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Remarks on Some Series Considered by Ramanujan

Nora Brambilla, Emilio Montaldi

Presented by V. Kiryakova

Some characteristic infinite series identities, transformations and evaluations, appearing in Ramanujan's Collected Papers and Notebooks, are revisited and generalized.

1. Introduction

"Infinite series", one mathematician has written, "were Ramanujan's first love". They appear on virtually every page of his notebooks and papers; their study is still an endless source of mathematical delight, and fully justifies Hardy's declaration "It was his insight into algebraical formulae, transformations of infinite series, and so forth, that was most amazing".

In the present note, we first re-examine (and generalize) some series related to Riemann's zeta function and allied functions. This is done in Section 2, where the underlying idea is to write appropriate (and manageable) integral representations. In Section 3, a beautiful partial fraction decomposition is used to obtain a seemingly new expression for $\zeta(2n+1)$, n being a positive integer. Section 4 contains a simple proof of an identity involving two infinite series, by means of the Poisson summation formula. Finally, in Section 5, we revise four remarkable Lambert series and derive other results of a similar nature.

2. Series related to $\zeta(s)$, $\zeta(s,x)$ and $L(s)$

Ramanujan [10] defined

$$\phi(s,x)=\sum_{n=0}^{\infty}(\sqrt{x+n}+\sqrt{x+n+1})^{-s} \tag{2.1}$$

and obtained for this series a finite expression in terms of generalized zeta functions, when s is an odd integer greater than 1. The corresponding result for the series

$$\psi(s,x)=\sum_{n=0}^{\infty}\frac{(\sqrt{x+n}+\sqrt{x+n+1})^{-s}}{\sqrt{(x+n)(x+n+1)}} \tag{2.2}$$

follows immediately by noting that

$$\psi(s,x)=-\frac{2}{s}\frac{\partial}{\partial x}\phi(s,x). \tag{2.3}$$

More generally, we consider the series ($x,y>0$)

$$\phi(s,x,y)=\sum_{n=0}^{\infty}(\sqrt{x+n}+\sqrt{y+n})^{-s} \tag{2.4}$$

$$\psi(s,x,y)=\sum_{n=0}^{\infty}\frac{(\sqrt{x+n}+\sqrt{y+n})^{-s}}{\sqrt{(x+n)(y+n)}} \tag{2.5}$$

which, by using the standard Laplace transforms [9]

$$(\sqrt{x+n}+\sqrt{y+n})^{-s}=\frac{s}{2}(y-x)^{-\frac{s}{2}}\int_0^\infty t^{-1}e^{-nt-\frac{1}{2}(y+x)t}I_{\frac{s}{2}}(\frac{y-x}{2}t)dt \tag{2.6}$$

$$\frac{(\sqrt{x+n}+\sqrt{y+n})^{-s}}{\sqrt{(x+n)(y+n)}}=(y-x)^{-\frac{s}{2}}\int_0^\infty e^{-nt-\frac{1}{2}(y+x)t}I_{\frac{s}{2}}(\frac{y-x}{2}t)dt \tag{2.7}$$

($I_\nu(z)$ being the modified Bessel function of the first kind) can be rewritten as definite integrals (the interchange of series and integration is easily justified):

$$\phi(s,x,y)=\frac{s}{4}u^{-\frac{s}{2}}\int_0^\infty t^{-1}e^{-vt}\operatorname{csch}t\, I_{\frac{s}{2}}(ut)dt \tag{2.8}$$

$$\psi(s,x,y)=u^{-\frac{s}{2}}\int_0^\infty e^{-vt}\operatorname{csch}t\, I_{\frac{s}{2}}(ut)dt, \tag{2.9}$$

where

$$u=y-x \quad , \qquad v=y+x-1. \tag{2.10}$$

Now, from a formula [8] for the Hankel function of the first kind

$$H^{(1)}_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, i^{-(n+1)} e^{ix} \sum_{r=0}^{n} \frac{(n+r)!}{r!(n-r)!}\left(\frac{i}{2x}\right)^r \tag{2.11}$$

it is a straightforward exercise to deduce that

$$\begin{aligned} I_{2n+\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\Bigg[\sinh x \sum_{r=0}^{n} \frac{(2n+2r)!}{(2r)!(2n-2r)!}\left(\frac{1}{4x^2}\right)^r \\ &\quad -2x\cosh x \sum_{r=1}^{n} \frac{(2n+2r-1)!}{(2r-1)!(2n-2r+1)!}\left(\frac{1}{4x^2}\right)^r \end{aligned} \tag{2.12}$$

$$\begin{aligned} I_{2n+\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\Bigg[\cosh x \sum_{r=0}^{n} \frac{(2n+2r+1)!}{(2r)!(2n-2r+1)!}\left(\frac{1}{4x^2}\right)^r \\ &\quad -2x\sinh x \sum_{r=1}^{n+1} \frac{(2n+2r)!}{(2r-1)!(2n-2r+2)!}\left(\frac{1}{4x^2}\right)^r. \end{aligned} \tag{2.13}$$

