Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol.11, 1997, Fasc. 3-4

Every Riccati Equation Can Be Solved by Quadratures in a Wider Sense

D. Dimitrovski[1], *L. Stefanovska*[2], *M. Kujumdzieva-Nikolska*[3]

Presented by P. Kenderov

"Solved by quadratures in a wider sence" means that the solution of the analytical differential equation can be expressed by series of integrals of the coefficients of the equation. This is a natural generalization of the classic quadrature theory given by Liouville-Véssiot [1]. Hence, this paper proves that any Riccati equation can be solved by quadratures in a wider sense, so that the knowledge on particular solution is not requred at first.

1. Preliminaries

Let

$$y' = a(x)y^2 + b(x)y + c(x) \tag{1}$$

which is a general Riccati equation where it is supposed that the following conditions are satisfied for coefficients $a(x), b(x), c(x)$:

$H1$: $a(x)$, $b(x)$, $c(x)$ are defined on the interval $I \subseteq R$, which is symmetrical to the coordinate origin;

$H2$: $a(x)$, $b(x)$, $c(x)$ are analytical functions on I;

$H3$: $a(x) \neq 0$ for $x \in I$.

It is known that by the substitution

$$y = -\frac{1}{a(x)}z$$

equation (1) can be transformed into

$$z' = -z^2 + (b + \frac{a'}{a})z - ac$$

and by the substitution

$$z = \frac{u'}{u}$$

a linear differential equation of 2nd order is obtained

$$u'' - (b + \frac{a'}{a})u' + acu = 0.$$

By the substitution

$$u = e^{\frac{1}{2}\int(b+\frac{a'}{a})dx} . w$$

it can be transformed into a canonical equation of 2nd order

$$w'' + A(x)w = 0, \tag{2}$$

where

$$A(x) = \frac{1}{2}(b + \frac{a'}{a})' - \frac{1}{4}(b + \frac{a'}{a})^2 + ac. \tag{3}$$

2. Result: Quadrature in a wider sense

Based on hypotheses $H1, H2$ and $H3$, $A(x)$ is also an analytical function and it can be expanded in a convergent series

$$A(x) = \sum_{k=0}^{\infty} a_k x^k \tag{4}$$

where a_k are constants which depend on the terms in the power series of the functions $a(x)$, $b(x)$, and $c(x)$. Applying the Cauchy theorem a unique solution in the form of power series is obtained:

$$w(x) = \sum_{k=0}^{\infty} c_k x^k \tag{5}$$

where c_k are constants. Solving (2) by series, after substituting (4) and (5) in (2), we have

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} c_k x^k = 0$$

and using the method of unknown coefficients we get

$$1.2c_2 + a_o c_o = 0$$

$$2.3c_3 + a_o c_1 + a_1 c_o = 0$$

$$3.4c_4 + a_o c_2 + a_1 c_1 + a_1 c_o = 0$$

$$. \quad . \quad . \quad . \quad .$$

$$(k-1)kc_k + a_o c_{k-2} + a_1 c_{k-3} + \cdots + a_{k-2} c_o = 0$$

$$. \quad . \quad . \quad . \quad .$$

If we suppose that c_o and c_1 are arbitrary cofficients, we can find that all constants c_j $(i \geq 2)$ depend on c_o and c_1 as follows

$$
\begin{aligned}
c_2 &= -\tfrac{1}{2.1} a_o c_o \\
c_3 &= -\tfrac{1}{3.2}(a_o c_1 + a_1 c_o) \\
c_4 &= -\tfrac{1}{4.3}(a_o c_2 + a_1 c_1 + a_2 c_o) = \\
&= -\tfrac{1}{4.3}[a_o(-\tfrac{1}{2.1} a_o c_o) + a_1 c_1 + a_2 c_o] \\
&. \quad . \quad . \quad . \quad . \\
c_k &= -\tfrac{1}{(k-1)k}[a_o c_{k-2} + \cdots + a_{k-2} c_o] \\
&. \quad . \quad . \quad . \quad .
\end{aligned}
\tag{6}
$$

Substituting (6) in (5), the solution (5) is expanded in the series with numberical coefficients which depend on known a_k and two arbitrary constants c_o and c_1.

By rearranging the terms, we have

$$\begin{aligned} w(x) &= c_o[1 - \tfrac{1}{1.2}a_o x^2 - \tfrac{1}{2.3}a_1 x^3 - \tfrac{1}{3.4}a_2 x^4 - \tfrac{1}{4.5}a_3 x^5 - \ldots \\ &+a_o\left(\tfrac{a_o}{1.2.3.4}x^4 + \tfrac{a_1}{2.3.4.5}x^5 + \tfrac{a_2}{3.4.5.6}x^6 + \ldots\right) + \\ &+a_1\left(\tfrac{a_o}{1.2.4.5}x^5 + \tfrac{a_1}{2.3.5.6}x^6 + \ldots\right) + \\ &+a_2\left(\tfrac{a_o}{1.2.5.6}x^6 + \tfrac{a_1}{2.3.6.7}x^7 + \ldots\right) + \ldots] + \\ &+c_1[x - \tfrac{1}{2.3}a_o x^3 - \tfrac{1}{3.4}a_1 x^4 - \tfrac{1}{4.5}a_2 x^5 - \ldots \\ &+a_o\left(\tfrac{a_o}{2.3.4.5}x^5 + \tfrac{a_1}{3.4.5.6}x^6 + \ldots\right) + \\ &+a_1\left(\tfrac{a_o}{2.3.5.6}x^6 + \tfrac{a_1}{3.4.6.7}x^7 + \ldots\right) + \ldots] \end{aligned}$$

