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Some Properties of a Linear Operator

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A linear operator defined by a Hadamard product (or convolution) for analytic functions in the open unit disk is introduced. The object of the present paper is to derive some properties of this linear operator.

1. Introduction

Let A denote the class of functions of the from

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \qquad (a_1 = 1) \tag{1.1}$$

that are analytic in the open unit disk $U = \{z : |z| < 1\}$.

For functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{n+1,j} z^{n+1} \qquad (a_{1,j} = 1), \tag{1.2}$$

we denote the *Hadamard product (or convolution)* of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n+1,1} a_{n+1,2} z^{n+1}. \tag{1.3}$$

The confluent hypergeometric function $\phi(a, c; z)$

$$\phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \neq 0, -1, -2, \ldots), \tag{1.4}$$

can be defined also as:

$$\phi(a,c;z) = z\,{}_2F_1(1,a;c;z), \tag{1.5}$$

where

$$ {}_2F_1(1,a;c;z) = \sum_{n=0}^{\infty} \frac{(1)_n(a)_n}{(c)_n}\frac{z^n}{n!} \tag{1.6}$$

and $(x)_n$ is the Pochhammer symbol

$$(x)_n = \begin{cases} x(x+1)(x+2)\dots(x+n-1), & if\ n \in N = \{1,2,3,\dots\} \\ 1, & if\ n = 0. \end{cases} \tag{1.7}$$

Corresponding to the function $\phi(a,c;z)$, Carlson and Shaffer [1] defined a linear operator $L(a,c)$ on A by the convolution

$$L(a,c)f(z) = \phi(a,c;z) * f(z) \tag{1.8}$$

for $f(z) \in A$. Clearly, $L(a,c)$ maps A onto itself, and $L(c,a)$ is an inverse of $L(a,c)$, provided that $a \neq 0,-1,-2,\dots$.

Recently, Srivastava and Owa [4] have given some properties of $L(a,c)$ concerning with univalent functions in U. To derive our result for linear operator (1.8), we have to recall the following lemma due to Jack [2] (also due to Miller and Mocanu [3]).

Lemma 1. *Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then we can write*

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geq 1$.

2. Some properties of the operator L(a,c)

First, we show the following lemma.

Lemma 2. *If $f(z) \in A$, then*

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z), \tag{2.1}$$

where $c \neq 0,-1,-2,\dots$.

Proof. Note that

$$L(a,c)f(z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}a_{n+1}z^{n+1} \tag{2.2}$$

and

$$L(a+1,c)f(z)=\sum_{n=0}^{\infty}\frac{(a+1)_n}{(c)_n}a_{n+1}z^{n+1}. \tag{2.3}$$

This gives that

$$\begin{aligned} & aL(a+1,c)f(z)-(a-1)L(a,c)f(z) \\ &=\sum_{n=0}^{\infty}(a+n)\frac{(a)_n}{(c)_n}a_{n+1}z^{n+1}-\sum_{n=0}^{\infty}(a-1)\frac{(a)_n}{(c)_n}a_{n+1}z^{n+1} \\ &=\sum_{n=0}^{\infty}(n+1)\frac{(a)_n}{(c)_n}a_{n+1}z^{n+1}=z(L(a,c)f(z))'. \end{aligned} \tag{2.4}$$

■

Applying Lemma 1 and Lemma 2, we prove

Theorem 1. *If $f(z)\in A$ satisfies*

$$Re(L(a+1),c)f(z))>-\alpha \qquad (z\in U) \tag{2.5}$$

for $\alpha>0$, then

$$Re(L(a-j),c)f(z))>-\alpha\beta_j \qquad (z\in U), \tag{2.6}$$

where $a<0$ and

$$\beta_j=2\prod_{k=0}^{j}\left(\frac{a-k}{2a-2k-1}\right). \tag{2.7}$$

Proof. We define the function $p(z)$ by $p(z)=L(a,c)f(z)$. Then, Lemma 2 gives us that

