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On the Means of an Entire Function and its Derivatives

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In this paper we generalize some results of Rahman [4] and Jain [5] on the means of an entire function in the case when its order ρ is an arbitrary nonnegative integer.

Let $f(z)$ be an entire function on $\mathbb{C}$ and

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

We introduce the order ρ of a function f, as usually:

(1) $$\rho = \overline{\lim}_{r\to\infty} \frac{\ln\ln M_f(r)}{\ln r}.$$

For each fixed $\delta, 1 \leq \delta < \infty$ and each $k \in Z_+$, let

$$I_\delta(r) = I_\delta(r; f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \right\}^{1/\delta},$$

$$m_\delta(r) = m_\delta(r; f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta = (I_\delta(r))^\delta,$$

$$\mathcal{M}_{\delta,k}(r) = \mathcal{M}_{\delta,k}(r; f) = \frac{1}{\pi r^{k+1}} \int_0^r \int_0^{2\pi} |f(xe^{i\theta})|^\delta x^k d\theta dx.$$

For each $p \in Z_+$ let $I_\delta^{(p)}(r), m_\delta^{(p)}(r), \mathcal{M}_{\delta,k}^{(p)}(r)$ be the same characteristics but of the function $f^{(p)}(z) = d^p f/dz^p$, i.e.

$$I_\delta^{(p)}(r) = I_\delta(r; f^{(p)}(z)), \quad m_\delta^{(p)}(r) = m_\delta(r; f^{(p)}(z)), \quad \mathcal{M}_{\delta,k}^{(p)} = M_{\delta,k}(r; f^{(p)}(z)).$$

It is known ([2], Problem 66) that if $\delta = 2, k = 1$, then

$$\overline{\lim}_{r\to\infty} \frac{\ln\left\{\frac{m_2(r)}{\mathcal{M}_{2,1}(r)}\right\}}{\ln r} = \rho. \tag{2}$$

In 1956 Q. A. Rahman [4] proved for every δ, $1 \leq \delta < \infty, k \in Z_+$,

$$\overline{\lim}_{r\to\infty} \frac{\ln\left\{\frac{m_\delta(r)}{\mathcal{M}_{\delta,k}(r)}\right\}}{\ln r} = \rho. \tag{3}$$

In 1971, considering functions of two variables, but in fact solving one dimesional problem, P. K. Jain [5] showed that

$$\overline{\lim}_{r\to\infty} \frac{\ln\left\{\frac{I_\delta^{(1)}(r)}{I_\delta(r)}\right\}}{\ln r} = \rho. \tag{4}$$

In our paper we generalize the results of Rahman and Jain in the case when ρ is an arbitrary nonnegative integer.

First, let us consider some auxiliary propositions and remarks.

Since $|f|^\delta$ is a logarithmic subharmonic function, $I_\delta(r)$ is an increasing function of r and a logarithmic convex one with respect to $\ln r$ (see [3]). According to the general theory of the convex function, there exists an increasing, continuous except for at most countable set of points, function $\omega(r)$ (which we can define there as a function continuous on the left) such that

$$\ln I_\delta(r) = \ln I_\delta(r_0) + \int_{r_0}^{r} \frac{\omega(x)}{x}\, dx. \tag{5}$$

The same is true for $|f^{(p)}|^\delta$, so from expression (5) we get the next remarks.

Remark 1. For each fixed $p \in Z_+$ there exists an increasing continuous on the left function $\omega_p(r)$ such that

$$\ln I_\delta^{(p)}(r) = \ln I_\delta^{(p)}(r_0) + \int_{r_0}^{r} \frac{\omega_p(x)}{x}\, dx. \tag{6}$$

Remark 2. If r is sufficiently large and $r_0 \geq 1$, the functions $\omega_p(r)$ satisfy the estimates

$$\omega_p(r) \geq \frac{\ln I_\delta^{(p)}(r) - \ln I_\delta^{(p)}}{\ln r}, \quad p = 0, 1, \ldots \tag{7}$$

We prove the following three lemmas.