and we can see that the series in brackets behind c_o and c_1 are in fact double integrals of the terms of series (4), i.e.,

$$\begin{aligned} w(x) &= c_o[1 - \textstyle\int_0^x dx \int_0^x a_o dx - \int_0^x dx \int_0^x a_1 x dx - \int_0^x dx \int_0^x a_2 x^2 dx - \ldots \\ &+a_o\left(\textstyle\int_0^x dx \int_0^x dx \int_0^x dx \int_0^x a_o dx + \int_0^x dx \int_0^x dx \int_0^x dx \int_0^x a_1 x dx + \ldots\right) \\ &+a_1\left(\textstyle\int_0^x dx \int_0^x x dx \int_0^x dx \int_0^x a_o dx + \int_0^x dx \int_0^x x dx \int_0^x dx \int_0^x a_1 x dx + \ldots\right) + \ldots] \\ &+c_1[x - \textstyle\int_0^x dx \int_0^x x a_o dx - \int_0^x dx \int_0^x a_1 x^2 dx - \int_0^x dx \int_0^x a_2 x^3 dx - \ldots \\ &+a_o\left(\textstyle\int_0^x dx \int_0^x dx \int_0^x dx \int_0^x a_o x dx + \int_0^x dx \int_0^x dx \int_0^x dx \int_0^x a_1 x^2 dx + \ldots\right) \\ &+a_1\left(\textstyle\int_0^x dx \int_0^x x dx \int_0^x dx \int_0^x a_o x dx + \int_0^x dx \int_0^x x dx \int_0^x dx \int_0^x a_1 x^2 dx + \ldots\right) + \ldots \end{aligned}$$

or using (4) by condenzation,

$$\begin{aligned} w(x) &= c_o[1 - \textstyle\int_0^x dx \int_0^x A(x)dx + \int_0^x dx \int_0^x A(x)dx \int_0^x dx \int_0^x A(x)dx \\ &- \textstyle\int_0^x dx \int_0^x A(x)dx \int_0^x dx \int_0^x A(x)dx \int_0^x dx \int_0^x A(x)dx + \ldots] \\ &+c_1[x - \textstyle\int_0^x dx \int_0^x xA(x)dx + \int_0^x dx \int_0^x A(x)dx \int_0^x dx \int_0^x xA(x)dx \\ &- \textstyle\int_0^x dx \int_0^x A(x)dx \int_0^x dx \int_0^x A(x)dx \int_0^x dx \int_0^x xA(x)dx + \ldots] \\ &= c_o w_1 + c_1 w_2, \end{aligned}$$

where

$$w_1(x) = \sum_{k=0}^{\infty}(-1)^k \underbrace{\int_0^x \int_0^x A(x)dx^2 \; . \; . \; . \int_0^x \int_0^x A(x)dx^2}_{k} \tag{7}$$

$$w_2(x) = x + \sum_{k=1}^{\infty}(-1)^k \underbrace{\int_0^x \int_0^x A(x)dx^2 \; . \; . \; . \int_0^x \int_0^x xA(x)dx^2}_{k} \tag{8}$$

(the k-double integrals are k-fold double integrals).

Theorem 1. *Let:*
(i) hypotheses H1, H2 and H3 be valid;
(ii) function $A(x)$ be given by (3);
then:
1° functions $w_1(x)$ and $w_2(x)$ given by (7) and (8) are solutions of equation (2) for $x \in I$;
2° functions $w_1(x)$ and $w_2(x)$ are linearly independent for $x \in I$;
3° the general solution of equation (2) is

$$w(x) = c_o w_1(x) + c_1 w_2(x),$$

where c_o and c_1 are arbitrary constants.

Proof.
1° If we differentiate (7) two times, we get

$$w''_1(x) = -A(x) + A(x)\int_0^x dx \int_0^x A(x)dx - A(x)\int_0^x dx \int_0^x A(x)dx \int_0^x dx$$

$$\int_0^x A(x)dx + \ldots$$

$$= -A(x)[1 - \int_0^x dx \int_0^x A(x)dx + \int_0^x dx \int_0^x A(x)dx \int_0^x dx \int_0^x A(x)dx - \ldots]$$

$$= -A(x)w_1(x).$$

The proof that w_2 is another solution of (2) is the same.
2° If we take $w_1(0) = 1$, $w'_1(0) = 1$, $w_2(0) = 0$ and $w'_2(0) = 1$, Then the Wronskian determinant

$$W(w_1(0),\, w_2(0)) = \begin{vmatrix} w_1(0) & w_1'(0) \\ w_2(0) & w_2'(0) \end{vmatrix} = 1 \neq 0$$

and the solutions $w_1(x)$ and $w_2(x)$ are linearly independent. ∎

Theorem 2. *The two independent particular solutions of the Riccati equation (1) are*

