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A Note on Jensen's Functional

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In this note we investigate some estimates of integral $\int_0^{2\pi} \log|f(e^{i\theta})|\frac{d\theta}{2\pi}$, if the polynomial $f(z)$ has concentration at low degrees measured by l_p-quasi-norm, $p \in]0,1[$.

Let $f(z) = a_0 + a_1 z + a_2 z^2 + \dots$ be a polynomial with complex coefficients, and let d, $0 < d < 1$, $k \in N$. We say that $f(z)$ has concentration d at degrees at most k, if:

$$\sum_{j\leq k} |a_j| \geq d \sum_{j\geq 0} |a_j|. \tag{1}$$

Other ways of measuring such a concentration can be expressed. For instance,

$$\left(\sum_{j\leq k} |a_j|^2\right)^{\frac{1}{2}} \geq d \left(\sum_{j\geq 0} |a_j|^2\right)^{\frac{1}{2}}, \tag{2}$$

or

$$\sum_{j\leq k} |a_j| \geq d\|f\|_\infty, \tag{3}$$

where

$$\|f\|_\infty = \max_\theta |f(e^{i\theta})|.$$

It is not hard to see that (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2), as and (2) $\not\Longrightarrow$ (3) $\not\Longrightarrow$ (1). But if $\|f\|_\infty \leq 1$ and $1 < p < 2$, then the following conditions are equivalent:

$$\left|f\right|^k\Big|_1 = \sum_{j\leq k} |a_j| \geq d, \tag{1'}$$

$$(2') \qquad \left|f|^k\right|_p = \left(\sum_{j\le k} |a_j|^p\right)^{\frac{1}{p}} \ge d,$$

$$(3') \qquad \left|f|^k\right|_2 = \left(\sum_{j\le k} |a_j|^2\right)^{\frac{1}{2}} \ge d.$$

Indeed, since $|f|_2 \le |f|_p \le |f|_1$ for every $p \in]1,2[$, it follows that $(3') \Longrightarrow (2') \Longrightarrow (1')$. Because $\left|f|^k\right|_2 \ge \frac{1}{\sqrt{k+1}}\left|f|^k\right|_1 \ge \frac{d}{\sqrt{k+1}} = d_1 = d_1(k,d)$, we have that $(1') \Longrightarrow (3')$.

In this note we investigate some estimates of the Jensen's functional

$$J(f) = \int_0^{2\pi} \log|f(e^{i\theta})|\frac{d\theta}{2\pi},$$

if the polynomial $f(z)$ has concentration d at low degrees measured by l_p quasi-norm, $p \in]0,1[$, i.e. if $f(z)$ satisfies:

$$(4) \qquad \sum_{j\le k} |a_j|^p \ge d \sum_{j\ge 0} |a_j|^p.$$

In the sequel, we shall normalize $f(z)$ under l_p-quasi-norm and assume that

$$|f|_p = \sum_{j\ge 0} |a_j|^p = 1.$$

Similarly as in [2] and [4], we have the following results:

Theorem 1 *Let $f(z) = \sum_{j\ge 0} a_j z^j$ be a polynomial which satisfies (4) and $|f|_p = 1$. Then there exists constant $C(d,k,p)$ and the functions $f_{d,k,p}(t)$ and $g_{d,k,p}(t)$ such that:*

a) $J(f) \ge C(d,k,p)$.

b) $J(f) \ge \sup_{1<t\le 3} f_{d,k,p}(t)$, where

$$f_{d,k,p}(t) = t \log d^{\frac{1}{p}} \left(\frac{t-1}{t+1}\right)^{k+1}.$$

c) $J(f) \ge \sup_{t>1} g_{d,k,p}(t)$, where

$$g_{d,k,p}(t) = \frac{t}{p} \log d^{\frac{1}{p}} \left(\left(\frac{t+1}{t-1}\right)^p - 1\right) \Big/ \left(\left(\frac{t+1}{t-1}\right)^{p(k+1)} - 1\right).$$

For the proofs of the b) and c) we use the following well known facts:

1^0 $a_j = \int_0^{2\pi} \frac{f(re^{i\theta})}{r^j e^{ij\theta}} \frac{d\theta}{2\pi}$, if $0 < r < 1$.

2^0 $|a_j| \leq |f(z_0)| r^{-j}$, where $|f(z_0)| = \max_{|z_0|=r} |f(z)| = M(f, r)$.

