Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

New Series Vol.11, 1997, Fasc. 1-2

Some Colombeau Products of Distributions [1]

Blagovest P. Damyanov

Presented by Bl. Sendov

By "Colombeau product of distributions" we mean the product of some distributions as they are embedded in Colombeau algebra $G(\mathbb{R}^m)$, whenever the result can be evaluated in terms of distributions again. Here we propose some results on Colombeau product of the distributions $x_\pm^a$ and $\delta^{(p)}(x)$, x in $\mathbb{R}^m$, that have coinciding point singularities.

The class $G(\mathbb{R}^m)$ of generalized functions introduced by J.-F. Colombeau [1] is a most relevant multiplicative system of such functions: $G(\mathbb{R}^m)$ is differential $\mathbb{C}$-algebra that contains (a copy of) the distribution space $D'(\mathbb{R}^m)$ as a $\mathbb{C}$-vector subspace. A connection with the distribution theory is established via the concept of 'association', generalizing the equality of distributions in $D'(\mathbb{R}^m)$. This notion is particularly useful for the evaluation of certain products of distributions—as they are embedded in $G(\mathbb{R}^m)$—in terms of distributions again. In this note we propose some results of that kind concerning the widely used distributions $x_\pm^p$ and $\delta^{(p)}(x)$ $(x \in \mathbb{R}^m)$. They are also easily transformed into the setting of so-called model product in the classical distribution theory, in dimension one.

We first recall the basic definitions of Colombeau algebra $G(\mathbb{R}^m)$ following their recent presentation in [6], Ch. 3.

Notation. If $\mathbb{N}_0$ stands for the nonnegative integers and $p = (p_1, p_2, ..., p_m)$ is a multiindex in $\mathbb{N}_0^m$, we let $|p| = \sum_{i=1}^m p_i$ and $p! = p_1!...p_m!$. Then, if $x = (x_1, ..., x_m)$ is in $\mathbb{R}^m$, we shall denote by $x^p = (x_1^{p_1}, x_2^{p_2}, ..., x_m^{p_m})$ and $\partial_x^p = \partial^{|p|}/\partial x_1^{p_1}...\partial x_m^{p_m}$. Now for any q in $\mathbb{N}_0$, denote by $A_q(\mathbb{R}) = \{\varphi(x) \in D(\mathbb{R}) : \int_{\mathbb{R}} x^j \varphi(x)\, dx = \delta_{0j}$ for $0 \le j \le q$, where $\delta_{00} = 1, \delta_{0j} = 0$ for $j > 0\}$.

[1] Acknowledgements are due to the Ministry of Science and Education of Bulgaria for a financial help under NFSR Grant MM 610.

This also extends to $\mathbb{R}^m$ as an m-fold tensor product: $A_q(\mathbb{R}^m) = \{\varphi(x) \in D(\mathbb{R}^m) : \varphi(x_1, ..., x_m) = \prod_{i=1}^m \chi(x_i)$ for some χ in $A_q(\mathbb{R})$. Finally, we denote by $\varphi_\varepsilon = \varepsilon^{-m}\varphi(\varepsilon^{-1}x)$ for any φ in $A_q(\mathbb{R}^m)$ and $\varepsilon > 0$.

Let now $E[\mathbb{R}^m]$ stand for the set of functions $f(\varphi, x) : A_0(\mathbb{R}^m)\times\mathbb{R}^m \to \mathbb{C}$ that are C^∞-differentiable with respect to x by a fixed 'parameter' φ. Note that $E[\mathbb{R}^m]$ is a $\mathbb{C}$-algebra with the point-wise function operations. Then each generalized function of Colombeau is an element of the quotient algebra $G(\mathbb{R}^m) = E_M[\mathbb{R}^m] / I[\mathbb{R}^m]$. Here the subalgebra $E_M[\mathbb{R}^m]$ and the ideal $I[\mathbb{R}^m]$ of $E_M[\mathbb{R}^m]$ are the sets of functions $f(\varphi, x)$ in $E[\mathbb{R}^m]$ such that the derivatives $\partial_x^p f(\varphi_\varepsilon, x)$ satisfy certain asymptotic evaluations, as $\varepsilon \to 0$ [6], Ch. 3.