By inserting these expressions into (2.8),(2.9), and recalling [7] that

$$\int_0^\infty t^{\alpha-1} e^{-(2z-1)t} \operatorname{csch} t\, dt = 2^{1-\alpha}\Gamma(\alpha)\zeta(\alpha, z) \tag{2.14}$$

($\mathrm{Re}(\alpha) > 1, \mathrm{Re}(z) > 0$) we easily obtain

(2.15)

$$\begin{aligned}\phi(4n+1,x,y) &= -(4n+1)u^{-(2n+1)}\Bigg\{\sum_{r=0}^{n}\binom{2n+2r}{4r}\frac{16^r u^{-2r}}{4r+1}\Big[\zeta(-\tfrac{1}{2}-2r,x) \\ &-\zeta(-\tfrac{1}{2}-2r,y)\Big] + \tfrac{u}{4}\sum_{r=1}^{n}\binom{2n+2r-1}{4r-2}\frac{16^r u^{-2r}}{4r-1}\Big[\zeta(\tfrac{1}{2}-2r,x)+\zeta(\tfrac{1}{2}-2r,y)\Big]\Bigg\};\end{aligned}$$

(2.16)

$$\begin{aligned}\phi(4n+3,x,y) &= -(4n+3)u^{-2(n+1)}\Bigg\{\sum_{r=0}^{n}\binom{2n+2r+1}{4r}\frac{16^r u^{-2r}}{4r+1}\Big[\zeta(-\tfrac{1}{2}-2r,x) \\ &+\zeta(-\tfrac{1}{2}-2r,y) + \tfrac{u}{4}\sum_{r=1}^{n+1}\binom{2n+2r}{4r-2}\frac{16^r u^{-2r}}{4r-1}\zeta(\tfrac{1}{2}-2r,x)-\zeta(\tfrac{1}{2}-2r,y)\Big]\Bigg\};\end{aligned}$$

(2.17)

$$\begin{aligned}\psi(4n+1,x,y) &= u^{-(2n+1)}\sum_{r=0}^{n}\binom{2n+2r}{4r}16^r u^{-2r}\Big[\zeta(\tfrac{1}{2}-2r,x)-\zeta(\tfrac{1}{2}-2r,y)\Big] \\ &+\tfrac{1}{4}u^{-2n}\sum_{r=1}^{n}\binom{2n+2r-1}{4r-2}16^r u^{-2r}\Big[\zeta(\tfrac{3}{2}-2r,x)+\zeta(\tfrac{3}{2}-2r,y)\Big];\end{aligned}$$

(2.18)

$$\begin{aligned}\psi(4n+3,x,y) &= u^{-2(n+1)}\sum_{r=0}^{n}\binom{2n+2r+1}{4r}16^r u^{-2r}\Big[\zeta(\tfrac{1}{2}-2r,x)+\zeta(\tfrac{1}{2}-2r,y)\Big] \\ &+\tfrac{1}{4}u^{-(2n+1)}\sum_{r=1}^{n+1}\binom{2n+2r}{4r-2}16^r u^{-2r}\Big[\zeta(\tfrac{3}{2}-2r,x)-\zeta(\tfrac{3}{2}-2r,y)\Big].\end{aligned}$$

Remark . The condition $\mathrm{Re}(\alpha) > 1$, ensuring the convergence of the integral (2.14), has been waived by analytic continuation, since the integrands in (2.8), (2.9) are well–behaved.

With $y = x + 1$, Eqs.(2.15)–(2.18) become

$$\phi(4n+1,x) = -(4n+1)\Big\{ \sum_{r=0}^{n} \binom{2n+2r}{4r} \frac{16^r}{4r+1} x^{2r+\frac{1}{2}} + \frac{1}{4}\sum_{r=1}^{n} \binom{2n+2r-1}{4r-2} \frac{16^r}{4r-1} \Big[2\zeta(\tfrac{1}{2}-2r,x) - x^{2r-\frac{1}{2}}\Big] \Big\}; \tag{2.19}$$

$$\phi(4n+3,x) = -(4n+3)\Big\{ \sum_{r=0}^{n} \binom{2n+2r+1}{4r} \frac{16^r}{4r+1} \Big[2\zeta(-\tfrac{1}{2}-2r,x) - x^{2r+\frac{1}{2}}\Big] + \frac{1}{4}\sum_{r=1}^{n+1} \binom{2n+2r}{4r-2} \frac{16^r}{4r-1} x^{2r-\frac{1}{2}} \Big\}; \tag{2.20}$$

$$\psi(4n+1,x) = \sum_{r=0}^{n} \binom{2n+2r}{4r} 16^r x^{2r-\frac{1}{2}} + \frac{1}{4}\sum_{r=1}^{n} \binom{2n+2r-1}{4r-2} 16^r \Big[2\zeta(\tfrac{3}{2}-2r,x) - x^{2r-\frac{3}{2}}\Big]; \tag{2.21}$$

$$\psi(4n+3,x) = \sum_{r=0}^{n} \binom{2n+2r+1}{4r} 16^r \Big[2\zeta(\tfrac{1}{2}-2r,x) - x^{2r-\frac{1}{2}}\Big] + \frac{1}{4}\sum_{r=1}^{n+1} \binom{2n+2r}{4r-2} 16^r x^{2r-\frac{3}{2}}. \tag{2.22}$$