3^0 The classical Jensen's inequality and known transformation:

$$\log |f(z_0)| \leq \int_0^{2\pi} \log \left| f\left(\frac{z_0 + e^{i\theta}}{1 + \bar{z}_0 e^{i\theta}}\right)\right| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log |f(e^{i\theta})| \frac{1 - r^2}{|1 - z_0 e^{i\theta}|^2} \frac{d\theta}{2\pi},$$

where $|z_0| = r$. If $0 < r < 1$, then

$$\frac{1 - r}{1 + r} \leq \frac{1 - r^2}{|1 - \bar{z}_0 e^{i\theta}|^2} \leq \frac{1 + r}{1 - r}.$$

4^0 $J(f) = \int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi} < 0$, since $|f(e^{i\theta})| \leq |f|_1 \leq |f|_p = 1$, because the l_p-quasi-norm decreases with p.

Now, in the case of the b) (resp. c)) we have from (4)

$$d \leq \sum_{j \leq k} |a_j|^p \leq M^p(r, f) \sum_{j \leq k} r^{-pj} \leq M^p(r, f) r^{-p(k+1)}$$

for some $r \in]0, 2^{-1}]$, respectively

$$d \leq \sum_{j \leq k} |a_j|^p \leq M^p(r, f) \sum_{j \leq k} r^{-pj} = M^p(r, f) \frac{r^{-p(k+1)} - 1}{r^{-p} - 1}$$

for some $r \in]0, 1[$. Finally, we get the functions $f_{d,k,p}(t)$ and $d_{d,k,p}(t)$ after the change of variable:

$$t = \frac{1 + r}{1 - r}.$$

Now, since

$$f_{d,k,p}(t) = \frac{t}{p} \log d + t(k + 1) \log(t - 1) - t(k + 1) \log(t + 1),$$

we have that $\lim_{t \to 1+} f_{d,k,p}(t) = -\infty$ as and

$$f'_{d,k,p}(t) = \tfrac{1}{p} \log d + (k + 1)(\tfrac{1}{t-1} + \log \tfrac{t-1}{t+1}) \quad \text{and}$$
$$f''_{d,k,p}(t) = -\tfrac{4(k+1)}{(t^2-1)^2}.$$

It is clear that $\lim_{t\to 1+} f'_{d,k,p}(t) = +\infty$, i.e. $f'_{d,k,p}(t)$ decreases, because $f''_{d,k,p}(t) < 0$. Therefore, the function $f_{d,k,p}(t)$ has a maximum. This proves ((a) or (b)) that there exists constant $C(d,k,p)$ such that

$$J(f) \geq C(d,k,p), \quad \text{where}$$

$$C(d,k,p) = \max_{1<t\leq 3} f_{d,k,p}(t) \quad \text{or} \quad C(d,k,p) = \max_{t>1} g_{d,k,p}(t).$$

Taking $t = 2$,

$$C(d,k,p) \geq 2\log \frac{d^{\frac{1}{p}}}{3^{k+1}}, \quad \text{or} \quad C(d,k,p) \geq \frac{2}{p}\log 2 \cdot \frac{3^p - 1}{3^{p(k+1)} - 1}.$$

From the proofs of the a) and b) it follows that the estimate by the function $g_{d,k,p}(t)$ is better than the estimate by the function $f_{d,k,p}(t)$.

Otherwise, the precise value of $C(d,k,p)$ is unknown. However, B. Beauzamy showed in [2], that, for $d = \frac{1}{2}$,

$$C(\frac{1}{2}, k, 1) \leq -2k\log 2,$$

and that, asymptotically, when $k \to \infty$,

$$C(\frac{1}{2}, k, 1) \geq -2k.$$

The precise value of $C(d,k,p)$ has been computed for the class of Hurwitz polynomials. For this case, the best constant, denoted by $C^H(\frac{1}{2}, k, 1)$, is given by:

$$C^H(\frac{1}{2}, k, 1) = -(2k+1)\log 2, \quad k = 0, 1, 2... \quad \text{(see [5])}.$$

But outside this class of polynomials, nothing is known about the precise value of $C(d,k,1)$, even for small values of k (see [3]).

In the sequel, we have restricted ourselves to the best constant $C(d,k,p)$ for $p \in]1,2]$. The following theorem gives numerical asymptotic estimates for $C(d,k,p)$, $p \in]1,2]$ where $k \to +\infty$.

Theorem 2. *If the polynomial $f(z)$ has the concentration d at low degrees k and $|f|_p = 1, p \in]1,2]$, then $C(d,k,p) \geq -2k$ asymptotically, when $k \to +\infty$, where $J(f) \geq C(d,k,p)$.*

(See [2], Theorem 2 and Lemmas 4,5; [4], Proposition 1.)