Lemma 1. *For every entire function f of order ρ,*

$$\overline{\lim}_{r\to\infty}\frac{\ln\ln I_\delta(r)}{\ln r}=\rho. \tag{8}$$

Proof. It is evident that $I_\delta(r)\le M_f(r)$ and for arbitrary $\varepsilon>0$ and sufficiently large r, from (1) we get

$$\frac{\ln\ln I_\delta(r)}{\ln r}<\rho+\varepsilon. \tag{9}$$

On the other hand, the Poisson's formula for $|f|^\delta$ and Harnac's inequality give us

$$|f(z)|\le\left(\frac{R+r}{R-r}\right)^{1/\delta} I_\delta(R)$$

for $r=|z|<R$. So, if $R=2r$, since I_δ is an increasing function and tends to infinity when $r\to\infty$,

$$\begin{aligned}\ln\ln M_f(r)&=\ln\ln I_\delta(2r)+\ln\left[1+\tfrac{\ln 3}{\delta\ln I_\delta(2r)}\right]\\&\le\ln\ln I_\delta(2r)+o(1).\end{aligned}$$

Then, from (1) there exists a subsequence $r_j\to\infty$ when $j\to\infty$ such that

$$\frac{\ln\ln I_\delta(r_j)}{\ln r_j}>\rho-\varepsilon$$

and together with (9), this proves (8). ■

Lemma 2. *If f is an analytic function in the circle $|z|\le R, R>0$, then for every $p\in Z_+, r<R$:*

$$I_\delta^{(p)}(r)\le\frac{I_\delta(R)}{(R-r)^p}\,p!\,. \tag{10}$$

Proof. Since f is analytic, we can write the Cauchy's integral formula for its p-th derivative

$$f^{(p)}(z)=\frac{p!}{2\pi i}\int_{|\xi-z|=R-r}\frac{f(\xi)}{(\xi-z)^{p+1}}\,d\xi$$

and from the Hölder's inequality and Fubini's theorem for the multiple integral we obtain: there exists some $r', r < r' < R$ such that

$$I_\delta^{(p)}(r) \le \frac{p!}{(R-r)^p} \left\{ \frac{1}{2\pi} \int_0^{2\pi} I_\delta^\delta(r') d\phi \right\}^{1/\delta} \le \frac{p!}{(R-r)^p} I_\delta(R).$$

Thus, Lemma 2 is proved.

Lemma 3. *For each $p = 2, 3, \dots$ there exists an increasing continuous on the left function $\omega_{p-1}(r)$ such that*

$$I_\delta^{(p)} \ge \frac{I_\delta^{(p-1)}(r)}{r} \omega_{p-1}(r). \tag{11}$$

Proof. Using the definition of the derivative and Minkowski's inequality, we have

(12)

$$\begin{aligned} I_\delta^{(p)}(r) &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \lim_{\varepsilon\to 0} \frac{f^{(p-1)}(re^{i\theta}) - f^{(p-1)}((r-r\varepsilon)e^{i\theta})}{\varepsilon r e^{i\theta}} \right|^\delta d\theta \right\}^{1/\delta} \\ &\ge \lim_{\varepsilon\to 0} \frac{1}{\varepsilon r} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left[|f^{(p-1)}(re^{i\theta})| - |f^{(p-1)}((r-r\varepsilon)e^{i\theta})| \right]^\delta d\theta \right\}^{1/\delta} \\ &\ge \lim_{\varepsilon\to 0} \frac{1}{\varepsilon r} \left[I_\delta^{(p-1)}(r) - I_\delta^{(p-1)}(r - r\varepsilon) \right] \\ &\ge \lim_{\varepsilon\to 0} \frac{I_\delta^{(p-1)}(r-r\varepsilon)}{\varepsilon r} \left[\frac{I_\delta^{(p-1)}(r)}{I_\delta^{(p-1)}(r-r\varepsilon)} - 1 \right] \end{aligned}$$

and from (6) with $r_0 = r - r\varepsilon$, we obtain

$$\begin{aligned} \ln I_\delta^{(p-1)}(r) &\ge \ln I_\delta^{(p-1)}(r - r\varepsilon) + \omega_{p-1}(r - r\varepsilon) \ln \frac{1}{1-\varepsilon} \\ &= \ln \left\{ I_\delta^{(p-1)}(r - r\varepsilon) [1 + \varepsilon\omega_{p-1}(r - r\varepsilon) + o(\varepsilon)] \right\}, \end{aligned}$$

that is,

$$\frac{I_\delta^{(p-1)}(r)}{I_\delta^{(p-1)}(r - r\varepsilon)} \ge 1 + \varepsilon\omega_{p-1}(r - r\varepsilon) + o(\varepsilon).$$