In particular, we have

$$\phi(3,x) = -6\zeta(-\frac{1}{2},x) - (4x-3)\sqrt{x} \tag{2.23}$$

$$\phi(5,x) = -40\zeta(-\frac{3}{2},x) - (16x^2-20x+5)\sqrt{x} \tag{2.24}$$

$$\psi(1,x) = \frac{1}{\sqrt{x}} \tag{2.25}$$

$$\psi(3,x) = 2\zeta(\frac{1}{2},x) - \frac{1}{\sqrt{x}} + 4\sqrt{x} \tag{2.26}$$

$$\psi(5,x) = 24\zeta(-\frac{1}{2},x) + \frac{1}{\sqrt{x}} - 12\sqrt{x} + 16x\sqrt{x} \tag{2.27}$$

and, recalling the functional equation for $\zeta(s)$,

$$\phi(3,0) = \frac{3}{2\pi}\zeta(\frac{3}{2}) \tag{2.28}$$

$$\phi(5,0) = \frac{15}{2\pi^2}\zeta(\frac{5}{2}) \tag{2.29}$$

in agreement with Ramanujan [10]. He stated the formulas for $\phi(s,x)$ and $\psi(s,x)$ in a somewhat disguised form, because he followed a different (and very ingenious) procedure, which however does not apply to the more general cases (2.4), (2.5). If, in (2.4) and (2.5), the terms of the series have alternating signs, it suffices to observe that, for instance

$$\sum_{n=0}^{\infty}(-1)^n(\sqrt{x+n}+\sqrt{y+n})^{-s} = 2^{-\frac{s}{2}}\left[\phi(s,\frac{x}{2},\frac{y}{2}) - \phi(s,\frac{x+1}{2},\frac{y+1}{2})\right]. \tag{2.30}$$

In particular, defining

$$A(s) = \sum_{n=0}^{\infty}(-1)^n(\sqrt{2n+1}+\sqrt{2n+3})^{-s}, \tag{2.31}$$

we obtain

$$\begin{aligned} A(4n+1) &= -(4n+1)4^{-n-1}\Big\{2\sum_{r=0}^{n}\binom{2n+2r}{4r}\frac{4^r}{4r+1}\left[2L(-\tfrac{1}{2}-2r)-1\right] \\ &\quad + \sum_{r=1}^{n}\binom{2n+2r-1}{4r-2}\frac{4^r}{4r-1}\Big\}. \end{aligned} \tag{2.32}$$

$$\begin{aligned} A(4n+3) &= -(4n+3)4^{-n-1}\Big\{\sum_{r=0}^{n}\binom{2n+2r+1}{4r}\frac{4^r}{4r+1} \\ &\quad +\tfrac{1}{2}\sum_{r=1}^{n+1}\binom{2n+2r}{4r-2}\frac{4^r}{4r-1}\left[2L(\tfrac{1}{2}-2r)-1\right]\Big\}, \end{aligned} \tag{2.33}$$

where

$$L(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s} = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \tag{2.34}$$

$\chi(n)$ beeing the primitive character of modulus 4. Eqs. (2.32), (2.33) follow at once from (2.30), by noting that

$$\zeta(s,\frac{1}{4}) - \zeta(s,\frac{3}{4}) = 4^s L(s) \tag{2.35}$$

$$\zeta(s,\frac{3}{4})-\zeta(s,\frac{5}{4})=4^s[1-L(s)]; \tag{2.36}$$

alternatively, we can derive them from the integral representation

$$A(s)=s2^{-\frac{s+4}{2}}\int_0^\infty \frac{1}{t}e^{-t}\operatorname{sech} t\, I_{\frac{s}{2}}(t)dt \tag{2.37}$$

together with the formula

$$\int_0^\infty t^{\alpha-1}e^{-t}\tanh t dt=\Gamma(\alpha)[2L(\alpha)-1] \tag{2.38}$$

which is readily established by writing $\tanh t=1+2\sum_{n=1}^{\infty}(-1)^n e^{-2nt}$. If $n=0$, (2.32)–recalling the functional equation [7.35(29)] for $L(s)$– gives the formula

$$A(1)=\frac{1}{2}-L(-\frac{1}{2})=\frac{1}{2}-\frac{1}{\pi}L(\frac{3}{2}) \tag{2.39}$$

which is one of the results communicated by Ramanujan [10,XXVI] in his first letter to Hardy, dated January 16, 1913.