P r o o f. From [4], Theorem 2 it follows that $C(d,k,p) = \max_{t>1} f_{d,k,p}(t)$, where

$$f_{d,k,p}(t) = h_{d,p}(t) + g_k(t) - \frac{1}{p} t \log\left(1 - (\frac{t-1}{t+1})^{p(k+1)}\right), \text{ i.e.,}$$

$$h_{d,p}(t) = t\log d - \frac{1}{2}t^2 + \frac{t}{p}\log[(t+1)^p - (t-1)^p]$$

$$g_k(t) = kt\log(t-1) - (k+1)t\log(t+1)$$

Also, from [4], Proposition 1, we know that the function $h_{d,p}(t) + g_k(t)$ takes its maximum at a point (unique) t_k such that $t_k \to +\infty$, when $k \to +\infty$. This follows from the equality

$$h'_{d,p}(t) + g'_k(t) = 0, \quad \text{i.e.,}$$

$$k = \frac{(t^2-1)\log(t+1) + t^2 - t - (t^2-1)h'_{d,p}(t)}{2t + (t^2-1)\log(t-1) - (t^2-1)\log(t+1)}$$

from which we easily deduce that $t_k \to +\infty$. Writing $\log(t \pm 1) = \log t + \log(1 \pm 1/t)$ and by the Taylor expansions of order 3 for $\log(1 \pm 1/t)$, we have that $h'_{d,p}(t_k) \approx -t_k$ when $k \to +\infty$, that is

$$\begin{aligned} k &= \tfrac{(t^2-1)\log(t+1)+t^2-1}{2t+(t^2-1)\log(t-1)-(t^2-1)\log(t+1)} + \tfrac{1-t-(t^2-1)h'_{d,p}(t)}{2t+(t^2-1)\log(t-1)} \\ &\approx \tfrac{3}{4}t_k^3\log t_k + \tfrac{3}{4}t_k^4 \approx \tfrac{3}{4}t_k^4, \quad \text{when} \quad k \to +\infty. \end{aligned}$$

Because $f_{d,k,p}(t) > h_{d,p}(t) + g_k(t)$, $t > 1$ and $t_k \log\left(1 - \left(\frac{t_k-1}{t_k+1}\right)^{p(k+1)}\right) \to 0$, when $k \to +\infty$, the value of $f_{d,k,p}(t_k)$ and the value of $h_{d,p}(t_k) + g_k(t_k)$ are asymptotically the same. All we have to do now is to compute $f_{d,k,p}(t_k) \approx h_{d,p}(t_k) + g_k(t_k) \approx -2k$ when $k \to +\infty$. Indeed, by Taylor expansions of order 3 for $\log(1 \pm 1/t)$ we have

$$\begin{aligned} h_{d,p}(t_k) &= t_k\log d - \tfrac{1}{2}t_k^2 + \tfrac{1}{p}t_k\log[(t_k+1)^p - (t_k-1)^p] \\ &= t_k\log d - \tfrac{1}{2}t_k^2 + \tfrac{1}{9}t_k\log 2pt_k^{p-1} + \tfrac{1}{p}t_k\log(1+\varphi(t_k)) \\ &\approx -\tfrac{1}{2}t_k^2, \quad \text{where} \quad \varphi(t_k) \to 0 \quad \text{and} \\ g_k(t_k) &\approx -2k \quad \text{(as in [2]).} \end{aligned}$$

Hence, $f_{d,k,p}(t_k) \approx -\frac{1}{2}t_k^2 - 2k \approx -2k$ when $k \to +\infty$, that is,

$$C(d,k,p) \geq -2k, \quad \text{asymptotically, when} \quad k \to +\infty$$

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The proof of the theorem is completed. ■

In the case $p \geq 2$ we have the following result.

Theorem 3. *Let $f(z)$ be the polynomial with $\|f\|_p = 1$ $(p \geq 2)$ which has the concentration d at low degrees, i.e.*

$$\left| f \right|^k \Big|_p \geq d,$$

then there exists a constant $C(d,k,p)$ such that $J(f) \geq C(d,k,p)$, where

$$C(d,k,p) = \max_{t>1} \frac{t}{p} \log d^p \left((\frac{t+1}{t-1})^p - 1 \right) / \left((\frac{t+1}{t-1})^{p(k+1)} - 1 \right) - \frac{1}{p} t^2,$$

as and $C(d,k,p) \geq -2k$, asymptotically, when $k \to +\infty$.

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