$$zp'(z)=aL(a+1,c)f(z)-(a-1)p(z), \tag{2.8}$$

so that,

$$L(a+1,c)f(z) = \left(1-\frac{1}{a}\right)p(z) + \frac{1}{a}zp'(z). \tag{2.9}$$

Further, define the function $w(z)$ by

$$p(z) = \frac{-\gamma w(z)}{1-w(z)} \qquad (w(z) \neq 1), \tag{2.10}$$

where $\gamma = -4a\alpha/(2a-1)$. Then we have

$$zp'(z) = \frac{-\gamma z w'(z)}{(1-w(z))^2}. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$L(a+1,c)f(z) = -\gamma\left(1-\frac{1}{a}\right)\frac{w(z)}{1-w(z)} - \frac{1}{a}\gamma\frac{zw'(z)}{(1-w(z))^2}. \tag{2.12}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq 1).$$

Then, using Lemma 1, we have

$$z_0 w'(z_0) = k w(z_0) \qquad (k \geq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$, we obtain

$$\begin{aligned}
(2.13) \qquad & Re(L(a+1,c)f(z_0)) \\
&= Re\left\{-\gamma\left(1-\frac{1}{a}\right)\frac{w(z_0)}{1-w(z_0)} - \frac{1}{a}\gamma\frac{z_0 w'(z_0)}{(1-w(z_0))^2}\right\} \\
&= Re\left\{-\gamma\left(1-\frac{1}{a}\right)\frac{e^{i\theta}}{1-e^{i\theta}} - \frac{1}{a}\gamma\frac{ke^{i\theta}}{(1-e^{i\theta})^2}\right\} \\
&= \frac{\gamma}{2}\left(1-\frac{1}{a}\right) + \frac{1}{a}\gamma\frac{k}{2(1-\cos\theta)} \\
&\leq \frac{\gamma}{2}\left(1-\frac{1}{a}\right) + \frac{k\gamma}{4a} \leq \frac{\gamma}{2}\left(1-\frac{1}{a}\right) + \frac{\gamma}{4a} = -\alpha.
\end{aligned}$$

This contradicts our condition (2.5). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$, so that

$$Re(p(z)) > \frac{\gamma}{2} = -\frac{2a\alpha}{2a-1} \quad (z \in U).$$

Thus we have that

$$Re(L(a,c)f(z)) > -\alpha\beta_0 \quad (z \in U) \tag{2.14}$$

with $\beta_0 = 2a/(2a-1)$.

Further, repeating this manner, we prove that

$$Re(L(a-j,c)f(z)) > -\alpha\beta_j \quad (z \in U). \tag{2.15}$$

This completes the proof of Theorem 1. ■

Similarly, we have

Theorem 2. *If $f(z) \in A$ satisfies*

$$Re(L(a+1,c)f(z)) < \alpha \quad (z \in U) \tag{2.16}$$

for $\alpha > 0$, then

$$Re(L(a-j,c)f(z)) < \alpha\beta_j \quad (z \in U), \tag{2.17}$$

where $a > 0$ and

$$\beta_j = 2\prod_{k=0}^{j}\left(\frac{a-k}{2a-ak-1}\right). \tag{2.18}$$

P r o o f. Let $p(z) = L(a,c)f(z)$ and let $p(z) = -\gamma w(z)/(1-w(z))$ with $\gamma = 4a\alpha/(2a-1)$. If there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1),$$

then we have

$$\begin{aligned} Re(L(a+1,c)f(z_0)) &= \frac{\gamma}{2}\left(1-\frac{1}{a}\right) + \frac{1}{a}\gamma\frac{k}{2(1-\cos\theta)} \\ &\geq \frac{\gamma}{2}\left(1-\frac{1}{a}\right) + \frac{k\gamma}{4a} \geq \frac{\gamma}{2}\left(1-\frac{1}{a}\right) + \frac{\gamma}{4a} = \alpha. \end{aligned} \tag{2.19}$$

It follows from (2.19) that

$$Re(L(a,c)f(z)) < \alpha\beta_0 \qquad (z \in U). \tag{2.20}$$

■

Repeating this step, we prove Theorem 2.

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