The series

$$K_l=\sum_{n=1}^{\infty}\frac{1}{n^l(n+1)^l} \tag{2.40}$$

appearing at the end of Chapter 9 of Ramanujan's second notebook [2], can be dealt with along similar lines. More generally, we consider $(x,y>0)$

$$S_{\alpha,\beta}(x,y)=\sum_{n=0}^{\infty}(x+n)^{-\alpha}(y+n)^{-\beta}. \tag{2.41}$$

By writing

$$(x+n)^{-\alpha}(y+n)^{-\beta}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\frac{t^{\alpha-1}(1-t)^{\beta-1}}{[xt+y(1-t)+n]^{\alpha+\beta}}dt \tag{2.42}$$

we first have

$$S_{\alpha,\beta}(x,y)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\zeta(\alpha+\beta,xt+y(1-t))dt \tag{2.43}$$

Next, we replace ζ by its integral representation, Eq.(2.14), and then integrate over t, [6]. This gives

$$S_{\alpha,\beta}=\frac{1}{\Gamma(\alpha+\beta)}\int_0^\infty r^{\alpha+\beta-1}\frac{e^{-xr}}{1-e^{-r}}\,{}_1F_1(\beta;\alpha+\beta;-ur)dr. \tag{2.44}$$

When $\beta = \alpha$, recalling that [6,208]

$$
{}_1F_1(\alpha; 2\alpha; -z) = \Gamma(\alpha + \frac{1}{2})(\frac{z}{4})^{-\alpha+\frac{1}{2}} e^{-\frac{z}{2}} I_{\alpha-\frac{1}{2}}(\frac{z}{2}), \tag{2.45}
$$

Eq.(2.44) becomes

$$
\sum_{n=0}^{\infty} \frac{1}{[(x+n)(y+n)]^{\alpha}} = \frac{\sqrt{\pi}}{\Gamma(\alpha)} u^{-\alpha+\frac{1}{2}} \int_0^{\infty} r^{\alpha-\frac{1}{2}} \frac{e^{-\frac{v+1}{2}r}}{1-e^{-r}} I_{\alpha-\frac{1}{2}}(\frac{ur}{2}) dr. \tag{2.46}
$$

With $x = 1$, $y = 2$, $\alpha = 2l + 1$ or $\alpha = 2l + 2$, the integral is easily evaluated by using (2.12) and (2.13), conveniently rewritten as

$$
\begin{aligned}
I_{2l+\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \Bigg[\sinh x \sum_{r=0}^{2l} \frac{(2l+r)!}{r!(2l-r)!} (\frac{1}{2x})^r \\
&\quad -2xe^{x} \sum_{r=1}^{l} \frac{(2l+2r-1)!}{(2r-1)!(2l-2r+1)!} (\frac{1}{4x^2})^r,
\end{aligned} \tag{2.47}
$$

$$
\begin{aligned}
I_{2l+\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \Bigg[\sinh x \sum_{r=0}^{2l+1} \frac{(2l+r+1)!}{r]!(2l-r+1)!} (-\frac{1}{2x})^r \\
&\quad +e^{-x} \sum_{r=0}^{l} \frac{(2l+2r+1)!}{(2r)!(2l-2r+1)!} (\frac{1}{4x^2})^r.
\end{aligned} \tag{2.48}
$$

It is also useful to recall the elementary formula

$$
\sum_{r=0}^{n} \frac{(a)_r}{r!} = \frac{(a+1)_n}{n!}, \tag{2.49}
$$

which is readily verified by induction. The final result

$$
\begin{aligned}
K_l &= \sum_{r=0}^{l} (-1)^r \binom{l+r-1}{l-1} A_{l-r} \\
A_r &= [1 + (-1)^r]\zeta(r), \qquad A_1 \equiv 0
\end{aligned} \tag{2.50}
$$

is, of course, well-known [2]. Nevertheless, the above derivation is noteworthy because of its elegance and simplicity.

3. A formula for $\zeta(2k+1)$

A large number of formulas appearing in Chapter 14 of Ramanujan's second notebook arise from partial fraction decomposition. In particular, we have [3]

$$
\pi^2 z^2 \frac{\cosh(\pi z\sqrt{2}) + \cos(\pi z\sqrt{2})}{\cosh(\pi z\sqrt{2}) - \cos(\pi z\sqrt{2})} = 1 + 4\pi z^4 \sum_{n=1}^{\infty} \frac{n\coth(n\pi)}{n^4 + z^4}. \tag{3.1}
$$

This result can be rewritten as

$$\phi(\alpha) \equiv \frac{\cos(\sqrt{\alpha})}{\cosh(\sqrt{\alpha})-\cos(\sqrt{\alpha})} = -\tfrac{1}{2} + \tfrac{1}{\alpha} + \tfrac{4}{\beta}\sum_{n=1}^{\infty}\frac{n\coth(n\pi)}{n^4+\lambda}. \tag{3.2}$$
$$\alpha,\beta > 0 \qquad \alpha\beta = 4\pi^3, \qquad \lambda = (\tfrac{2\pi}{\beta})^2 .$$

Replacing α by $r\alpha$, dividing by r^{2k+1} (k being a positive integer) and summing over r, we deduce that

$$\sum_{r=1}^{\infty}\frac{\phi(r\alpha)}{r^{2k+1}} = -\frac{1}{2}\zeta(2k+1) + \frac{1}{\alpha}\zeta(2k+2) + \frac{4}{\beta}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}\frac{n\coth(n\pi)}{r^{2k}(n^4+\lambda r^2)}. \tag{3.3}$$