Thus, by the continuity of $I_\delta^{(p-1)}(r)$ and the left-continuity of $\omega_{p-1}(r)$ when $\varepsilon \to 0$, we obtain

$$I_\delta^{(p)}(r) \ge \frac{I_\delta^{(p-1)}(r)}{r} \omega_{p-1}(r).$$

This completes the proof. ■

Corollary 1. *For each $p = 2, 3, \ldots$*

(13) $$I_\delta^{(p)}(r) \geq \frac{I_\delta(r)}{r^p}\, \omega_{p-1}(r) \ldots \omega_0(r),$$

where all the functions $\omega_j(r), j = p-1, \ldots, 0$ are increasing and continuous on the left.

For each of the functions $I_\delta^{(j)}, j = p-1, \ldots, 0$ we use Lemma 3 and then (13) follows from (12).

Corollary 2. *For every $r < R, R > 0$ the quotient $I_\delta^{(p)}(r)/I_\delta(r)$ satisfies the estimate*

(14) $$\ln \frac{I_\delta^{(p)}(r)}{I_\delta(r)} \leq p\, \ln \omega_0(R) - p\, \ln R + 0(1).$$

Indeed, from Lemma 3 and formula (5) when $r_0 = r, r = R$ we have

$$\ln \frac{I_\delta^{(p)}(r)}{I_\delta(r)} \leq p\, \ln \frac{1}{R-r} + \omega_0(R)\, \ln \frac{R}{r} + 0(1)$$

and if the function f is not a polynomial (this is the case $\rho = 0$ and then all the results follow immediately from the definitions), the function $\omega_0(r)$ is increasing $\to \infty$ and we can find a large R, so that the equality

$$\frac{1}{R-r} = \frac{\omega_0(R)}{R}$$

holds. From here,

$$R = \frac{r}{1 - 1/\omega_0(R)} \leq \frac{r}{1 - 1/\omega_0(R)}$$

and since $\omega_0(r) \to \infty$ when $r \to \infty$, we can put $R = r\,(1 + o(1))$.

Therefore, $\alpha := 1/\omega_0(R) < 1$, $\ln(1-\alpha) = -\alpha + o(\alpha)$ and

$$\ln \frac{I_\delta^{(p)}(r)}{I_\delta(r)} \leq p\, \ln \omega_0(R) - p\, \ln R + 0(1).$$

Corollary 3. *The functions $\omega_j(r)$ satisfy the following lower estimations:*

(15) $$\ln \omega_j(r) \geq \ln \ln I_\delta(r) + o(\ln r), \;\; j = 0, 1, \ldots, p-1.$$

Since $\ln I_\delta^{(j)}$ is a subharmonic and tends to ∞ when $|z| \to \infty$, from Remark 2 it folows that

$$
\begin{aligned}
\ln \omega_j(r) &\geq \ln \left[\ln I_\delta^{(j)}(r)\left(1-\frac{c_j}{\ln I_\delta^{(j)}(r)}\right)\right]+o(\ln r) \\
&= \ln\ln I_\delta^{(j)}(r)+o(1)+o(\ln r) \\
&= \ln\ln I_\delta^{(j)}(r)+o(\ln r),
\end{aligned} \tag{16}
$$

where $c_j = I_\delta^{(j)}(r_0)$ is a constant.

Now the conclusion follows by induction. When $j = 0$, (15) is evident; assume it is true for each $\nu \leq j$. From Corollary 1, (16) and the hypothesis, we obtain

$$
\begin{aligned}
\ln \omega_{j+1}(r) &\geq \ln\ln I_\delta^{(j)}(r)+o(\ln r) \\
&\geq \ln\left[\ln I_\delta(r)+\sum_{\nu=0}^{j}\ln\omega_\nu(r)-j\ln r\right]+o(\ln r) \\
&\geq \ln\left[\ln I_\delta(r)+j\ln\ln I_\delta(r)+0(\ln r)+o(\ln r)\right]+o(\ln r) \\
&= \ln\left[\ln I_\delta(r)\left(1+j\frac{\ln\ln I_\delta(r)}{\ln I_\delta(r)}+\frac{0(\ln r)}{\ln I_\delta(r)}\right)\right]+o(\ln r) \\
&= \ln\ln I_\delta(r)+o(1)+o(\ln r).
\end{aligned}
$$

This completes the proof.

