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On one Cubic Identity for KdV Tau-Functions [1]

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We obtain some specific relations for KdV τ-functions. These identities are cubic in τ relations in contrary to the famous Fay identity, which is a quadratic in τ relation.

0. Introduction

Let $\tau(t)$, $t \equiv (t_1, t_2, t_3, \ldots) \in \mathbb{C}^\infty$, $t_1 \equiv x$ be an arbitrary tau-function, related to the Kadomtzev-Petviashvili (shortly KP) hierarchy [AvM]. The following Fay identity is well known:

$$
\begin{aligned}
&(z_0 - z_1)(z_2 - z_3)\tau(t + [z_0] + [z_1])\tau(t + [z_2] + [z_3]) + \\
(1)\qquad &(z_0 - z_2)(z_3 - z_1)\tau(t + [z_0] + [z_2])\tau(t + [z_3] + [z_1]) + \\
&(z_0 - z_3)(z_1 - z_2)\tau(t + [z_0] + [z_3])\tau(t + [z_1] + [z_2]) = 0,
\end{aligned}
$$

where z_0, z_1, z_2, $z_3 \in \mathbb{C}$ and for given $z \in \mathbb{C}$ we have defined:

$$
\begin{aligned}
[z] &:= (z, \tfrac{z^2}{2}, \tfrac{z^3}{3}, \ldots) \in \mathbb{C}^\infty, \\
t + [z] &:= (t_1 + z, t_2 + \tfrac{z^2}{2}, t_3 + \tfrac{z^3}{3}, \ldots) \in \mathbb{C}^\infty.
\end{aligned}
$$

The FI was firstly obtained [Fay] for theta-functions and in this case was important in the geometric tratement of the soliton equations [Mum]. Later FI was generelized for tau-functions [Shi]. Nowadays the FI is useful in different aspects of study of tau-(theta-) functions [AvM].

FI is fulfilled for tau-functions, related to n-th ($n = 2, 3, 4, \ldots$) Gel'fand-Dickey reductions of KP hierarchy. In the present paper we are interested only

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on tau-functions related to the $n = 2$ reduction, i.e. to the KdV hierarchy. Such functions will be called K d V t a u - f u n c t i o n s. It is well known [AvM] that they are characterized by the conditions:

$$\frac{\partial}{\partial t_{2k}}\tau(t) = 0 \quad , \; k = 1, 2, 3, \ldots$$

which imlpy for every $z \in \mathbb{C}$:

$$(2) \qquad \tau(t - [z]) = \tau(t + [-z]).$$

The main result of this paper is given in the following

Theorem 1. *Let $\tau(t)$, $t \in \mathbb{C}$ be an arbitrary KdV tau-function. Then for every μ, $\lambda \in \mathbb{C}$ the following identities hold:*

$$\begin{aligned}
(i) \quad & (\mu - \lambda)\,\{\tau(t + [\mu^{-1}] + [\lambda^{-1}])\,\tau(t - [\mu^{-1}])\,\tau(t - [\lambda^{-1}]) \\
& -\tau(t - [\mu^{-1}] - [\lambda^{-1}])\,\tau(t + [\mu^{-1}])\,\tau(t + [\lambda^{-1}])\} \\
& = (\mu + \lambda)\,\{\tau(t + [\mu^{-1}] - [\lambda^{-1}])\,\tau(t - [\mu^{-1}])\,\tau(t + [\lambda^{-1}]) \\
& -\tau(t - [\mu^{-1}] + [\lambda^{-1}])\,\tau(t + [\mu^{-1}])\,\tau(t - [\lambda^{-1}])\}. \\
(ii) \quad & \tau(t + 2[\lambda^{-1}])\,\tau^2(t - [\lambda^{-1}]) - \tau(t - 2[\lambda^{-1}])\,\tau^2(t + [\lambda^{-1}]) \\
& = 2\lambda\partial\,/\partial_\mu\{\tau(t + [\mu^{-1}] - [\lambda^{-1}])\quad \tau(t - [\mu^{-1}])\,\tau(t + [\lambda^{-1}]) \\
& -\tau(t - [\mu^{-1}] + [\lambda^{-1}])\,\tau(t + [\mu^{-1}])\,\tau(t - [\lambda^{-1})\}\,|_{\mu=\lambda}.
\end{aligned}$$

R e m a r k 1. We would like to mention the fact that these identities are cubic in τ-relations (in contrary to the FI(1), which is a quadratic in τ-relation) and they are specific only for KdV tau-functions. The proof of Theorem 1 is based only on the following three results:

(i) The FI (1) (which is common for all the tau-functions);
(ii) The property (2) (which is specific only for KdV tau-functions);
(iii) The obvious identity for Wronskians ($W(f, g) := fg' - f'g$, where $'$ denotes ∂_x)

$$(3) \qquad \begin{aligned} W(f_1 f_2, g_1 g_2) &= f_1 g_1\, W(f_2, g_2) + f_2 g_2\, W(f_1, g_1) \\ &= f_1 g_2\, W(f_2, g_1) + f_2 g_1\, W(f_1, g_2), \end{aligned}$$

where f_1, f_2, g_1 and g_2 are arbitrary functions.

R e m a r k 2. From the results of Theorem 1 corresponding c u b i c identities for theta-functions follow.

R e m a r k 3. In a forthcomming paper we will investigate the geometric interpretation of the results of Theorem 1.