Now

$$\frac{1}{r^{2k}(n^4+\lambda r^2)} = \frac{1}{r^{2k}}\sum_{l=1}^{k}(-1)^{l-1}\frac{(\lambda r^2)^{l-1}}{n^{4l}} + (-1)^k\frac{\lambda^k}{n^{4k}(n^4+\lambda r^2)}. \tag{3.4}$$

Therefore, putting

$$S_l = \sum_{n=1}^{\infty}\frac{\coth(n\pi)}{n^{4l-1}} \tag{3.5}$$

and recalling the well-known formula

$$\sum_{r=1}^{\infty}\frac{1}{n^4+\lambda r^2} = -\frac{1}{2n^4} + \frac{\beta}{4n^2}\coth(\frac{1}{2}\beta n^2), \tag{3.6}$$

Eq.(3.3) takes the form

(3.7)
$$\sum_{r=1}^{\infty}\frac{\phi(r\alpha)}{r^{2k+1}} = -\tfrac{1}{2}\zeta(2k+1) + \tfrac{1}{\alpha}\zeta(2k+2) + \tfrac{4}{\beta}\textstyle\sum_{l=1}^{k+1}(-\lambda)^{l-1}\zeta(2k-2l+2)S_l$$
$$+(-\lambda)^k\sum_{n=1}^{\infty}\frac{\coth(n\pi)\coth(\frac{1}{2}\beta n^2)}{n^{4k+1}}.$$

The term $-\frac{1}{2n^4}$, appearing in the r.h.s. of (3.6), is accounted for by the upper limit $l = k+1$ (recall that $\zeta(0) = -\frac{1}{2}$).

In particular, taking $\alpha = 2\pi^2$, we have

(3.8)
$$\sum_{r=1}^{\infty}\frac{\phi(2r\pi^2)}{r^{2k+1}} = -\tfrac{1}{2}\zeta(2k+1) + \tfrac{1}{2\pi^2}\zeta(2k+2) + \tfrac{2}{\pi}\sum_{l=1}^{k+1}(-1)^{l-1}\zeta(2k-2l+2)$$
$$\times S_l + (-1)^k\sum_{n=1}^{\infty}\frac{\coth(n\pi)\coth(n^2\pi)}{n^{4k+1}}.$$

We remark that the S_l's, Eq. (3.5), are explicitly known [3,293], for instance

$$S_1 = \frac{7\pi^3}{180} \quad , \qquad S_2 = \frac{19\pi^7}{56700} \tag{3.9}$$

and so on. Furthermore, if we write

$$\sum_{n=1}^{\infty} \frac{\coth(n\pi)\coth(n^2\pi)}{n^{4k+1}} = \zeta(4k+1) + \sum_{n=1}^{\infty} \frac{\coth(n\pi)\coth(n^2\pi) - 1}{n^{4k+1}}, \tag{3.10}$$

we see that (3.8), as a matter of fact, expresses $-\frac{1}{2}\zeta(2k+1) + (-1)^k\zeta(4k+1)$ by means of known quantities plus two series whose terms are exponentially decreasing. Finally, for the series $\sum_{r=1}^{\infty} \frac{1}{r}\phi(r\alpha)$ – corresponding to $k = 0$ in the l.h.s. of (3.3) – Ramanujan [3,280] gives a beautiful transformation formula which, however, cannot be obtained from (3.7) with $k = 0$, because the r.h.s. becomes meaningless.

Several other formulas [3,275] like (3.1) can be manipulated, in a similar way, to produce further expressions for $\zeta(2k+1)$ analogous to (3.7). However, we don't insist on this point here.

4. An infinite series identity

Let $\alpha, \beta > 0$ with $\alpha\beta = \pi$, and let n be real with $|n| < \frac{\beta}{2}$. Then ([3], Entry 11, p.258),

$$\alpha\left[\frac{1}{4}\sec(\alpha n) + \sum_{k=1}^{\infty} \chi(k)\frac{\cos(\alpha nk)}{e^{\alpha^2 k} - 1}\right] = \beta\left[\frac{1}{4} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{\cosh(2\beta nk)}{\cosh(\beta^2 k)}\right]. \tag{4.1}$$

Berndt proves this result by using a transformation formula given by himself in a beautiful paper [5] on Eisenstein series with characters. As a matter of fact, Eq. (4.1) can be easily obtained by means of the familiar Poisson summation formula

$$\frac{1}{2}f(0) + \sum_{k=1}^{\infty} f(k) = \int_0^{\infty} f(x)dx + 2\sum_{k=1}^{\infty}\int_0^{\infty} f(x)\cos(2k\pi x)dx. \tag{4.2)]}$$

