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On a Bibasic Transformation Formula

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In this paper, we derive a general transformation formula which involve two different bases and also discuss a few interesting special and limiting cases of this transformation, including two classes of identities of the Rogers - Ramanujan type.

1. Introduction

As is well-known, there is an innate relationship between the transformation theory of basic hypergeometric series and hypergeometric identities. The celebrated Rogers-Ramanujan identities are the most well-known illustration of this relationship in the unibasic case. The pioneering work of Andrews [3] and Agarwal and Verma ([1], [2]) on basic hypergeometric series involving two unrelated bases has provided the impetus for an extensive study of transformations connecting basic hypergeometric series with two different bases (see, for example, Verma and Jain ([12], [13], [14]), Gasper [6], Gasper and Rahman [7], and Rahman [9]). The relationship between transformations connecting basic hypergeometric series involving two different bases and identities of the Rogers-Ramanujan type has been studied in some detail by Verma and Jain ([12], [14]). Their work has led in turn to a whole new wide class of identities of the Rogers-Ramanujan **type** involving either moduli greater that five or double series or both.

In this paper, we shall first derive a general transformation formula involving two different bases, and then discuss a few interesting special cases, including two classes of identities of the Rogers-Ramanujan type.

2. Notation

For $|p| < 1$, let ([11];p.90)

$$(a;p)_0 = 1,\ (a;p)_n = (1-a)...(1-ap^{n-1}),\ n \geq 1;$$

$$((a_A);p)_n = (a_1;p)_n...(a_A;p_n);(a;p)_\infty = \prod[a;p] = \prod_{n=1}^{\infty}(1-ap^{n-1}).$$

A generalized basic hypergeometric series is defined ([8]; p.4) by

$$ {}_A\phi_B\left[\begin{matrix}(a_A); & p; & z\\ (b_B) & & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{((a_A);p)_n z^n}{(p;p)_n((b_B);p)_n}\{(-1)^n p^{\binom{n}{2}}\}^{1+B-A}$$

with $\binom{n}{2} = n(n-1)/2$, where $p \neq 0$ when $A > B+1$. Further, let

$$E_n = \frac{((\alpha_p);q)_n((\beta_R);\ q^2)_n}{((\gamma_Q);q)_n((\delta_S);\ q^2)_n},$$

and

$$F_n(a,b) = \frac{(a,b;q)_n q^{\binom{n+1}{2}}}{(q;q)_n(abq;\ q^2)_n}.$$

3. We shall prove the following general bibasic transformation formula:

$$\sum_{j=0}^{\infty}F_j(a,b)\sum_{n=0}^{[j/2]}\frac{(q^{-j};q)_{2n}}{(q;q)_n}q^{\lambda\binom{n+1}{2}}E_n\theta_n \tag{1}$$

$$= \frac{(-q;q)_\infty(aq,bq;q^2)_\infty}{(abq;q^1)_\infty}\sum_{n=0}^{\infty}\frac{(a,b;q^2)_n}{(q;q)_n}q^{\lambda\binom{n+1}{2}}E_n\theta_{n'}$$

where θ_n is any sequence of numbers, real or complex.

P r o o f. In view of the q-analog of Gauss's second summation theorem ([4]; eq. (1.8)), we have

$$\sum_{r=0}^{\infty} F_r(aq^{2n}, bq^{2n}) = \prod[-q;q] \prod \left[\begin{matrix} aq^{2n+1}, & bq^{2n+1}; q^2 \\ abq^{4n+1} & \end{matrix} \right]. \tag{2}$$

Multiplying both sides of (2) by $E_n \theta_n q^{\lambda \binom{n+1}{2}}$ and summing from $n = 0$ to ∞, we get

$$\sum_{n=0}^{\infty} E_n \theta_n q^{\lambda \binom{n+1}{2}} \sum_{r=0}^{\infty} F_r(aq^{2n}, bq^{2n})$$

$$= \prod[-q;q] \sum_{n=0}^{\infty} E_n \theta_n q^{\lambda \binom{n+1}{2}} \prod \left[\begin{matrix} aq^{2n+1}, & bq^{2n+1}; q^2 \\ abq^{4n+1} & \end{matrix} \right].$$

