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Separable Commuting with the Generalized Hardy-Littlewood Operator for Several Complex Variables

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Results of previous publications are generalized for several complex variables. Descriptions of the commutants of the generalized Hardy-Littlewood operator are made in the case of separable commuting. Different cases are considered: when the components of the operator preserve the powers, increase the powers, and a mixed variant - some components increase, while other preserve the powers.

The question about the separable commuting with the generalized Hardi-Littlewood operator for several complex variables was posed and partially considered in [1]. Later in [2] a possibility was shown to define the commutant in the multidimensional case when the joint commuting was considered. Descriptions of commutants of different operators in the multidimensional case are made by other mathematicions, too (see for example [4], [5], [6], where further refferences exist).

In this note the separable commuting with the generalized Hardi-Littlewood operator is considered in the case when the powers are not increased by the action of the operator, and a miscelaneous variant of separable commuting is given, too.

As in our previous papers let A_0 be the space of functions $y(z)$, $z = (z_1, z_2, \ldots, z_s) \in \mathbb{C}^s$, analytic in neighbourhoods of the origin. The standard multiindices $m = (m_1, \ldots, m_s)$, $n = (n_1, \ldots, n_s), k = (k_1, \ldots, k_s)$ and $\alpha := (\alpha_1, \ldots, \alpha_s)$, where $\alpha_l = n_l - m_l + 1$, $1 \leq l \leq s$, will be used as well as the multipowers $z^k = z_1^{k_1} \ldots z_s^{k_s}$.

Denote by $B_l = B_{m_l, n_l}$, $1 \leq l \leq s$, the operator

$$(1) \qquad B_l y(z_1, \ldots, z_l, \ldots, z_s) = \frac{1}{z_l^{m_l}} \int_0^z t_l^{n_l} y(z_1, \ldots, t_l, \ldots, z_s)\, dt_l,$$

where $n_l = 0, 1, 2, \ldots$ and $m_l = 0, 1, 2, \ldots, n_l + 1$. It is onedimensional generalized Hardy-Littlewood operator with respect to the variable z_l. For every variable z_l, $1 \le l \le s$, a fixed positive integer power p_l of the operator B_l can be considered. The commutant of $B_l^{p_l}$ will be denoted by $K_{B_l^{p_l}}$.

Fix an arbitrary l, $1 \le l \le s$, and consider the case when the action of the operator B_l preserves the powers, i.e. $\alpha_l := n_l - m_l + 1 = 0$ (α shows the change of the powers).

Let $L : A_0 \to A_0$ be a continuous linear operator which commutes with the operator $B_l^{p_l}$. Consider $Lz^k = L \prod_{l=1}^{s} z_1^{k_1}$ as a power series $\sum_{|i|=0}^{\infty} \lambda_{k,i} z^i$ with unknown coefficients $\lambda_{k,i} = \lambda_{k_1,\ldots,k_s,i_1,\ldots,i_s}$.

Theorem 1. *Let l be an integer, $1 \le l \le s$, $\alpha := n_l - m_l + 1 = 0$, and p_l be a positive integer. A continuous linear operator $L : A_0 \to A_0$ belongs to the commutant $K_{B_l^{p_l}}$ if and only if it has the form*

$$
\begin{aligned}
&Ly(z_1, \ldots, z_l, \ldots, z_s) = \\
&\sum_{k_1=0}^{\infty} \cdot\cdot \sum_{k_s=0}^{\infty} a_{k_1,\ldots,k_s} \sum_{i_1=0}^{\infty} \cdot\cdot \sum_{i_{l-1}=0}^{\infty} \times \\
&\times \sum_{i_{l+1}=0}^{\infty} \cdot\cdot \sum_{i_s=0}^{\infty} d_{k_1,\ldots,k_s,i_1,\ldots,i_{l-1},i_{l+1},\ldots,i_s} z^{i_1} \ldots z^{i_{l-1}} z^{k_l} z^{i_{l+1}} \ldots z^{i_s},
\end{aligned}
\tag{2}
$$

where $y(z) = \sum_{|k|=0}^{\infty} a_k z^k \in A_0$, α_l, p_l, n_l, m_l are the l-th components of the multiindices α, p, n, m and $d_{k_1,\ldots,k_s,i_1,\ldots,i_{l-1},i_{l+1},\ldots,i_s}$ are arbitrary complex numbers, such that the series in (2) are convergent in a neighbourhood of the origin.

