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On Almost Summable Sequences and Core of a Sequence in Nonarchimedean Spaces

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Presented by P. Kenderov

S h e r b a k o f f [15] generalized the notion of the core of a bounded complex sequence by introducing the idea of the generalized α-core $\chi^{(\alpha)}(x)$ of a bounded complex sequence. In [15] N a t a r a j a n introduced a definition analogous to the core in the classical case, but when K is a complete, locally compact, nontrivially valued, non-archimedean field. The purpose of this paper is to improve the result of N a t a r a j a n by using almost regular matrix transformation.

1. Introduction

In this paper, K denotes a complete, locally compact, non-trivially valued, non-archmedean field. After M o n n a [6] gave a complete characterization of regular infinite matrices with coefficients in a complete non-archimedean field K, the investigation in a summability theory started developing over complete non-archimedean fields, see the works cited in [6],[10],[11],[12],[13],[14].

If $A = (a_{nk})$, $n = 0,1,2,\ldots$, $k = 0,1,2,\ldots$ is a infinite matrix with coefficients in K. The *A-transform* Ax of a sequence $x = (x_k)$, $x \in K$, $k = 0,1,2,\ldots$ is a defined by the sequence $(A_n(x))$, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k, \quad n = 0,1,2,\ldots,$$

assuming that the series on the right hand side converges. The sequence x is said to be *summable by the matrix summability method A*, or shortly: *A-summable* to s, if $A_n(x) \mapsto s$ as $n \mapsto \infty$.

It may be noted here that L o r e n t z [5] introduced the concept of almost convergence for real or complex sequences and K u r t z [3] studied almost convergent vector sequences. In K i n g [2], D u r a n [1] and N a n d a [7], some matrix transformations of almost convergent sequences of real or complex numbers have been characterized. Further K u r t z [4] has discussed some matrix transformations for almost convergent vector sequences.

N a n d a [8] introduced and studied the concept of $\hat{c}$ of all the almost convergent sequences of elements of K.

The matrix A is said to be *almost regular*, if whenever $x = (x_k)$ converges to some L in K, the transformed sequences $(A_n(x))$ also almost converge to the same L.

In [8], the following conditions are given as necessary and sufficient for A to be almost regular:

1) $\sup_{n,k,m} |a(n,k,m)| = \sup_{n,k} |a_{nk}| < \infty$,

2) $\lim_m a(n,k,m) = 0$ uniformly in n for all k,

3) $\lim_m \sum_k a(n,k,m) = 1$ uniformly in n,

where $a(n,k,m) = \frac{1}{m+1} \sum_{i=0}^{m} a_{n+i,k}$ $(m > 0,\ n,k \geq 0)$, $a(n,k,0) = a_{nk}$.

We quote some definitions and theorems here for our future reference.

Definition 1.1 ([9] p.55). Let X be a topological linear space over a non-trivially valued, non-archimedean fields K. A subset S of X is said to be *absolutely* K-convex, if S is a V module, i.e. $VS + VS \subset S$, when V is the valuation ring of K. S is said to be K-*convex*, if S is absolutely K-convex or a translate of an absolutely K-convex set.

Theorem 1.1. ([9],p.27) *The only proper K-convex subsets of K are spheres.*

The following result gives some equivalent formulations of the K-convexity.

Theorem 1.2. ([9],p.56) *Let X be a topological linear space over a non-trivially valued, non-archimedean field K.*

i) *$S \subset X$ is K-convex if and only if for all $\lambda, \mu, \gamma \in K$ with $\lambda + \mu + \gamma = 1$, we have $\lambda x + \mu y + \gamma z \in S$ for all $x, y, z \in S$.*

ii) *If the characteristic of the residue class field V/P of K is not 2, then $S \subset X$ is K-convex if and only if for all $x, y \in S$, $\lambda x + (1-\lambda)y \in S$, i.e. the non-archimedean K-convexity is analogous to the classical convexity when characteristic of $V/P \neq 2$.*

Let us define the core of a sequence in K analogously to the definition of a core in the classical case.

Definition 1.2. ([13]) If $x = (x_n)$, $x_n \in K$, $n = 0, 1, 2, \ldots$ we denote by $K_n(x)$, $n = 0, 1, 2, \ldots$ the smallest K-convex set containing x_n, $x_{n+1}, \ldots$ and call

$$\mathcal{K}(x) = \bigcap_{n=0}^{\infty} K_n(x)$$

the *core of* x.

Any limit point of a bounded sequence x is in $\mathcal{K}(x)$, since if z is such a limit point, then $z = \lim_i x_{n(i)}$ for some sequence $n(i)$ of positive integers and so, $z \in K_p(x)$, $p = 0, 1, 2, \ldots$ and consequencetly, $z \in \mathcal{K}(x)$. On the other hand, $\mathcal{K}(x) = K$ if and only if x is unbounded, in view of Theorem 1.1. Also if x is a bounded sequence, $\mathcal{K}(x)$ is the smallest closed K-convex set containing the limit points of x. It now follows that if two bounded sequences have the same set of limit points, their cores are the same. There are, however, bounded sequences having different sets of limit points but the same core. From what has been said above, it is clear that $\mathcal{K}(x)$ is the singleton $\{p\}$ if and only if x converges to p.