We take

$$f(x) = \frac{\cosh(2\beta nx)}{\cosh(\beta^2 x)}. \tag{4.3}$$

Then [7,11(27)]

$$\begin{aligned} \int_0^{\infty} f(x)\cos(2k\pi x)dx &= \tfrac{\alpha}{4\beta}[\operatorname{sech}(k\alpha^2 - in\alpha) + \operatorname{sech}(k\alpha^2 + in\alpha)] \\ &= \tfrac{\alpha}{2\beta}\sec(n\alpha) \quad k = 0. \end{aligned} \tag{4.4}$$

Now, it is straightforwardly shown that

$$\operatorname{sech} x = 2\sum_{r=1}^{\infty} \chi(r) e^{-rx} \tag{4.5}$$

Thus

$$\operatorname{sech}(k\alpha^2 - in\alpha) + \operatorname{sech}(k\alpha^2 + in\alpha) = 4\sum_{r=1}^{\infty} \chi(r) e^{-rk\alpha^2} \cos(nr\alpha), \tag{4.6}$$

whence

$$\sum_{k=1}^{\infty} [\operatorname{sech}(k\alpha^2 - in\alpha) + \operatorname{sech}(k\alpha^2 + in\alpha)] = 4\sum_{r=1}^{\infty} \chi(r) \frac{\cos(\alpha nr)}{e^{\alpha^2 r} - 1} \tag{4.7}$$

and Eq. (4.1) follows at once.

5. Comments on four Lambert series identities

In his second notebook, on page 264, Ramanujan states four remarkable identities which, following Berndt [4], we write compactly as

$$L_j(q) = R_j(q) \quad , \qquad j = 1, \ldots, 4 \tag{5.1}$$

where, for $|q| < 1$

$$L_1 = \frac{1}{\varphi^2(-q)} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2 q^{\frac{1}{2}n(n+1)}}{1+q^n} \tag{5.2}$$

$$L_2 = \frac{1}{\psi^2(q)} \sum_{n=1}^{\infty} (-1)^{n-1} (2n-1)^2 q^{n(n-1)} \frac{1+q^{2n-1}}{1-q^{2n-1}} \tag{5.3}$$

$$L_3 = \frac{1}{\psi^2(q)} \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1) q^{n(n+1)-1}}{(1-q^{2n-1})^2} \tag{5.4}$$

$$L_4 = \frac{1}{\varphi^2(-q)} \sum_{n=1}^{\infty} (-1)^{n-1} n q^{\frac{1}{2}n(n+1)} \frac{1-q^n}{(1+q^n)^2} \tag{5.5}$$

$$R_1 = \sum_{n=1}^{\infty} q^{n(2n-1)} \frac{1+q^{2n-1}}{(1-q^{2n-1})^2} \tag{5.6}$$

(5.7) $$R_2 = 1 + 8\sum_{n=1}^{\infty}(-1)^n \frac{q^{n(n+1)}}{(1+q^n)^2}$$

(5.8) $$R_3 = \sum_{n=1}^{\infty}(-1)^n n q^{n^2}\frac{1+q^{2n}}{1-q^{2n}}$$

(5.9) $$R_4 = \sum_{n=1}^{\infty}\frac{nq^{\frac{1}{2}n(n+1)}}{1-q^n},$$

in (5.2)–(5.5),

(5.10) $$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} = \frac{1}{2}q^{-\frac{1}{8}}\vartheta_2(\sqrt{q})$$

and

(5.11) $$\varphi(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \vartheta_3(q)$$

are theta functions.

G. E. Andrews [1] has given brilliant proofs of these identities employing tools borrowed from the realm of basic hypergeometric series, in particular Bailey pairs and Bailey's non terminating extension of the q-analogue of Whipple's theorem. However, as Berndt [4,155] remarks, Andrews' proofs are probably unlike those found by Ramanujan, who undoubtedly possessed methods that we are unable to discern.

We first point out that Andrews' analysis can be somewhat simplified. As a matter of fact, it suffices to prove only two of the four identities, for instance $L_3 = R_3$ and $L_4 = R_4$, since it is not difficult to see that

(5.12) $$L_1 + L_4 = R_1 + R_4$$

and

(5.13) $$L_2 - 8L_3 = R_2 - 8R_3$$

Of course, (5.12) and (5.13) do not imply $L_1 = R_1$, $L_4 = R_4$ and $L_2 = R_2$, $L_3 = R_3$; perhaps, Ramanujan was aware of (5.12), (5.13), and was led to this identification by examinating the first few terms of the expansions of both sides.

In order to establish (5.12) and (5.13), we need the Lambert series ([4], p. 152-153, Entries (18) and (20))

$$\varphi^2(-q) = 1 + 4\sum_{n=1}^{\infty}(-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{1+q^n} \tag{5.14}$$

and

$$\psi^2(q) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \frac{1+q^{2n+1}}{1-q^{2n+1}} \tag{5.15}$$

together with the formula

$$u(x,y,\alpha) \equiv \sum_{n=0}^{\infty} \frac{x^n}{1-\alpha y^n} = \sum_{n=0}^{\infty} (\alpha x)^n y^{n^2} \left(\frac{1}{1-\alpha y^n} + \frac{x y^n}{1-x y^n}\right) \tag{5.16}$$

which is readily verified by comparing the coefficients of α^k on both sides.