Proof. Let

$$
Lz_1^{k_1} \ldots z_l^{k_l} \ldots z_s^{k_s} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_l=0}^{\infty} \cdot\cdot \sum_{i_s=0}^{\infty} \lambda_{k_1,\ldots,k_s,i_1,\ldots,i_l,\ldots,i_s} . z^{i_1} \ldots z^{i_l} \ldots z^{i_s}. \tag{3}
$$

Taking into account the action of the operators $B_l^{p_l}$ and L, and expressing $LB_l^{p_l} z^k = B_l^{p_l} Lz^k$ for an arbitrarily fixed multipower k, one compares the coefficients of the equal multipowers $i = (i_1, \ldots, i_l, \ldots, i_s)$ and gets the equalities

$$
\frac{\lambda_{k_1,\ldots,k_s,i_1,\ldots,i_l,\ldots,i_s}}{(n_l + k_l + 1)^{p_l}} = \frac{\lambda_{k_1,\ldots,k_s,i_1,\ldots,i_l,\ldots,i_s}}{(n_l + i_l + 1)^{p_l}}.
$$

From here it is clear that

$$
\lambda_{k_1,\ldots,k_s,i_1,\ldots,i_l,\ldots,i_s} = \begin{cases} 0, & \text{for } i_l \ne k_l \\ d_{k_1,\ldots,k_s,i_1,\ldots,i_{l-1},i_{l+1},\ldots,i_s}, & \text{for } i_l = k_l. \end{cases}
$$

The replacment in (3) gives

$$L z_1^{k_1} \dots z_l^{k_l} \dots z_s^{k_s} =$$

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_{l-1}=0}^{\infty} \sum_{i_{l+1}=0}^{\infty} \cdots \sum_{i_s=0}^{\infty} d_{k_1,\dots,k_s,i_1,\dots,i_{l-1},i_{l-1},\dots,i_s} \cdot z_1^{i_1} \dots z_{l-1}^{i_{l-1}} z_l^{k_l} z_{l+1}^{i_{l+1}} \dots z^{i_s}.$$

and it remains only to apply the operator L to an arbitrary function $y \in A_0$ with the power expansion $y(z) = \sum_{k=0}^{\infty} a_k z^k$, in order to obtain the description (2). Thus the necessity of (2) for commuting with $B_l^{p_l}$ is proved. The verification of the sufficiency of (2) will be omitted because it is a direct calculation of $LB_l^{p_l} z^k$ and $B_l^{p_l} L z^k$. ■

Now we give a corollary of Theorem 1 for operators commuting simultaneously with arbitrary powers of an arbitrary set, say $B_{l_1}^{p_{l_1}}, \dots, B_{l_q}^{p_{l_q}}$, $1 \le l_1 < \dots < l_q \le s$, of operators B_l. To make the result more readable, we suppose, without loss of generality, that the commutting is with the first q operators.

Corollary. *Let q be an integer, $1 \le q \le s$. Let $\alpha_l := n_l - m_l + 1 = 0$ and p_l be positive integers, $1 \le l \le q$. A continuous linear operator $L : A_0 \to A_0$ commutes separably with each of the operators $B_l^{p_l}$, $1 \le l \le q$, if and only if it has the form*

$$(4) \quad Ly(z) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_s=0}^{\infty} \sum_{i_{q+1}=0}^{\infty} \cdots \sum_{i_s=0}^{\infty} d_{k_1,\dots,k_s,i_{q+1},\dots,i_s} z_1^{k_1} \dots z_q^{k_q} z_{q+1}^{i_{q+1}} \dots z_s^{i_s},$$

where $d_{k_1,\dots,k_s,i_{q+1},\dots,i_s}$ are arbitrary complex numbers, such that the series in (4) are convergent in a neighbourhood of the origin.

If $q = s$, then (4) becomes

$$Ly(z) = \sum_{|k|=0}^{\infty} a_k d_k z^k.$$

In the onedimensional case this is the result of Theorem 2 in [3].

Proof. It is sufficient to mention that in fact here the intersection $\cap_{l=1}^{q} K_{B_l^{p_l}}$ of the commutants of the operators $B_l^{p_l}$, $1 \le l \le q$, should be taken, using the form (2) of each of the commutants. ■

In [1] the separable commutting is considered in the case when the operators B_l increace the powers , i.e. $\alpha_l := n_l - m_l + 1 \ge 1$. A theorem is given there without an explicit description. Here we give such description in order to formulate a corollary analogous to the above one.