On replacing r by $j - 2n$ here and changing the order of summation, we get the following transformation:

$$\sum_{j=0}^{\infty} F_j(a,b) \sum_{n=0}^{[j/2]} \frac{(q^{-j};q)_{2n}(abq;q^2)_{2n}}{(a,b;q)_{2n}} q^{\lambda \binom{n+1}{2}} E_n \theta_n \tag{3}$$

$$= \prod[-q;q] \sum_{n=0}^{\infty} E_n \theta_n q^{\lambda \binom{n+1}{2}} \prod \left[\begin{matrix} aq^{2n+1}, & bq^{2n+1}; q^2 \\ abq^{4n+1} & \end{matrix} \right].$$

Replacing R, S and Q now by $R + 4$, $S + 4$ and $Q + 1$, respectively, and suitably choosing the nine new additional parameters, we easily get (1) after some simplification.

We shall now discuss a few special cases of (1). We first deduce four summation theorems.

(i) Let us take $P = 0 = R$, $Q = 1 = S$, $\gamma_1 = -q$, $\delta_1 = c$, $\lambda = 0$ and $\theta_n = (c/ab)^n$ in (1). Then, summing the series on the right of the resulting transformation by the q-analogue of Gauss's theorem ([11]; eq. (3.3..2.5)), we get

$$\sum_{j=0}^{\infty} F_j(a,b)\, {}_2\phi_1 \left[\begin{matrix} q^{-j}, & q^{-j+1}; & q^2; & c/ab \\ c & & & \end{matrix} \right] \tag{4}$$

$$= \prod \begin{bmatrix} c/a, & c/b, & aq, & bq; & q^2 \\ c, & c/ab, & abq & & \end{bmatrix} \prod[-q;q].$$

(ii) Let us take $P = 0 = R$, $Q = 1$, $S = 2$, $\gamma_1 = -q$, $\delta_1 = -q^2$, $\delta_2 = c$, $\lambda = 2$, $\theta_n = (c/q^2)^n$ and $a = q^2/b$ in (1). Then, summing the resulting series on the right of (1) by the q-analogue of Bailey's theorem ([4]; eq. (1.9)), we get

$$\text{(5)} \qquad \sum_{j=0}^{\infty} F_j(q^2/b, b)\,{}_2\phi_2 \begin{bmatrix} q^{-j}, & q^{-j+1}; & q^2; & -c \\ -q^2, & c & & \end{bmatrix}$$

$$= \prod \begin{bmatrix} q\sqrt{c/b}, & -q\sqrt{c/b}, & \sqrt{bc}, & -\sqrt{bc}, & q^3/b, & bq; & q^2 \\ c, & q^3 & & & & & \end{bmatrix} \prod[-q;q].$$

(iii) If we take $P = 0 = R$, $Q = 1 = S$, $\gamma_1 = -q$, $\delta_1 = aq^2/b$, $\lambda = 0$ and $\theta_n = (-q^2/b)^n$ in (1) and sum the resulting series on the right of (1) by the q-analogue of Kummer's theorem ([4]; eq. (1.7)), we get

$$\text{(6)} \qquad \sum_{j=0}^{\infty} F_j(a,b)\,{}_2\phi_1 \begin{bmatrix} q^{-j}, & q^{-j+1}; & q^2; & -q^2/b \\ aq^2/b & & & \end{bmatrix}$$

$$= \prod \begin{bmatrix} q\sqrt{a}, & -q\sqrt{a}, & q^2\sqrt{a}/b, & -q^2\sqrt{a}/b, & -q^2, & aq, & bq; & q^2 \\ aq^2/b, & -q^2/b, & abq & & & & & \end{bmatrix} \prod[-q;q].$$