The Colombeau algebra $G(\mathbb{R}^m)$ contains all distributions (and C^∞-differentiable functions) on $\mathbb{R}^m$, canonically embedded as a $\mathbb{C}$-vector subspace (respectively, a subalgebra) by the map $i : D'(\mathbb{R}^m) \to G(\mathbb{R}^m) : u \mapsto \tilde{u} = [\tilde{u}(\varphi, x)]$. The representatives here are given by $\tilde{u}(\varphi, x) = (u * \check{\varphi})(x)$, where $\check{\varphi}(x) = \varphi(-x)$ and φ is running the set $A_q(\mathbb{R}^m)$. Equivalently, one writes $\tilde{u}(\varphi, x) = \langle u_y, \varphi(y - x)\rangle$.

Further, a generalized function f in $G(\mathbb{R}^m)$ is said to admit some u in $D'(\mathbb{R}^m)$ as *associated distribution*, which is denoted by $f \approx u$, if f has a representative $f(\varphi_\varepsilon, x)$ in $\mathcal{E}_M[\mathbb{R}^m]$ such that for any $\psi(x)$ in $D(\mathbb{R}^m)$ there exists q in $\mathbb{N}_0$ so that, for all $\varphi(x)$ in $A_q(\mathbb{R}^m)$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^m} f(\varphi_\varepsilon, x)\psi(x)dx = \langle u, \psi\rangle. \tag{1}$$

This definition is independent of the representative chosen and the distribution associated is unique if it exists; the image in $G(\mathbb{R}^m)$ of every distribution is associated with that distribution [6], Ch. 3. The $\approx$-association is thus a faithful generalization of the equality of distributions in $D'(\mathbb{R}^m)$.

Then by "Colombeau product of distributions" we denote the product of some distributions, as they are embedded in Colombeau algebra $G(\mathbb{R}^m)$, whenever the result admits an associated distribution [4]. We now proceed with some results on Colombeau product of distributions. In what follows, we shall use the following.

Lemma 1. *Let u, v be distributions in $D'(\mathbb{R}^m)$ such that $u(x) = \prod_{i=1}^m u^i(x_i), v(x) = \prod_{i=1}^m v^i(x_i)$ with each u^i and v^i in $D'(R)$, and suppose that their embeddings in $G(\mathbb{R}^m)(R)$ satisfy $\tilde{u}^i.\tilde{v}^i \approx w^i$, for $i = 1, \dots, m$. Then $\tilde{u}.\tilde{v} \approx w$, where $w = \prod_{i=1}^m w^i(x_i)$.*

Proof. Suppose we have confined ourselves to the subspace of test-functions $\psi(x) = \prod_{i=1}^m \psi_i(x_i)$, with each ψ_i in $D(\mathbb{R})$. In view of the tensor-product structure of the distributions u, v in $D'(\mathbb{R})^m$ and that of the elements φ of $A_0(\mathbb{R}^m)$, on applying a Fubini-type theorem for tensor-product distributions (see [3], §4.3), we get:

$$\langle \tilde{u}(\varphi_\varepsilon, x)\tilde{v}(\varphi_\varepsilon, x), \psi(x)\rangle = \prod_{i=1}^{m} \langle \tilde{u}^i(\chi_\varepsilon, x_i)\tilde{v}^i(\chi_\varepsilon, x_i), \psi_i(x_i)\rangle =$$
$$= \prod_{i=1}^{m} \Big(\langle w^i(x_i), \psi_i(x_i)\rangle + f^i(\varepsilon)\Big).$$