Theorem 2. *Let l, $1 \le l \le s$ be arbitrarily fixed, and $p_l\alpha_l > 0$ with p_l a positive integer and $\alpha_l := n_l - m_l + 1 \ge 1$. A continuous linear operator $L : A_0 \to A_0$ commutes with the operator $B_l^{p_l}$ if and only if for every $y(z) = \sum_{k=0}^{\infty} a_k z^k$ it has the form*

$$(5)\qquad Ly(z) = \sum_{k_1=0}^{\infty}\cdots\sum_{k_l=0}^{p_l\alpha_l-1}\cdots\sum_{k_s=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_l=0}^{\infty}\cdots\sum_{i_s=0}^{\infty} a_k\lambda_{k,i}z^i +$$

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_l=p_l\alpha_l}^{\infty}\cdots\sum_{k_s=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_l=\left[\frac{k_l}{p_l\alpha_l}\right]p_l\alpha_l}^{\infty}\cdots\sum_{i_s=0}^{\infty} a_k \prod_{\mu_l=0}^{\left[\frac{k_l}{p_l\alpha_l}\right]p_l-1}\frac{m_l+k_l-\mu_l\alpha_l}{m_l+i_l-\mu_l\alpha_l}\cdot$$

$$\lambda_{k_1,\dots,k_l-\left[\frac{k_l}{p_l\alpha_l}\right]p_l\alpha_l,\dots,k_s,i_1,\dots,i_l-\left[\frac{k_l}{p_l\alpha_l}\right]p_l\alpha_l,\dots,i_s}$$

where $\alpha_l, p_l, k_l, i_l, m_l, \mu_l$ are the l-th components of the corresponding multi-indices, and $\lambda_{k_1,\dots,k_l-\left[\frac{k_l}{p_l\alpha_l}\right]p_l\alpha_l,\dots,k_s,i_1,\dots,i_l-\left[\frac{k_l}{p_l\alpha_l}\right]p_l\alpha_l,\dots,i_s}$ are arbitrary complex numbers, such that the series in (5) are convergent in a neighbourhood of the origin.

After a suitable renumbering of the variables and a convention about some products the following holds:

Corollary. *Let q be a positive integer, $1 \le q \le s$ and let $\alpha_l := n_l - m_l + 1 \ge 1$, $p_l \ge 1$, $1 \le l \le q$. A continuous linear operator $L : A_0 \to A_0$ commutes separably with each of the operators $B_l^{p_l}$, $1 \le l \le q$, if and only if for every $y(z) = \sum_{k=0}^{\infty} a_k z^k$ it has the form*

$$(6)\qquad Ly(z) = \sum_{k_1=0}^{\infty}\cdots\sum_{k_s=0}^{\infty}\sum_{i_1=\left[\frac{k_1}{p_1\alpha_1}\right]p_1\alpha_1}^{\infty}\cdots\sum_{i_q=\left[\frac{k_q}{p_q\alpha_q}\right]p_q\alpha_q}^{\infty}\sum_{i_{q+1}=0}^{\infty}\cdots\sum_{i_s=0}^{\infty}\cdot$$

$$a_k\prod_{\mu_1=0}^{\left[\frac{k_1}{p_1\alpha_1}\right]p_1-1}\frac{m_1+k_1-\mu_1\alpha_1}{m_1+i_1-\mu_1\alpha_1}\cdot\ldots\cdot\prod_{\mu_q=0}^{\left[\frac{k_q}{p_q\alpha_q}\right]p_q-1}\frac{m_q+k_q-\mu_q\alpha_q}{m_q+i_q-\mu_q\alpha_q}\cdot$$

$$\lambda_{k_1-\left[\frac{k_1}{p_1\alpha_1}\right]p_1\alpha_1,\dots,k_q-\left[\frac{k_q}{p_q\alpha_q}\right]p_q\alpha_q,k_{q+1},\dots,k_s,i_1-\left[\frac{i_1}{p_1\alpha_1}\right]p_1\alpha_1,\dots,i_q-\left[\frac{i_q}{p_q\alpha_q}\right]p_q\alpha_q,i_{q+1},\dots,i_s}z^i,$$

where the complex numbers $\lambda_{r_1,\dots,r_q,k_{q+1},\dots,k_s,i_1,\dots,i_s}$, $0 \le r_l < p_l\alpha_l, 1 \le l \le q$ *are arbitrary, but such that the series in* (6) *are convergent, and the products*

$$\prod_{\mu_l=0}^{\left[\frac{k_l}{p_l\alpha_l}\right]p_l-1} \frac{m_l + k_l - \mu_l\alpha_l}{m_l + i_l - \mu_l\alpha_l}$$

are considered to be equal to 1 *for* $0 \le k_l \le p_l\alpha_l$, $1 \le l \le q$, *to avoid the splitting of the sum* $\sum_{k_l=0}^{\infty}$ *into* $\sum_{k_l=0}^{p_l\alpha_l-1}$ *and* $\sum_{k_l=p_l\alpha_l}^{\infty}$.