Differentiating (5.14) and (5.15), we find

$$L_1 + L_4 = -q\frac{d}{dq}\log\varphi(-q) \tag{5.17}$$

$$L_2 - 8L_3 = 1 + 8q\frac{d}{dq}\log\psi(q) \tag{5.18}$$

that is, by using the well-known infinite products for φ and ψ

$$L_1 + L_4 = 2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} = 2\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} \tag{5.19}$$

$$L_2 - 8L_3 = 1 + 8\sum_{n=1}^{\infty}(-1)^{n-1}\frac{nq^n}{1-q^n} = 1 + 8\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}. \tag{5.20}$$

On the other hand, from (5.16)

(5.21)

$$\sum_{n=1}^{\infty}\frac{x^n}{(1-\alpha y^n)^2} = \sum_{n=1}^{\infty}(\alpha x)^n y^{n(n-1)}\left[\frac{n}{\alpha}\left(\frac{1}{1-\alpha y^n} + \frac{x y^{n-1}}{1-x y^{n-1}}\right) + \frac{y^n}{(1-\alpha y^n)^2}\right].$$

Taking $x = y = q^2$, $\alpha = q$, this gives the identity

$$2\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - \sum_{n=1}^{\infty} q^{n(2n-1)}\frac{1+q^{2n-1}}{(1-q^{2n-1})^2} = \sum_{n=1}^{\infty}\frac{nq^{\frac{1}{2}n(n+1)}}{1-q^n} \tag{5.22}$$

which, by (5.19), (5.6) and (5.9), is just (5.12).

Taking $x = y = q$, $\alpha = -1$ in (5.21), we get

$$-\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} + \sum_{n=1}^{\infty}(-1)^n \frac{q^{n(n+1)}}{(1+q^n)^2} = \sum_{n=1}^{\infty}(-1)^n n q^{n^2} \frac{1+q^{2n}}{1-q^{2n}} \tag{5.23}$$

equivalent, by (5.20), (5.7) and (5.8), to (5.13). Incidentally, since - as shown by Andrews - the r.h.s. of (5.22) coincides with L_4, we have a straightforward proof of the identity $L_4 = R_4$. Similarly, (5.23) amounts to $L_3 = R_3$.

Next, we derive some other results of this type. From the Fourier series for the Jacobian elliptic function cn [11],

$$cn(\vartheta_3^2 x) = \frac{4}{\vartheta_2^2} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}} \cos(2n+1)x, \tag{5.24}$$

recalling that $cnu = 1 - \frac{u^2}{2} + O(u^4)$, we have

$$\vartheta_2^2 \vartheta_3^4 = 4 \sum_{n=0}^{\infty} (2n+1)^2 \frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}}, \tag{5.25}$$

whence, with $q \to -q$

$$\vartheta_2^2 \vartheta_4^4 = 4 \sum_{n=0}^{\infty} (-1)^n (2n+1)^2 \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}}, \tag{5.26}$$

that is, in Ramanujan's notation

$$\psi^2(q)\varphi^4(\sqrt{q}) = \sum_{n=0}^{\infty}(2n+1)^2 \frac{q^{\frac{n}{2}}}{1+q^{n+\frac{1}{2}}} = (2x\frac{d}{dx}+1)^2 u(x,q,-q^{\frac{1}{2}})\Big|_{x=q^{\frac{1}{2}}} \tag{5.27}$$

(5.28)

$$\psi^2(q)\varphi^4(-\sqrt{q}) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^2 \frac{q^{\frac{n}{2}}}{1-q^{n+\frac{1}{2}}} = (2x\frac{d}{dx}+1)^2 u(x,q,q^{\frac{1}{2}})\Big|_{x=-q^{\frac{1}{2}}},$$

u being defined by (5.16). Performing the indicated differentiations, we obtain the formulas

$$\psi^2(q)\varphi^4(\sqrt{q}) = L_2\psi^2(q) + 8q^{\frac{1}{2}} \sum_{n=0}^{\infty}(-1)^n q^{n(n+2)} \frac{n(1-q^{n+\frac{1}{2}})+1}{(1-q^{n+\frac{1}{2}})^3}, \tag{5.29}$$

$$\psi^2(q)\varphi^4(-\sqrt{q}) = L_2\psi^2(q) - 8q^{\frac{1}{2}} \sum_{n=0}^{\infty}(-1)^n q^{n(n+2)} \frac{n(1+q^{n+\frac{1}{2}})+1}{(1+q^{n+\frac{1}{2}})^3}, \tag{5.30}$$

which, amazingly, involve the series $L_2(q)$. From these, observing that $\varphi^4(\sqrt{q}) - \varphi^4(-\sqrt{q}) = 16q^{\frac{1}{2}}\psi^4(q)$ (this is nothing but the standard theta function identity $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$), we obtain a Lambert series for $\psi^6(q)$

$$\psi^6(q) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+2)}[n\frac{1+q^{2n+1}}{(1-q^{2n+1})^2} + \frac{1+3q^{2n+1}}{(1-q^{2n+1})^3}], \tag{5.31}$$

which, as far as we know, does not appear in Ramanujan's notebooks. On the other hand, adding (5.29) and (5.30) and expressing L_2 in terms of L_3 through (5.20), we get, after a straightforward simplification

$$\frac{1}{\psi^2(q)}\sum_{n=1}^{\infty}(-1)^n q^{n(n+1)}\frac{1+q^{2n-1}}{(1-q^{2n-1})^3} = \frac{1}{2}q^2\frac{d}{dq}\log\varphi(-q). \tag{5.32}$$