Here, by assumption, one has the asymptotic evaluation $f^i(\varepsilon) = o(1)(\varepsilon \to 0)$ for each $i = 1, \ldots, m$. Thus

$$\lim_{\varepsilon\to 0} \langle \tilde{u}(\varphi_\varepsilon, x)\tilde{v}(\varphi_\varepsilon, x), \psi(x)\rangle = \prod_{i=1}^{m} \langle w^i, \psi^i\rangle = \langle w, \psi\rangle,$$

where $w = \prod_{i=1}^{m} w^i(x_i)$ is uniquely determined distribution in $\mathcal{D}'(\mathbb{R}^m)$. Moreover, since $\psi(x) = \prod_{i=1}^{m} \psi_i(x_i)$ is running a dense subset of $\mathcal{D}(\mathbb{R}^m)$ [3], §4.3, it follows by (1) that the product $\tilde{u}.\tilde{v}$ in $\mathcal{G}(\mathbb{R}^m)$ admits w as associated distribution. ■

Proposition 1. *For any p in $\mathbb{N}_0^m$, let $\tilde{\delta}^{(p)}(x)$ and $\tilde{x}_+^p$ be the embeddings in $\mathcal{G}(\mathbb{R}^m)$ of the distributions $\delta^{(p)}(x)$ and $x_+^p = \{x^p$ for $x \geq 0$, $= 0$ for $x < 0\}$ on $\mathbb{R}^m$. Then*

$$\tilde{x}_+^p \,.\, \tilde{\delta}^{(p)}(x) \approx \frac{(-1)^{|p|} p!}{2^m}\, \delta(x). \tag{2}$$

P r o o f. In the one-variable case, the first multiplier in (2) is represented by

$$\tilde{x}_+^p(\varphi_\varepsilon, x) = \varepsilon^{-1} \int_0^\infty y^p \,\varphi((y-x)/\varepsilon)\, dy = \int_{-x/\varepsilon}^\infty (x + \varepsilon t)^p \,\varphi(t)\, dt,$$

where the substitution $(y - x)/\varepsilon = t$ is made. Also, on differentiation in $\mathcal{D}'(\mathbb{R})$, we have

$$\tilde{\delta}^{(p)}(\varphi_\varepsilon, x) = (-1)^p \varepsilon^{-p-1} \langle \delta_y, \varphi^{(p)}((y-x)/\varepsilon)\rangle = (-1)^p \varepsilon^{-p-1} \varphi^{(p)}(-x/\varepsilon).$$

Now if $\operatorname{supp} \varphi(x) \subseteq [a, b]$ for some a, b in $\mathbb{R}$, then $\operatorname{supp} \varphi(-x/\varepsilon) \subseteq [-\varepsilon b, -\varepsilon a]$. Thus, replacing $x \to y = -x/\varepsilon$, we get for any $\psi(x)$ in $\mathcal{D}(\mathbb{R})$

$$\langle \tilde{x}_+^p(\varphi_\varepsilon, x)\, \tilde{\delta}^{(p)}(\varphi_\varepsilon, x), \psi(x)\rangle = \tfrac{(-1)^p}{\varepsilon^{p+1}} \int_{-b\varepsilon}^{-a\varepsilon} \Big(\int_{-x/\varepsilon}^{b} (x + \varepsilon t)^p \varphi(t)\, dt \Big)$$
$$\varphi^{(p)}(\tfrac{-x}{\varepsilon}) \psi(x)\, dx = \int_a^b \psi(-\varepsilon y)\, \varphi^{(p)}(y) \int_y^b (y - t)^p \varphi(t)\, dt\, dy.$$

By the Taylor theorem, we have $\psi(-\varepsilon y) = \psi(0) + (-\varepsilon y)\psi'(\eta y)$ for some $\eta \in [0,1]$. Now the integrand function in the latter equation, that reads

$$y\psi'(\eta y)\varphi^{(p)}(y)\int_y^b (y-t)^p\varphi(t)\,dt = y\psi'(\eta y)\varphi^{(p)}(y)\frac{1}{p+1}\left(t^{p+1} * \varphi(t)\right)(y),$$

is clearly a product of differentiable functions and is thus integrable on the finite interval $[a,b]$. Therefore, on taking the limit as $\varepsilon \to 0$ and applying the Dirichlet formula for changing the order of integration (which is permissible here), we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle \tilde{x}_+^p(\varphi_\varepsilon, x)\,\tilde{\delta}^{(p)}(\varphi_\varepsilon, x), \psi(x)\rangle = &\ \textstyle\int_a^b \psi(0)\varphi^{(p)}(y)\int_y^b (y-t)^p\varphi(t)\,dt\,dy = \\ = \psi(0) &\ \textstyle\int_a^b \varphi(t)\int_a^t (y-t)^p\varphi^{(p)}(y)\,dy\,dt \equiv \psi(0)\,I_p.\end{aligned}$$