(iv) Lastly, let us take $P = 0$, $R = 2$, $Q = 1$ $S = 3$, $\beta_1 = \sqrt{c}$, $\beta_2 = -\sqrt{c}$, $\gamma_1 = q\sqrt{ab}$, $\delta_2 = -q\sqrt{ab}$, $\delta_3 = c$, $\lambda = 0$ and $\theta_n = q^{2n}$ in (1) and assume that b is of the form q^{-2N} (when N is a positive integer). Then, summing the series on the right of the resulting transformation by the q-analogue of Watson's theorem [5], we get the following summation theorem:

$$\text{(7)} \qquad \sum_{j=0}^{\infty} F_j(a,b)\,{}_4\phi_3 \begin{bmatrix} \sqrt{c}, & -\sqrt{c}, & q^{-j}, & q^{-j+1}; & q^2; & q^2 \\ q\sqrt{ab}, & -q\sqrt{ab}, & c & & & \end{bmatrix}$$

$$= a^{N/2} \prod \begin{bmatrix} aq^2, & bq^2, & cq^2/a, & cq^2/b; & q^4 \\ q^2, & abq^2, & cq^2, & cq^2/ab & \end{bmatrix} \prod \begin{bmatrix} aq, & bq; & q^2 \\ abq & & \end{bmatrix} \prod[-q;q].$$

4. Finally, we consider a particular case of (1) and deduce from it two classes of identities of the Rogers-Ramanujan type.

Let us take $P = 0$, $R = 4$, $Q = 1$, $S = 5$, $\beta_1 = q^2\sqrt{a}$, $\beta_2 = -q^2\sqrt{a}$, $\beta_3 = c$, $\beta_4 = d$, $\gamma_1 = -q$, $\delta_1 = \sqrt{a}$, $\delta_2 = -\sqrt{a}$, $\delta_3 = aq^2/b$, $\delta_4 = aq^2/c$,

$\delta_5 = aq^2/d$, $\theta_n = (a^2/bcd)^n$ and $\lambda = 2m$, $m = 1, 2, ..$ in (1). Then (1) becomes

$$(8)\quad \sum_{j=0}^{\infty} F_j(a,b) \sum_{n=0}^{[j/2]} \frac{(q^2\sqrt{a}, -q^2\sqrt{a}, c, d; q^2)_n (q^{-j}; q)_{2n} q^{2m\binom{n+1}{2}}}{(q^2, \sqrt{a}, -\sqrt{a}, aq^2/c, aq^2/d, aq^2/b; q^2)_n} (a^2/bcd)^n$$

$$= \frac{(-q;q)_\infty (aq, bq; q^2)_\infty}{(abq; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(a, q^2\sqrt{a}, -q^2\sqrt{a}, b, c, d; q^2)_n q^{2m\binom{n+1}{2}}}{(q^2, \sqrt{a}, -\sqrt{a}, aq^2/b, aq^2/c, aq^2/d; q^2)_n} \times$$
$$\times (a^2/bcd)^n.$$

If we now make $b \to 0$ and $c, d \to \infty$ in (8), we get the following transformation:

$$(9)\quad \sum_{j=0}^{\infty} F_j(a,0) \sum_{n=0}^{[j/2]} \frac{(1 - aq^{4n})(q^{-j}; q)_{2n}}{(q^2; q^2)_n} (-a)^n q^{(m+1)n^2 + (m-3)n}$$

$$= (-q;q)_\infty (aq; q^2)_\infty \sum_{n=0}^{\infty} \frac{(a;q^2)_n (1 - aq^{4n})}{(q^2;q^2)_n} (-a)^n q^{(m+1)n^2 + (m-3)n}.$$