Here the proof will be omitted, because it is similar to the proof of the previous corollary of Theorem 1.

Now a mixed variant of the separable commutting will be considered, when a part of the numbers $\alpha_l := n_l - m_l + 1$ are positive integers, while the others are equal to 0.

Theorem 3. *Let* q *and* t *be integer,* $1 \le q < t \le s$. *Let* p_l *be positive integers for* $1 \le l \le t$, *and* $p_l = 0$ *for* $t+1 \le l \le s$. *Let* $\alpha_l := n_l - m_l + 1 \ge 1$ *for* $1 \le l \le q$, *and* $\alpha_l = 0$ *for* $q+1 \le l \le t$. *A continuous linear operator* $L : A_0 \to A_0$ *commutes separably with each of the operators* $B_l^{p_l}$, $1 \le l \le t$, *if and only if for every* $y(z) = \sum_{k=0}^{\infty} a_k z^k$ *it has the form*

$$Ly(z) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_s=0}^{\infty} \sum_{i_1=\left[\frac{k_1}{p_1\alpha_1}\right]p_1\alpha_1}^{\infty} \cdots \sum_{i_q=\left[\frac{k_q}{p_q\alpha_q}\right]p_q\alpha_q}^{\infty} \sum_{i_{t+1}=0}^{\infty} \cdot \sum_{i_s=0}^{\infty} \cdot \tag{7}$$

$$a_k \prod_{\mu_1=0}^{\left[\frac{k_1}{p_1\alpha_1}\right]p_1-1} \frac{m_1+k_1-\mu_1\alpha_1}{m_1+i_1-\mu_1\alpha_1} \cdot \ldots \cdot \prod_{\mu_q=0}^{\left[\frac{k_q}{p_q\alpha_q}\right]p_q-1} \frac{m_q+k_q-\mu_q\alpha_q}{m_q+i_q-\mu_q\alpha_q} \cdot$$

$$d_{k_1-\left[\frac{k_1}{p_1\alpha_1}\right]p_1\alpha_1,\dots,k_q-\left[\frac{k_q}{p_q\alpha_q}\right]p_q\alpha_q,k_{q+1},\dots,k_s,i_1-\left[\frac{i_1}{p_1\alpha_1}\right]p_1\alpha_1,\dots,i_q-\left[\frac{i_q}{p_q\alpha_q}\right]p_q\alpha_q,i_{t+1},\dots,i_s} \cdot$$

$$z_1^{i_1}\ldots z_q^{i_q}, z_{q+1}^{k_{q+1}}\ldots z_t^{k_t} z_{t+1}^{i_{t+1}}\ldots z_s^{i_s},$$

where $d_{r_1,\dots,r_q,k_{q+1},\dots,k_s,i_1,\dots,i_q,i_{t+1},\dots,i_s}$ $0 \le r_l \le p_l\alpha_l$, $1 \le l \le q$, *are arbitrary complex numbers, such that the series in* (7) *are convergent in a neighbourhood of the origin, and the products*

$$\prod_{\mu_l=0}^{\left[\frac{k_l}{p_l\alpha_l}\right]p_l-1} \frac{m_l + k_l - \mu_l\alpha_l}{m_l + i_l - \mu_l\alpha_l}$$

of the origin, and the products

$$\prod_{\mu_l=0}^{\left[\frac{k_l}{p_l\alpha_l}\right]p_l-1} \frac{m_l+k_l-\mu_l\alpha_l}{m_l+i_l-\mu_l\alpha_l}$$

are considered to be equal to 1 *for* $0 \le k_l < p_l\alpha_l$, $1 \le l \le q$, *to avoid the splitting of the sum* $\sum_{k_l=0}^{\infty}$ *into* $\sum_{k_l=0}^{p_l\alpha_l-1}$ *and* $\sum_{k_l=p_l\alpha_l}^{\infty}$.

Proof. The formula (7) can be obtained as a consequence of Theorem 1 and Theorem 2 taking the intersection of the commutants $K_{B_l}^{p_l}$, $1 \le l \le q$, described by (2), and $K_{B_l}^{p_l}$, $q+1 \le l \le t$, described by (5). ∎

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