It is also wortwhile to note that

$$\mathcal{K}(x) = \bigcap_{n=0}^{\infty} C_{r_n}(x_n), \quad r_n = \sup_{k \geq n} |x_k - x_n|$$

and $\mathcal{K}(x) = C_r(\alpha)$, where α is any limit point of x and

$$r = \inf_{n \geq 0} r_n.$$

Following S h e r b a k o f f [15] we can define for $\alpha > 0$, the generalized α-core of a sequence $x = (x_n)$ by

$$\mathcal{K}^{(\alpha)}(x) = \begin{cases} \bigcap_{z \in K} C_{\alpha \lim_n |z - x_n|}(z), & \text{if } x \text{ is bounded}, \\ K, & \text{if } x \text{ is unbounded}. \end{cases} \tag{1.1}$$

For $\alpha = 1$, $\mathcal{K}^{(\alpha)}(x)$ reduces to the usual core $\mathcal{K}(x)$.

2. Main theorem

We are ready to establish the following theorem concerning almost regular matrices.

Theorem 2.1. *An infinite matrix* $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$ *is such that* $\mathcal{K}(A_{mn}(x)) \subset \mathcal{K}^{(\alpha)}(x)$ *for any sequence* x *if and only if* A *is almost regular and satisfies*

$$\overline{\lim_m}\left(\sup_{n,k} |a(n, k, m)|\right) \leq \alpha, \tag{2.1}$$

where $A_{mn}(x) = \sum_k a(n,k,m)x_k$ *is such that* $a(n,k,m) = \frac{1}{m+1}\sum_{i=0}^{m} a_{n+i,k}$.

Proof. It is sufficient to consider bounded sequences. Let $x = (x_n)$ be a bounded sequence. If $y \in \mathcal{K}(A_{mn}(x))$ for any $x \in K$,

$$|y - z| \leq \overline{\lim_m}|z - (A_{mn}(x))|.$$

If A is almost regular and satisfies (2.1),

$$\begin{aligned} |y - z| &\leq \overline{\lim_m}|z - (A_{mn}(x))| \\ &\leq \overline{\lim_m}\left|\sum_{k=0}^{\infty} a(n,k,m)(z - x_k)\right| \leq \alpha\, \overline{\lim_k}|z - x_k|, \end{aligned}$$

i.e. $y \in C_{\alpha \overline{\lim}_k |z - x_k|}(z)$ for any $z \in K$, which implies that

$$\mathcal{K}(A_{mn}(x)) \subset \mathcal{K}^{(\alpha)}(x).$$

Conversely, if $\mathcal{K}(A_{mn}(x)) \subset \mathcal{K}^{(\alpha)}(x)$, then it is clear that A is almost regular, by considering convergent sequences $x = (x_n)$ for which $\mathcal{K}^{(\alpha)}(x) = \{\lim_n x_n\}$.

It remains to prove that (2.1) holds. Let, i fpossible, for each n

$$\overline{\lim_m}(\sup_{n,k}|a(n,k,m)|) > \alpha. \tag{2.2}$$

Using (2.2) and the fact that A is almost regular, we can choose two strictly increasing sequences $((m(i)),(k(m(i))))$ of positive integers such that

$$\begin{aligned} &\sup_{0\leq k\leq k(m(i-1))} |a(n,k,m(i))| < \alpha \\ &|a(n,k,m(i))| > \alpha, \\ &\sup_{k\geq k(m(i+1))} |a(n,k,m(i))| < \alpha. \end{aligned}$$

Define the sequence $x = (x_k)$ by

$$x_k = \left.\begin{array}{ll} 1, & k = k(m(i)) \\ 0, & k \neq k(m(i)) \end{array}\right\}, \quad i = 1,2,\ldots.$$

For this sequence x, $\mathcal{K}^{(\alpha)}(x) \subset C_\alpha(0)$, which follows from (1.1). However,

$$\begin{aligned}(A_{m(i),n}(x)) &= \sum_{k=0}^{k(m(i-1))} a(n,k,m(i))x_k \\ &+ \sum_{k=k(m(i-1))+1}^{k(m(i))} a(n,k,m(i))x_k + \sum_{k=k(m(i+1))}^{\infty} a(n,k,m(i))x_k \\ &= \sum_{k=0}^{k(m(i-1))} a(n,k,m(i))x_k + a(n,k,m(i)) \\ &+ \sum_{k=k(m(i+1))}^{\infty} a(n,k,m(i))x_k.\end{aligned}$$

So,

$$\begin{aligned}&\alpha < |a(n,k(m(i)),m(i))| \\ &\leq \max\{|A_{m(i),n}(x)|, \sup_{0\leq k\leq k(m(i-1))} |a(n,k,m(i))|, \sup_{k\geq k(m(i+1))} |a(n,k,m(i))|\} \\ &< \max\{|(A_{m(i),n}(x))|, \alpha, \alpha\}.\end{aligned}$$

Consequently,

$$|A_{m(i),n}(x)| > \alpha, \ i = 1, 2, \ldots.$$

The almost regularity of A implies that $(A_{m(i),n}(x))_{i=1}^{\infty}$ is a bounded sequence. Since K is locally compact, it has a convergent subsequence whose limit can not be in $C_\alpha(0)$, because of (2.3). This leads to a contradiction with the fact that $\mathcal{K}(A_{mn}(x)) \subset \mathcal{K}^{(\alpha)}(x)$. Hence,

$$\overline{\lim_m}(\sup_{n,k} |a(n,k,m)|) \leq \alpha,$$

which completes the proof of the theorem. ■

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