For $a = q^2$, the series on the right of (9) can be seen to be transformable into a bilateral series which can be summed with the help of Jacobi's triple product identity ([11]; eq. (3.5.8)). We thus get the following identity, which is believed to be new:

$$(10)\quad \sum_{j=0}^{\infty} (1 - q^{j+1}) q^{\binom{j+1}{2}} \sum_{n=0}^{[j/2]} \frac{(1 - q^{4n+2})(q^{-j}; q)_{2n}}{(q^2; q^2)_n} (-1)^n q^{mn^2 + (m-2)n}$$

$$= \prod_{n=1}^{\infty} (1 - q^{2mn-2})(1 - q^{2mn-2m+2})(1 - q^{2mn})(1 + q^n)(1 - q^{2n-1}),$$

where $m = 2, 3....$

In particular, for $m = 3$, we have

$$(11)\quad \sum_{j=0}^{\infty} (1 - q^{j+1}) q^{\binom{j+1}{2}} \sum_{n=0}^{[j/2]} \frac{(1 - q^{4n+2})(q^{-j}; q)_{2n}}{(q^2; q^2)_n} (-1)^n q^{3n^2+n}$$

$$= \prod_{n=1}^{\infty}(1-q^{6n-2})(1-q^{6n-4})(1-q^{6n})(1+q^n)(1-q^{2n-1}).$$

It may be noted that if we compare (10) with the following generalized hypergeometric identity ([10]; Theorem 2) of the Schur-Gleissberg type (for $r=2$):

$$(12) \qquad \prod_{n=1}^{\infty}\frac{(1-q^{2mn-r})(1-q^{2mn-2m+r})(1-q^{2mn})}{(1-q^n)} =$$

$$= \left\{\prod[-q;q]\right\}^{-1}\sum_{t=0}^{\infty}\frac{B_{m,t}(r,q)}{(q;q)_t},$$

where $1 \leq r \leq m$ and $B_{m,t}(r,q)$ are certain polynomials ([10]; Theorem 2), we get the following interesting transformation:

$$(13) \quad \sum_{j=0}^{\infty}(1-q^{j+1})q^{\binom{j+1}{2}}\sum_{n=0}^{[j/2]}\frac{(1-q^{4n+2})(q^{-j};q)_{2n}}{(q^2;q^2)_n}(-1)^n q^{mn^2+(m-2)n}$$

$$= \prod_{n=1}^{\infty}(1-q^n)(1-q^{2n-1})\sum_{t=0}^{\infty}\frac{B_{m,t}(2,q)}{(q;q)_t},$$

where $m = 2,3,\ldots$.

Again, if we take $c = q$ in (8) and make $b \to 0$, $d \to \infty$, we get the following transformation:

$$(14) \qquad \sum_{j=0}^{\infty}F_j(a,0)\sum_{n=0}^{[j/2]}\frac{(1-aq^{4n})(q;q^2)_n(q^{-j};q)_{2n}}{(q^2,aq;q^2)_n}a^n q^{mn^2+(m-3)n}$$

$$= (-q;q)_\infty(aq;q^2)_\infty\sum_{n=0}^{\infty}\frac{(a;q^2)_n(1-aq^{4n})(q;q^2)_n}{(q^2;q^2)_n(aq;q^2)_n}a^n q^{mn^2+(m-3)n}.$$

Proceeding as in the case of (10) above, we now get the following identity, which is also believed to be new:

$$(15) \quad \sum_{j=0}^{\infty}(1-q^{j+1})q^{\binom{j+1}{2}}\sum_{n=0}^{[j/2]}\frac{(1+q^{2n+1})(q^{-j};q)_{2n}}{(q^2;q^2)_n}q^{mn^2+(m-1)n}$$

$$= \prod_{n=1}^{\infty}(1+q^{2mn-1})(1+q^{2mn-2m+1})(1-q^{2mn})(1+q^{n})(1-q^{2n-1}),$$

where $m = 1, 2, \ldots$.

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