit point at infinity.*

In the process of proving theorem 1.7 the method of linearization of pencil $A + kB + K^2C - w^2I$ and a well known method of perturbation of spectra are used.

If $A = A^*$ with a discrete spectrum, then for any small $\epsilon > 0$ spectrum of operator $A + L$, except the finite number, belongs to corners

$$\epsilon < arg\lambda < \epsilon, \ \pi - \epsilon < arg\lambda < \pi + \epsilon,$$

where $A^{\{-\frac{1}{2}\}}LA^{\{-\frac{1}{2}\}}$ is a completely continuous operator, i.e $A^{\{-\frac{1}{2}\}}LA^{\{-\frac{1}{2}\}} \in S_\infty$. Indeed, virtue of condition 3^0)$kB + k^2C$ is A - completely continuous operator.

In physical application great interest attaches to the functions $ue^{(wt-kx)}$ where k and w are real. From physical arguments it also follows that a set of running waves in given frequency $|w| \geq |\mu|$ must be finite. However, in frames of condition $1^0 - 4^0$ a dynamical equation (1.1), generally speaking, has an infinite quantity of running waves. Such an example will be illustrated below. On the other hand, there is the following condition guaranteeing the finiteness of running waves in given frequency:

$|(Bu, u)| \leq 2\epsilon(A_\mu u, u)^{\frac{1}{2}}.(Cu, u)^{\frac{1}{2}}, 0 < \epsilon < 1$, i.e condition 4^0 is fulfilled if we substitute C to ϵC.

(1.13) $(Au, u) + k(Bu, u) + \epsilon^2k^2(Cu, u) \geq \mu^2(u, u)$

This is called the energetic stability condition, which is valid for the majority of real physical problems.

2. The structure of real spectrum

In previous item it was noted that at fulfillment of condition,
$|(Bu, u)| \leq 2\epsilon(A_\mu u, u)^{\frac{1}{2}}.(Cu, u), 0 < \epsilon < 1$, a set of running waves on given frequency w is finite, i.e among $M_1(w)$ there is a finite number of real points of spectrum. Introduce the following functionals:

$$p(u, w) = \frac{-(Bu, u) + \sqrt{(Bu, u)^2 - 4(A_w u, u).(Cu, u)}}{2(Cu, u)}$$

Let $d(u, w) = (Bu, u)^2 - 4(A_w u, u).(Cu, u)$, $G(w) = \{u, d(u, w) > 0\}$,

$$G'(w) = \{u, d(u, w) \geq 0\}$$

The sets $G(w)$ and $G'(w)$ are cones in H. In [1] the following numbers are defined $k'_-(w) = \min p_-(x)$ on $G'(w), k'_+(w) = \max p_+$, on

$$G'(w), k_-(w) = \min P_-(x), x \in G(w), k_+ = \max p_+(x), x \in G(w),$$

$$\delta_-(w) = \min p_+(x),\ x \in G(w),\ \delta_+ = \max p_+(x), x \in G(w).$$

It is obvious that $k'_-(w) \leq k_-(w) \leq \delta_- \leq \delta_+ \leq k_+ \leq k'(w)$. Besides, all real spectrum on given frequency w belongs to segment $[k_-(w), k_+(w)]$. Corresponding to partition $[k'_-, k'_+]$ the spectrum σ_r is divided in intervals: $[k', k), [k_-, \delta), [\delta_-, \delta_+], [\delta_+, K_+], (k_+, k_+]$ and are denoted correspondingly by $\sigma'_-, \sigma_-, \sigma_0, \sigma_+, \sigma'_+$. Let w be fixed. Consider $\mathcal{L}(w, k)$ as one parametric pencil. We say that a pair k and u is a pair of the first (or second) genus if $\mathcal{L}(w, k) = 0$ and $(\mathcal{L}'(w, k)u, u) > 0$ $(\mathcal{L}'(w, k)u, u) < 0)$, where $\mathcal{L}'(w, k) = 2kC + B$. If $(\mathcal{L}'(w, k)u, u) = 0$ Then it is called neutral. The wave number is called of the first (or second) genus if for any vector $u \in Ker(\mathcal{L}(w, k))$ the pair k and u is a pair of the first (or second) genus. Neutral wave number is defined analogously.

In [1] the following theorem is proved.

2.1. **Theorem.** *a) $\sigma_+(w)$ consists of wave number of the first genus and σ_- of the second genus. b) σ'_- and σ'_+ consist of neutral wave numbers, whose eigenvectors have adjoin vectors.*

Now consider the structure of $\sigma_R(w)$ and $w(k)$ for a concrete example in frames of abstract mathematical model.