To evaluate further the factor I_p , we expand the term $(y-t)^p$ in the integrand function and then proceed with a multiple integrating by parts:

$$\begin{aligned} I_p &= \sum_{j=0}^{p}(-1)^j\binom{p}{j}\int_a^b t^j\varphi(t)\int_a^t y^{p-j}\varphi^{(p)}(y)\,dy\,dt = \\ &= \sum_{j=0}^{p}(-1)^j\binom{p}{j}\int_a^b t^j\varphi(t)\sum_{k=o}^{p-j}(-1)^k\frac{(p-j)!}{(p-j-k)!}t^{p-j-k}\varphi^{(p-k-1)}(t)\,dt = \\ &= \sum_{j=0}^{p}\sum_{k=o}^{p-j}(-1)^{j+k}\frac{p!}{j!(p-j-k)!}\int_a^b t^{p-k}\varphi(t)\varphi^{(p-k-1)}(t)\,dt = \\ &= \sum_{k=0}^{p}(-1)^k\frac{p!}{(p-k)!}J_{p-k}\sum_{j=0}^{p-k}(-1)^j\binom{p}{j}. \end{aligned}$$

We have denoted here $J_{p-k} = \int_a^b t^{p-k}\varphi(t)\varphi^{(p-k-1)}(t)\,dt$, where, if $k=p$, $\varphi^{(-1)}(t)$ stands for $\int_a^t \varphi(y)\,dy$. For any $q = p-k > 0$, however, it holds [5], §21.5–1(b): $\sum_{j=0}^q (-1)^j \binom{q}{j} = 0$. Whence, $I_p = (-1)^p p!\,J_0$. We further get, by our assumption, for the remaining term with $p-k=j=0$

$$J_0 = \int_a^b \varphi(t)\left(\int_a^t \varphi(y)\,dy\right)dt = \frac{1}{2}\left(\int_a^t \varphi(y)\,dy\right)^2\Bigg|_a^b = \frac{1}{2}.$$

Thus, for any p in $\mathbb{N}_0$, we obtain

$$\tilde{x}_+^p\,.\,\tilde{\delta}^{(p)}(x) \approx \frac{(-1)^p p!}{2}\delta(x). \tag{3}$$

Finally, in the many-variable case, in view of the tensor-product structure of the distributions x_+^p and $\delta^{(p)}(x)$ in $D'(\mathbb{R}^m)$, we can employ Lemma 1, which yields

$$\tilde{x}_+^p.\tilde{\delta}^{(p)}(x) = \prod_{i=1}^{m} \tilde{x}_{i+}^{p_i}.\tilde{\delta}^{(p_i)}(x_i) \approx \prod_{i=1}^{m} \frac{(-1)^{p_i} p_i!}{2}\, \delta(x_i) = \frac{(-1)^{|p|} p!}{2^m}\, \delta(x).$$

This completes the proof. ■

Proposition 2. *For any p in $\mathbb{N}_0^m$, let $\tilde{x}_-^p$ be the embedding in $G(\mathbb{R}^m)$ of the distribution $x_-^p = \{0$ for $x > 0, = |x|^p$ for $x \leq 0)\}$. Then it holds*

$$\tilde{x}_-^p\,.\,\tilde{\delta}^{(p)}(x) \;\approx\; \frac{p!}{2}\,\delta(x). \tag{4}$$

Proof. One has $x_-^p = (-x)_+^p$ and $\delta^{(p)}(-x) = (-1)^p \delta^{(p)}(x)$, for any p in $\mathbb{N}_0$, and x in $\mathbb{R}$. The result therefore follows on replacing $x \to -x$ in (3) and then again—on employing Lemma 1.

