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On the Roots of a Polynomial

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We give here an upper bound of the roots of a polynomial, found by the convenient "summand-setting" in the polynomial. This bound is given in a very general form so that many special cases can be derived from it.

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Theorem. *Let the polynomial*

$$P = a_1x^{s_1} + \ldots + a_kx^{s_k} - b_1x^{t_1} - \ldots - b_lx^{t_l}, \tag{1}$$

be given, where the coefficients a_i, b_j are positive real numbers with $s_1 > \ldots s_k$, $t_1 > \ldots > t_l$, $s_1 > t_1$, $s_kt_l = 0$ and all the powers differ: $s_i \neq t_j$.

Then the real roots (if any) of (1) are bounded from above by

$$M = \max(M_1, M_2), \tag{2}$$

where

$$M_1 = \inf_{m_i} \max_{i} \inf_{n_{iw}} \max_{w} \left(\frac{m_ib_w}{a_in_{iw}}\right) \tag{3}$$

(4)

$$(i = 1, \ldots, i_0;\ w = 1, \ldots, j_0;\ \forall i\ \forall w\ m_i > 0, n_{iw} > 0;\ \sum_{i=1}^{i_0} m_i = 1;\ \sum_{w=1}^{j_0} n_{iw} = 1)$$

and

$$M_2 = \inf_{m_j} \max_{j} \inf_{n_{jv}} \max_{v} \left(\frac{m_jb_v}{a_jn_{jv}}\right)^{1/(s_j - t_v)} \tag{5}$$

(6)

$$(j = i_0{+}1, \ldots, k;\ v = j_0, \ldots, l;\ \forall j\ \forall v\ m_j > 0, n_{jv} > 0;\ \sum_{j=i_0+1}^{k} m_j = 1;\ \sum_{v=j_0}^{l} n_{jv} = 1)$$

with $i_0 = \max\{i | s_i > t_1\}$ *and* $j_0 = \min\{j | s_k > t_j\}$.

(For the case where $i_0 = k$, i.e. $j_0 = 1$, see (7) below.)

Proof. Indeed, (1) can be represented in the form $P = P_1 + P_2$, where

$$P_1 = a_1 x^{s_1} + \ldots + a_{i_0} x^{s_{i_0}} - b_1 x^{t_1} - \ldots - b_{j_0-1} x^{t_{j_0-1}},\ s_{i_0} > t_1$$

and

$$P_2 = a_{i_0+1} x^{s_{j_0+1}} + \ldots + a_k x^{t_{j_0}} - \ldots - b_l x^{t_l},\ s_k > t_{j_0}.$$

If we choose positive numbers m_i such that $\sum_{i=1}^{i_0} m_i = 1$, then

$$P_1 = a_1 \left(x^{s_1} - \frac{m_1 b_1}{a_1} x^{t_1} - \ldots - \frac{m_1 b_{j_0-1}}{a_1} x^{t_{j_0-1}} \right) + \ldots$$

$$+ a_{i_0} \left(x^{s_0} - \frac{m_{i_0} b_1}{a_{i_0}} x^{t_1} - \ldots - \frac{m_{i_0} b_{j_0-1}}{a_{i_0}} x^{t_{j_0-1}} \right) = P_{11} + P_{12} + \ldots + P_{1 i_0},$$

and if $\forall i = 1, \ldots, i_0$, we choose positive numbers n_{iw} such that $\sum_{w=1}^{j_0-1} n_{iw} = 1$, then further transformation of the expression above gives $\forall i = 1, \ldots, i_0$,

$$P_{1i} = a_i \left(x^{t_i} \left(n_{i1} x^{s_i - t_1} - \frac{m_i b_1}{a_i} \right) + \ldots + x^{t_{j_0}-1} \left(n_{i, j_0-1} x^{s_i - t_{j_0-1}-1} - \frac{m_i b_{j_0-1}}{a_i} \right) \right).$$

Clearly, $P_{1i} \geq 0$ whenever $x \geq \max_w \left(\frac{m_1 b_w}{a_i n_{iw}} \right)^{1/(s_i - t_w)}$ $(w = 1, \ldots, j_0 - 1)$ and, since the n_{iw} are any numbers satisfying (4), then P_{1i} has positive roots bounded from above by

$$\inf_{n_{iw}} \max_{w} \left(\frac{m_i b_w}{a_i n_{iw}} \right)^{1/(s_i - t_w)}.$$

This is true $\forall i = 1, \ldots, i_0$ and as m_i are any reals satisfying (4), then the polynomial P_1 has upper bound M_1 for its roots, given by (3) and (4).