2. Proof of the main result.

Instead of FI (1) we will use the differential Fay identity [AvM] ($\mu, \lambda \in \mathbb{C}$):

$$W(\tau(t-[\mu^{-1}]),\ \tau(t-[\lambda^{-1}])) = (\mu-\lambda)\{\ \tau(t-[\mu^{-1}])\,\tau(t-[\lambda^{-1}]) - \tau(t)\,\tau(t-[\mu^{-1}]-[\lambda^{-1}])\ \}$$

which is equivalent (after a change of notations) to the identity

$$W(\tau(t-[\mu^{-1}]+[\lambda^{-1}]),\ \tau(t)) = (\mu-\lambda)\{\ \tau(t)\,\tau(t-[\mu^{-1}]+[\lambda^{-1}]) - \tau(t-[\mu^{-1}])\,\tau(t+[\lambda^{-1}])\ \}$$

Now we specify these relations for the KdV tau-functions.

Lemma 1.1. *Let $\tau(t)$ be an arbitrary KdV tau-function. Then we have* ($\mu, \lambda \in \mathbb{C}$)

(i) $W(\tau(t+[\mu^{-1}]),\ \tau(t+[\lambda^{-1}]))$
$= -(\mu-\lambda)\{\ \tau(t+[\mu^{-1}])\,\tau(t+[\lambda^{-1}]) - \tau(t)\,\tau(t+[\mu^{-1}]+[\lambda^{-1}])\ \}$,

(ii) $W(\tau(t-[\mu^{-1}]),\ \tau(t+[\lambda^{-1}]))$
$= (\mu+\lambda)\{\ \tau(t-[\mu^{-1}])\,\tau(t+[\lambda^{-1}]) - \tau(t)\,\tau(t-[\mu^{-1}]+[\lambda^{-1}])\ \}$,

(iii) $W(\tau(t+[\mu^{-1}]),\ \tau(t-[\lambda^{-1}]))$
$= -(\mu+\lambda)\{\ \tau(t+[\mu^{-1}])\,\tau(t-[\lambda^{-1}]) - \tau(t)\,\tau(t+[\mu^{-1}]-[\lambda^{-1}])\ \}$,

(iv) $W(\tau(t-[\mu^{-1}]-[\lambda^{-1}]),\tau(t))$
$= (\mu+\lambda)\{\ \tau(t)\,\tau(t-[\mu^{-1}]-[\lambda^{-1}]) - \tau(t-[\mu^{-1}])\,\tau(t-[\lambda^{-1}])\ \}$,

(v) $W(\tau(t+[\mu^{-1}]+[\lambda^{-1}]),\ \tau(t))$
$= -(\mu+\lambda)\{\ \tau(t)\,\tau(t+[\mu^{-1}]+[\lambda^{-1}]) - \tau(t+[\mu^{-1}])\,\tau(t+[\lambda^{-1}])\ \}$.

Proof. We will explain only the proof of (i). The proofs of the other relations are similar. Using the differential Fay identity and property (2) we have:

$$W(\tau(t+[\mu^{-1}]),\ \tau(t+[\lambda^{-1}])) = W(\tau(t-[-\mu^{-1}]),\ \tau(t-[-\lambda^{-1}]))$$

$$((-\mu)-(-\lambda))\{\ \tau(t-[-\mu^{-1}])\,\tau(t-[-\lambda^{-1}]) - \tau(t)\,\tau(t-[-\mu^{-1}]-[-\lambda^{-1}])\ \}$$
$$= -(\mu-\lambda)\{\ \tau(t+[\mu^{-1}])\,\tau(t+[\lambda^{-1}]) - \tau(t)\,\tau(t+[\mu^{-1}]+[\lambda^{-1}])\ \}.$$

■

Proof of Theorem 1.
Using the obvious identity for Wronskians:

$$W(f_1/g, f_2/g) = W(f_1, f_2)/g^2,$$

we have

$$W(\tau(t-[\mu^{-1}])\,\tau(t+[\mu^{-1}])\,/\tau^2(t),\ \tau(t-[\lambda^{-1}])\,\tau(t+[\lambda^{-1}])\,/\tau^2(t))$$
$$=\frac{1}{\tau^4(t)}\,W(\tau(t-[\mu^{-1}])\,\tau(t+[\mu^{-1}]),\ \tau(t-[\lambda^{-1}])\,\tau(t+[\lambda^{-1}])).$$

Then, using the two expressions in (3) and the relations from Lemma 1 we obtain for the above Wronskian the following expressions:

$$(\mu-\lambda)\,/\tau^3(t)\quad \{\,\tau(t+[\mu^{-1}]+[\lambda^{-1}])\,\tau(t-[\mu^{-1}])\,\tau(t-[\lambda^{-1}])$$
$$-\tau(t-[\mu^{-1}]-[\lambda^{-1}])\,\tau(t+[\mu^{-1}])\,\tau(t+[\lambda^{-1}])\},$$

and

$$(\mu+\lambda)\,/\tau^3(t)\quad \{\,\tau(t+[\mu^{-1}]-[\lambda^{-1}])\,\tau(t-[\mu^{-1}])\,\tau(t+[\lambda^{-1}])$$
$$-\tau(t-[\mu^{-1}]+[\lambda^{-1}])\,\tau(t+[\mu^{-1}])\,\tau(t-[\lambda^{-1}])\,\}.$$

■

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