Now we can formulate the following main result:

Theorem. *For each entire function f of order ρ and each δ, $1 \leq \delta < \infty$, $p \in Z_+$ the following limit equalities hold:*

i) $\overline{\lim}_{r\to\infty} \dfrac{\ln\left\{r^p\left(I_\delta^{(p)}(r)/I_\delta(r)\right)\right\}}{p\ \ln r} = \rho;$

ii) $\overline{\lim}_{r\to\infty} \dfrac{\ln\left\{r^{\delta p}\left(m_\delta^{(p)}(r)/\mathcal{M}_\delta(r)\right)\right\}}{\delta p\ \ln r} = \rho;$

iii) $\overline{\lim}_{r\to\infty} \dfrac{\ln\left(m_\delta^{(p)}(r)/\mathcal{M}_{\delta,k}^{(p)}(r)\right)}{\ln r} = \rho.$

Proof. From Lemma 2 and Corollary 2 of Lemma 3 we have

$$
\ln\left\{r^p\,\frac{I_\delta^{(p)}(r)}{I_\delta(r)}\right\} \leq p\ \ln\omega_0(R)+0(1).
$$

Putting in (6) $r_0 = R, r = eR$ in the case $p = 0$ and using the monotonicity of $\omega_0(R)$, we obtain

$$\ln I_\delta(eR) \geq \ln I_\delta(R) + \omega_0(R).$$

But since $I_\delta(R)$ increases when $R \to \infty$, then $\ln I_\delta(R)$ is greater than zero and

$$\ln I_\delta(eR) \geq \omega_0(R),$$

so

$$\ln(eR)\ \frac{\ln\ln I_\delta(eR)}{\ln(eR)} \geq \ln \omega_0(R).$$

Then from Lemma 1 there exists r_0 such that for every $r \geq r_0$,

$$\begin{aligned}\ln \omega_0(R) &< (\rho + \tfrac{\varepsilon}{2}) \ln R + 0(1)\\ &\leq (\rho + \tfrac{\varepsilon}{2}) \ln r + 0(1) + o(1)\end{aligned}$$

and if $r > 1$,

$$\frac{\ln\left\{z^p\ \left(I_\delta^{(p)}(r)/I_\delta(r)\right)\right\}}{\ln r} < p\ (\rho + \frac{\varepsilon}{2}) + \frac{0(1)}{\ln r},$$

that is,

$$\frac{\ln\left\{r^p\ \left(I_\delta^{(p)}(r)/I_\delta(r)\right)\right\}}{p \ln r} < \rho + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \rho + \varepsilon.$$

On the other hand, from Corollary 3 of Lemma 3,

$$\ln\left(r^p\ \frac{I_\delta^{(p)}(r)}{I_\delta(r)}\right) \geq p\ \ln\ln I_\delta(r) + o(\ln r)$$

and if $r > 1$,

$$\frac{\ln\left\{r^p\ \left(I_\delta^{(p)}(r)/I_\delta(r)\right)\right\}}{p\ \ln r} \geq \frac{\ln\ln I_\delta(r)}{\ln r} + 0(1). \tag{17}$$

From Lemma 1, for every ε there exists a sequence $r_j \to \infty$ such that

$$\frac{\ln\ln I_\delta(r_j)}{\ln r_j} > \rho - \frac{\varepsilon}{2}$$

and in (17) we get

$$\frac{\ln\left\{r_j^p\ \left(I_\delta^{(p)}(r_j)/I_\delta(r_j)\right)\right\}}{p\ \ln r_j} > \rho - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \rho - \varepsilon.$$

This completes the proof in case (i).

Case (ii) follows from the relations

$$m_\delta(r) = (I_\delta(r))^\delta, \qquad m_\delta^{(p)}(r) = \left(I_\delta^{(p)}(r)\right)^\delta$$

and from the Rahman's results.

Case (iii) follows from the definitions of functions $m_\delta^{(p)}(r)$ and $\mathcal{M}_{\delta,k}^{(p)}(r)$.

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