A problem on oscillation of countable number of noninteresting strings. Define the coefficients in equation $V_{tt} - CV_{xx} + iBV_x + AV = 0$ as

$$A\varphi_n = v_n^2,\ B\varphi_n = 2\beta_n\varphi_n,\ C\varphi_n = \gamma_n^2\varphi_n$$

where $\{\varphi\}_n$ is an orthonormal basis. For the fulfillment of condition $1^0) - 4^0)$ the following are sufficient:

$$1)\ \lim_{n\to\infty} v_n^2 = \infty\ ,$$

$$2)\ 0 < C_- = \inf m\gamma_n^2, \sup m\gamma_n^2 = C_+ < \infty\ ,$$

$$3)\ \lim_{n\to\infty} \frac{\beta_n}{v^2+1} = 0\ ,$$

$$4)\ v_n^2 + 2\beta_n k + \gamma_n^2 k^2 \geq \mu^2,\ \mu \geq 0\ .$$

Condition $1^0, 2^0$ and 4^0 are obvious. Let's explain only condition 3^0. Indeed the spectrum of operator $(A+B)^{-\frac{1}{2}}\ B(A+I)^{-\frac{1}{2}}$ is discrete and consist of number $\frac{2\beta_n}{v_n^2+1}$. Then it is known that ([2]) condition

$(A+I)^{-\frac{1}{2}}B(A+I)^{-\frac{1}{2}} \in S_\infty$ is equivalent to $\frac{2\beta_n}{v_n^2+1} \to 0$ for $n \to \infty$.

In our case, pencil $\mathcal{L}(w, k) = A + kB + k^2C - w^2I$ in basis $(\varphi_n)_0{}^\infty$ is given by diagonal matrix, where the elements $\gamma_n^2k^2 + 2\beta_{nk} + v_n^2 - w^2$ are on diagonals. Let φ is an eigenvector. Then $\varphi = \sum_{n=0}^\infty C_n\varphi_n$ and

$$(A + kB + k^2C - w^2I)\varphi = 0.$$

Hence

$$(A + kB + k^2C - w^2)\varphi = \sum_{n=0}^{\infty} C_n(A + kB + k^2C - w^2I)\varphi_n =$$

$$\sum_{n=0}^{\infty}(v_n^2 + 2\beta_{nk} + \gamma_n^2k^2 - w^2)C_n\varphi_n = 0$$

Consequently,

$C_n\,(v_n^2 + 2\beta_n k + k^2\gamma_n^2 - w^2) = 0$. Since $\varphi \not\equiv 0$ then $(k, w) \in M$ if $v_n^2 + 2\beta_n k + k^2\gamma_n^2 - w^2 = 0$. From this equality expressing w by k we obtain a dispersing curves equation:

$$w_n(k) = \pm\sqrt{\gamma_n^2k^2 + 2\beta_{nk} + v_n^2}. \tag{2.1}$$

$\pm w_n(k)$ are symmetric. For this reason we shall investigate only

$$w_n(k) = \sqrt{\gamma_n^2k^2 + 2\beta_n k + v_n^2}. \tag{2.2}$$

$w_n(k) = \sqrt{(\gamma_n k + \frac{\beta_n}{\gamma_n})^2 + v_n^2 - \frac{\beta_n^2}{\gamma_n^2}}$. Hence we obtain the coordinates of vertex of parabola: $(-\frac{\beta_n}{\gamma_n^2}, v_n^2 - \frac{\beta_n^2}{\gamma_n^2})$.

Note that under fulfillment of condition $1^0 - 4^0$ $\mathcal{L}(w,k)$ has an infinite number of real wave numbers. We shall show it in the given example. Put $\gamma_n^2 = 1$, $\beta_n^2 = v_n^2 - \alpha$. Then the vertices of parabolas will be $(-\beta_n, \alpha)$. Therefore, the straight line $w = w_0 > \alpha$ intersects the dispersing curves at an infinite number of points. If condition (1.13) is satisfied i.e

$|\beta_n| \leq \varepsilon \gamma_n \sqrt{v_n^2 - \mu_n^2}$, $n = 0, 1, 2, \ldots$ then finiteness of $\sigma_r(w)$ is proved.

We consider only this case. First of all note that if we introduce the function

$V_n(x,t) = (V(x,t), \varphi_n)$ then $V_n(x,t)$ satisfies the following equation

$$\frac{d^2V_n}{dt^2} - \gamma_n^2 \frac{d^2V_n}{dx^2} + 2i\beta_n \frac{dV_n}{dx} + v_n^2 V_n = 0 \tag{2.3}$$

Equation (2.3) is obtained by scalar multiplying the equation

$V_{tt} - CV_{xx} + iBV_x + AV = 0$ with φ_n. And (2.3) is a generalized equation of oscillation of countable number of nonintersecting strings.

The following theorem shows that the structure of $\sigma_R(w)$ is completely defined by interarrangement of dispersing curves.