Remark. Equations (2) and (4) are in consistency with the known formula

$$x^p\,\delta^{(p)}(x) = (-1)^{|p|} p!\,\delta(x) \quad (p \in \mathbb{N}_0^m), \tag{5}$$

in the space $D'(\mathbb{R}^m)$. Indeed, taking into account that $x^p = x_+^p + (-1)^{|p|} x_-^p$, the equations in consideration combine to give (5).

Consider now the "even" and "odd" sums of the distributions x_+^p, x_-^p (for any p in $\mathbb{N}_0^m$,defined by: $|x|^p = x_+^p + x_-^p$, $|x|^p \operatorname{sgn} x = x_+^p - x_-^p$. The Colombeau products below are then straightforward from (2) and (4).

Corollary 1. *Let $|\tilde{x}|^p$ and $|\tilde{x}|^p \operatorname{sgn} x$ be the embeddings in $G(\mathbb{R}^m)$ the distributions $|x|^p$ and $|x|^p \operatorname{sgn} x$ in $D'(\mathbb{R}^m)$. Then it holds*

$$|\tilde{x}|^p.\tilde{\delta}^{(p)}(x) \approx ((-1)^p + 1)\,\frac{p!}{2^m}\delta(x) = \begin{cases} 0, & p = 1, 3, \ldots \\ 2^{-m} p!\,\delta(x), & p = 0, 2, 4, \ldots \end{cases}$$

$$|\tilde{x}|^p \operatorname{sgn} x.\tilde{\delta}^{(p)}(x) \approx ((-1)^p - 1)\,\frac{p!}{2^m}\delta(x) = \begin{cases} -2^{-m} p!\,\delta(x), & p = 1, 3, \ldots \\ 0, & p = 0, 2, 4, \ldots \end{cases}$$

Remark. The proof of the above results can be modified—in dimension one only—so as to obtain the same formulas for model product of the corresponding distributions (denoted by [,]; see [6], Ch. 2). This is due to the fact that, replacing $\varphi(x) \to \rho(-x)$, where φ is in $A_0(\mathbb{R})$ (which requirement on φ we have only used), we get for any ψ in $D(\mathbb{R})$: $\lim_{\varepsilon\to 0}\langle \tilde{u}(\varphi_{\nu\varepsilon}, x)\tilde{v}(\varphi_\varepsilon, x), \psi(x)\rangle = \lim_{\varepsilon\to 0}\langle (u * \rho_\varepsilon)(v * \rho_\varepsilon), \psi\rangle = \langle [u, v], \psi\rangle$; where ρ will satisfy the requirements imposed on the mollifiers for model products. We finally note that the same equations for the distributions in $D'(\mathbb{R})$ were derived in [2] with the particular choice of the mollifiers $\rho(x)$ to be even functions.

References

[1] J.-F. Colombeau. *New Generalized Functions and Multiplication of Distributions*, North Holland Math. Studies 84, Amsterdam, 1984.

[2] B. Fisher. The product of distributions, *Quart. J. Oxford*, **22**, 1971, 291-298.

[3] F. G. Friedlander. *Introduction to the Theory of Distributions*, Cambridge Univ. Press, Cambridge, 1982.

[4] J. Jelinek. Characterization of the Colombeau product of distributions. *Comment. Math. Univ. Carolinae*, **27**, 1986, 377-394.

[5] G. A. Korn, T. M. Korn. *Mathematical Handbook*. McGraw-Hill Book Company, New York, 1968.

[6] M. Oberguggenberger. *Multiplication of Distributions and Applications to Partial Differential Equations*. Longman Scientific and Technical, New York, 1992.

Bulgarian Academy of Sciences,
INRNE Theory Group
72, Tzarigradsko Shosse
1784 Sofia, BULGARIA

Received 21.06.1995