We also observe that, from (5.24), we have

$$\psi^2(q) = \sum_{n=0}^{\infty}\frac{q^{\frac{n}{2}}}{1+q^{n+\frac{1}{2}}} = u(q^{\frac{1}{2}}, q, -q^{\frac{1}{2}}), \tag{5.33}$$

that is, a Lambert series simpler than (5.15), which indeed follows from (5.33) by applying the transformation (5.16). Differentiating (5.33), we find

$$2q\psi(q)\psi'(q) = \frac{1}{2}(P-Q) - N, \tag{5.34}$$

where

$$N = q^{-\frac{1}{2}}\sum_{n=1}^{\infty}(-1)^n\frac{nq^{\frac{n}{2}}}{(1-q^{n-\frac{1}{2}})^2} = -q^{-\frac{1}{2}}\sum_{n=1}^{\infty}\frac{nq^{\frac{n}{2}}}{(1+q^{n-\frac{1}{2}})^2}, \tag{5.35}$$

$$P = q^{-\frac{1}{2}}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n}{2}}}{(1-q^{n-\frac{1}{2}})^2} = -q^{-\frac{1}{2}}\sum_{n=1}^{\infty}\frac{nq^{\frac{n}{2}}}{1+q^{n-\frac{1}{2}}}. \tag{5.36}$$

$$Q = q^{-\frac{1}{2}}\sum_{n=1}^{\infty}\frac{q^{\frac{n}{2}}}{(1+q^{n-\frac{1}{2}})^2} = -q^{-\frac{1}{2}}\sum_{n=1}^{\infty}(-1)^n\frac{nq^{\frac{n}{2}}}{1-q^{n-\frac{1}{2}}}. \tag{5.37}$$

Not surprisingly, these quantities are related to

$$A_r = \sum_{n=1}^{\infty}(-1)^{n-1}n^r q^{n(n-1)}\frac{1+q^{2n-1}}{1-q^{2n-1}}, \tag{5.38}$$

$$B_r = \sum_{n=1}^{\infty}(-1)^n n^r \frac{q^{n(n+1)-1}}{(1-q^{2n-1})^2}, \tag{5.39}$$

that is, to the various pieces of

$$L_2\psi^2(q) = 4(A_2 - A_1) + A_0, \tag{5.40}$$

$$L_3\psi^2(q) = 2B_1 - B_0. \tag{5.41}$$

Indeed, from (see (5.21))

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{nx^n}{(1-\alpha y^n)^2} &= \sum_{n=1}^{\infty}(\alpha x)^n y^{n(n-1)} \Big[\frac{n^2}{\alpha}\Big(\frac{1}{1-\alpha y^n} + \frac{xy^{n-1}}{1-xy^{n-1}}\Big) \\ &\quad + \frac{ny^n}{(1-\alpha y^n)^2} + \frac{n}{\alpha}\frac{xy^{n-1}}{(1-xy^{n-1})^2}\Big]\end{aligned} \tag{5.42}$$

by appropriately specializing the parameters x, y and α, we get

$$N = -A_2 + 4B_1 \tag{5.43}$$

$$P - Q = -2A_1 + 4B_0, \tag{5.44}$$

furthermore, with

$$M = q^{\frac{1}{2}}\sum_{n=1}^{\infty}(-1)^n \frac{nq^{\frac{3}{2}n}}{(1-q^{n+\frac{1}{2}})^2} = -q^{\frac{1}{2}}\sum_{n=1}^{\infty} \frac{nq^{\frac{3}{2}n}}{(1+q^{n+\frac{1}{2}})^2} \tag{5.45}$$

we find

$$\frac{M-N}{\psi^2(q)} = 1 + 4\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}. \tag{5.46}$$

Many other results can be obtained by considering the Lambert series

$$\varphi^2(-q) = -1 + 4\sum_{n=0}^{\infty}(-1)^n\frac{q^n}{1+q^{2n}} = -1 + 4u(-q, q^2, -1), \tag{5.47}$$

simpler than (5.14) and related to it through (5.16), as well as formulas like (5.25), arising from the Fourier series for the various Jacobian elliptic functions. However, we do not investigate this possibility in the context of the present paper.

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Dipartimento di Fisica, Università di Milano
Istituto Nazionale di Fisica Nucleare, Sezione di Milano
Via Celoria 16,20133 Milano, ITALY

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