2.2. **Theorem.** 1) $k_0 \in \sigma_+(w_0)$ $(\sigma_-(w_0))$ *if and only if at point* k_0 *all functions* $w_n(k)$ *for which* $w_n(k_0) = w_0$ *has a derivative*

$$w'_n(k_0) > 0 \; (w'_n(k_0) < 0)$$

2)$k_0 \in \sigma'_+(w_0)$ $(\sigma'_-(w_0))$ *if and only if from the condition* $w_n(k_0) = 0$ *it follows that* (k_0, w_0) *is critical point of curve* $w_n(k)$ *and for any point* $k \in A = \{\lambda; w_n(\lambda) = w_0, w'_n(\lambda) = \odot\}$ *is valid* $k < k_0$ $(k > k_0)$

$$3) \dim \ker L(w_0, k_0) = \sum_{n_i} 1, \; w_{n_i}(k_0) = w_0$$

P r o o f. Consider a set of dispersing curves

$$w_n(k) = \sqrt{\gamma_n^2 + 2\beta_n k + v_n^2}, n = 0, 1, 2, 3, \ldots$$

The curve $w_n(k)$ passes through a point $(0, v_n)$ and to every point of this curve corresponds an eigenvector φ_n. If through point (k_0, w_0) pass curves $w_{n_1}(k), w_{n_2}(k), \ldots, w_{n_s}(k)$ then to the pair (k_0, w_0) corresponds $\varphi_{n_1}, \varphi_{n_2}, \ldots \varphi_{n_s}$ and there are no other eigenvectors. indeed, if φ corresponds to (k_0, w_0) then

$$k_0^2 C\varphi + k_0 B\varphi + A\varphi_{w_0^2} = 0, \; \varphi = \sum_{n=0}^{\infty} C_n \varphi_n$$

Hence,

$$\sum_{n=0}^{\infty} C_n(\gamma_n^2 + 2\beta_n k_0 + v_n^2 - w_0^2)\varphi_n = 0 \text{ or}$$

$$c_n((\gamma_n^2 + 2\beta_n k_0 + v_n^2 - w_0^2) = 0, \; n = 0, 1, 2 \ldots$$

Consequently, $c_n = 0$ only for $n = n_1, n_2, \ldots, n_s$ i.e. $\varphi = C_1\varphi_{n_1} + C_2\varphi_{n_2} + \ldots + C_s\varphi_{n_s}$. Peculiarity 3) is proved.

Now if $k_0 \in \sigma_+(w_0)$ then

$$(\mathcal{L}'(k_0, w_0)\, u, u) > 0, u \in ker\mathcal{L}(w_0, k_0)\, . \,(\mathcal{L}'(w_0, k_0) = (\,(2k_0C + B)\, u, u).$$

Let (w_0, k_0) be on curves $W_{n_1}, W_{n_2}, \ldots, W_{n_s}$. Then $Ker\mathcal{L}(w_0, k_0) = lin(\varphi_{n_i})_1^2$ is a linear combination of $\varphi_{n_1}, \ldots, \varphi_{n_s}$.

Therefore

$$(\mathcal{L}'(k_0, w_0)\, u, u) > 0 \Leftrightarrow (\mathcal{L}'(k_0, w_0)\varphi_{n_i}, \varphi_{n_1}) > 0 \Leftrightarrow w_{n_i}(k_0) > 0, \; i = 1, 2, \ldots$$

The property 2^0 follows from theorem 2.1. But for application of this theorem we are to prove that if $k_0 \in \sigma'_\pm$ then every eigenvector has an adjoint vector.

Indeed, the neutrality of k_0 follows from $w'_n(k_0) = 0$, $w_n(k_0) = w_0$. Let the eigenvector u_0 has an adjoint vector u_1. Then

$$\mathcal{L}(w_0, k_0)\, u_0 = 0\,,$$

$$\mathcal{L}'(w_0, k_0) \; + \mathcal{L}(w_0, k_0)\, u_1 = 0\,,. \text{ Hence}$$

$$\mathcal{L}(w_0, k_0)\, u_1 = \mathcal{L}'(w_0, k_0)\, u_0 \tag{2.4}$$

For solvability of (2.4) there must be $\mathcal{L}'(k_0, w_0)\, u_0 \in R(\mathcal{L}(k_0, w_0)$.

Thus $H = Ker\mathcal{L}(w_0, k_0) \oplus R(\mathcal{L}(w_0, k_0))$. Then for solvability of problem (2.4) we obtain the following condition:

$$(\mathcal{L}'(k_0, w_0)\, u_0, u^*) = 0, \; u^* \in Ker(\mathcal{L}(k_0, w_0)\,) \tag{2.5}$$

Let $Ker(\mathcal{L}(k_0, w_0)\,) = Lin(\varphi_{n_1}, \varphi_{n_2}, \ldots \varphi_{n_k})$ if $u^* = \varphi_{n_i}, i = \overline{1, \ldots, k}$ then

$$(\mathcal{L}'(k_0, w_0)u_0, u^*) = (\mathcal{L}'(k_0, w_0)u_0, \varphi_{n_i}) = (u_0, \mathcal{L}'(k_0, w_0)\varphi_{n_i}) =$$

$$= (u_0, (2k_0C + B)\varphi_{n_i}) = (u_0, (2k_0\gamma_{n_i} + 2\beta_{n_i})(u_0, \varphi_{n_i} = 0$$

Hence $w_{n_i}{}'(k_0) = 0$, $i = 1, 2, \ldots, k$

The theorem is proved. ∎

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