(9) $$y_1 = -\frac{1}{a(x)}.\frac{u_1'}{u_1} \quad \textit{and} \quad y_2 = -\frac{1}{a(x)}.\frac{u_2'}{u_2},$$

where

(10) $$u_1 = e^{\frac{1}{2}\int(b+\frac{a'}{a})dx}.\sum_{k=0}^{\infty}(-1)^k \underbrace{\int_0^x \int_0^x A(x)dx^2 \;.\;\;.\;\;. \int_0^x \int_0^x A(x)dx^2}_{\rightarrow\; k \;\leftarrow},$$

(11)
$$u_2 = e^{\frac{1}{2}\int(b+\frac{a'}{a})dx}.(x + \sum_{k=1}^{\infty}(-1)^k \underbrace{\int_0^x \int_0^x A(x)dx^2 \;.\;\;.\;\;. \int_0^x \int_0^x xA(x)dx^2}_{\rightarrow\; k \;\leftarrow}),$$

and $A(x)$ is given by (3).

If we adopt that "solved by quadratures in a wider sense" means series of integrals of the coefficients of the equation, we have the following theorem.

Theorem 3. *Every Riccati equation (1) can be solved by quadratures in a wider sense for any case of analytical coefficients $a(x)$, $b(x)$, $c(x)$; $a(x) \neq 0$.*

Proof. According to the general theory, if we know two particular solutions of equation (1), then the general solution can be found by one quadrature

$$\frac{y - y_1}{y - y_2} = Ce^{-\int a(x)(y_1 - y_2)dx}$$

hence,

$$\frac{y - y_1}{y - y_2} = C\frac{u_1}{u_2},$$

where y_1, y_2 and u_1, u_2 are given by (9),(10) and (11). ■

3. Example

The Riccati equation

$$y' = -y^2 - xy - 2 \tag{12}$$

by the substitution

$$y = \frac{u'}{u}$$

is transformed into a linear differential equation of 2nd order

$$u'' + xu' + 2u = 0 \tag{13}$$

and by the substitution

$$u = e^{-\frac{1}{2}\int x dx}\, w = e^{-\frac{x^2}{4}} w$$

is transformed into a canonical type

$$w'' + (\frac{3}{2} - \frac{x^2}{4})w = 0$$

with a particular solution according (8):

$$\begin{aligned}
w_2 &= x - \int_0^x \int_0^x x(\tfrac{3}{2} - \tfrac{x^2}{4})dx^2 + \int_0^x \int_0^x (\tfrac{3}{2} - \tfrac{x^2}{4})dx^2 \int_0^x \int_0^x x(\tfrac{3}{2} - \tfrac{x^2}{4})dx^2 - \ldots \\
&= x - \left(\tfrac{x^3}{4} - \tfrac{x^5}{5.4^2}\right) + \left(\tfrac{3}{2^3.2^2.5}x^5 - \tfrac{13}{7.3.2^6.5}x^7 + \tfrac{1}{9.8.4^3.5}x^9\right) \\
&- \left(\tfrac{9}{7.6.8.4.5.2}x^7 + \ldots\right) + \ldots \\
&= x[1 - \tfrac{x^2}{4} + \tfrac{x^4}{2^4}\left(\tfrac{1}{5} + \tfrac{3}{2.5}\right) - \tfrac{x^6}{2^6}\left(\tfrac{13}{7.3.5} + \tfrac{9}{7.3.5.2}\right) + \ldots] \\
&= x[1 - (\tfrac{x}{2})^2 + \tfrac{[(\frac{x}{2})^2]^2}{2!} - \tfrac{[(\frac{x}{2})^2]^3}{3!} + \ldots] = xe^{-(\frac{x}{2})^2} = xe^{-\frac{x^2}{4}}.
\end{aligned}$$

So,

$$u_2 = e^{-\frac{x^2}{4}} w_2 = e^{-\frac{x^2}{4}} xe^{-\frac{x^2}{4}} = xe^{-\frac{x^2}{2}}$$

is a particular solution of the linear equation of 2nd order (13), and

$$y = \frac{u_2'}{u_2} = \frac{1 - x^2}{x}$$

is a particular solution of the Riccati equation (12).

References

[1] I. K a p l a n s k i i, *Introduction in Differential Algebra* (In Russian). Moscow, I. L., 1959.
[2] M. K u r e n s k y, *Atti Accad.Lincei (6)* **9** (1929).
[3] L. T c h a k a l o f f, *Giornale Mat.* **63** (1925), 139.

[1] *Institute of Mathematics*
Faculty of Science
Skopje, MACEDONIA

Received: 20.01.1994

[2] *Faculty of Technology and Metallurgy*
Skopje, MACEDONIA

[3] *Faculty of Electrical Engineering*
Skopje, MACEDONIA