Similarly, we get M_2 as an upper bound of the roots of P_2. This finally gives M in (2) as an upper bound for the roots of the polynomial P in (1).

If $s_k > t_1$, we get that P is actually of the form P_1, thus its roots are $\leq M'$, where

(7)
$$M' = \inf_{m_i} \max_{i} \inf_{n_{iw}} \max_{w} \left(\frac{m_i b_w}{a_i n_{iw}} \right)^{1/(s_i - t_w)}$$

$(i = 1, \dots, k;\ w = 1, \dots, l;\ \forall i\ \forall w\ m_i > 0, n_{iw} > 0;\ \sum_{i=1}^{k} m_i = 1;\ \sum_{w=1}^{l} n_{iw}) = 1).$

If $k = 1,\ s_1 > t_1$, then a little inspection of our proof shows that, in this case, we do not use either m_i or i; hence the roots of the polynomial

$$P' = a_1 x^{s_1} - b_1 x^{t_1} - \dots - b_l x^{t_l} \tag{8}$$

are bounded above by

$$M'' = \inf_{n_w} \max_{w} \left(\frac{b_w}{a_1 n_w} \right)^{1/(s_1 - t_w)},\ (w = 1, \dots, l;\ \forall w\ n_w > 0;\ \sum_{w=1}^{l} n_w = 1). \tag{9}$$

The results (8)-(9) is in fact due to Cauchy, but it was first pointed out by Fujiwara [1] (it can be also found in [3], p.338).

Now, using Cauchy's theorem (see [2], pp.122-123), all the zeros of the polynomial

$$P(z) = a_n z^n + \dots + a_1 z + a_0,\ a_n \neq 0 \tag{10}$$

lie in the disk $|z| \leq x_0$, where x_0 is the positive root of the equation $|a_n|z^n - |a_{n-1}z^{n-1} - \dots - |a_1|z - |a_0| = 0$ and we can bind x_0 by using (9).

Using standard tricks (replacing x by $-x$ and $\pm(1/x)$), we obtain a lower bound for the negative roots as well as a (positive) lower bound for the positive roots and a (negative) upper bound for the negative roots. This could help us to get regions of complex roots of the polynomial (10).

Various formulas can be obtained from our main result (2), choosing suitable coefficients (satisfying (4) and (6)). As a matter of fact, easy inspection of the proof of our Theorem shows that, instead of taking coefficients m, n such that $\sum m = 1, \sum n = 1$, we could choose the coefficients so that $\sum m > 1, \sum n < 1$ as in Example 2) below. This corresponds to "balancing" only parts of summands of the polynomial (1).

E x a m p l e s: $f \in \{m, n\}$ may have some of the following forms $(u \in \{i, j, iw, jv\}, c \in \{i_0, j_0 - 1, k - i_0, l - j_0 + 1\})$:

$$1) f_u = 1/c,\quad 2) f_u = 1/2^u,\quad 3) f_{uc} = \binom{c}{u} (\sqrt[c]{2} - 1)^u,$$

$$4) f_{uc} = \binom{c}{u} 1/(2^c - 1),\quad 5) f_u = d_u / (\sum_{i=1}^{c} d_i)$$

(d_i- real numbers which may depend on coefficients of the polynomial (1)).

The following problems are open and seem to be of interest:

Problem 1. *Under what conditions will the number M in (2) be the greatest root of the polynomial P in (1)?*

Problem 2. *Give a method for "effective" calculation of the constant M.*

Of interest are also computational aspects of our method, but we leave that discussion for another occasion.

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