Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol.11, 1997, Fasc. 3-4

On Schwarz's Inequality in Hilbert Space

B. Mond and J. E. Pečarić***

Presented by Ž. Mijajlocić

The authors find a Schwarz's inequality for self-adjoint operators on a Hilbert space to itself.

1. Schwarz's inequality

The Schwarz inequality [3, p.198] in a Hilbert space (with real or complex scalars) implies that

$$|(Ax, Bx)|^2 \le (Ax, Ax)(Bx, Bx), \tag{1}$$

where x is an arbitrary vector of the Hilbert space H and A and B are self-adjoint operators on H to itself.

Moreover, in the case of several vectors x_i and several self-adjoint operators $A_i, B_i (i = 1, \dots, n)$, we have by the triangle inequality, by (1) and by Schwarz's inequality for real numbers, respectively,

$$\begin{aligned} |\textstyle\sum_{i=1}^n (A_i x_i, B_i x_i)|^2 &\le (\textstyle\sum_{i=1}^n |(A_i x_i, B_i x_i)|)^2 \\ &\le \left(\textstyle\sum_{i=1}^n (A_i x_i, A_i x_i)^{1/2} (B_i x_i, B_i x_i)^{1/2}\right)^2 \\ &\le (\textstyle\sum_{i=1}^n (A_i x_i, A_i x_i))(\textstyle\sum_{i=1}^n (B_i x_i, B_i x_i)), \end{aligned}$$

i.e., we have proved

$$|\sum_{i=1}^n (A_i x_i, B_i x_i)|^2 \le \left(\sum_{i=1}^n (A_i x_i, A_i x_i)\right)\left(\sum_{i=1}^n (B_i x_i, B_i x_i)\right), \tag{2}$$

a Schwarz's inequality in Hilbert space for several operators.

If $x_1 = \ldots = x_n = x$, we obtain

$$|\sum_{i=1}^{n}(A_ix, B_ix)|^2 \leq \left(\sum_{i=1}^{n}(A_ix, A_ix)\right)\left(\sum_{i=1}^{n} B_ix, B_ix)\right). \tag{3}$$

Another version of Schwarz's inequality with several operators is given in [1], where the following is proved:

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be permutable self-adjoint operators on H to itself. Suppose that $(\sum_{k=1}^{n} A_k^2)^{-1}$ exists. Then the following operator inequality holds

$$\left(\sum_{k=1}^{n} A_kB_k\right)^2 \leq \sum_{k=1}^{n} A_k^2 \sum_{k=1}^{n} B_k^2. \tag{4}$$

Equality holds in (4) if and only if, for each k, one has

$$\left(\sum_{i=1}^{n} A_i^2\right) B_k = \left(\sum_{i=1}^{n} A_iB_i\right) A_k.$$

2. Complementary inequalities

The following complementary inequality of (3) and (4) is given in [1].

Lemma 1. *Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be self-adjoint operators on H to itself satisfying $A_kB_k = B_kA_k$ for $k = 1, \ldots, n$. Suppose that A_k^{-1} exists and that*

$$mE \leq B_kA_k^{-1} \leq ME, \quad k = 1, \ldots, n, \tag{5}$$

where E is the identify operator on H. Then the following operator inequality holds,

$$\sum_{k=1}^{n} B_k^2 + mM \sum_{k=1}^{n} A_k^2 \leq (M+m) \sum_{k=1}^{n} A_kB_k, \tag{6}$$

that is,

$$\sum_{k=1}^{n}(B_kx, B_kx) + mM \sum_{k=1}^{n}(A_kx, A_kx) \leq (M+m) \sum_{k=1}^{n}(A_kx, B_kx) \tag{7}$$

for all $x \in H$. We have an equality sign in the operator inequality (6) if and only if $(B_kA_k^{-1} - mE)(ME - B_kA_k^{-1})$ is the zero operator for each $k = 1, \ldots, n$.

Equality holds in (7) for a vector x if and only if, for each k, $x = x_{1k} + x_{2k}$ with $x_{1k} \perp x_{2k}$, where $B_k x_{1k} = mA_k x_{1k}$ and $B_k x_{2k} = MA_k x_{2k}$. Furthermore, equality holds in (6) if and only if the equality conditions just written hold for every x in H.

The following result also holds.

Theorem 1. *Let $A_i, B_i, i = 1, \ldots, n$ be defined as in Lemma 1. Then for $x_i \in H$, $i = 1, \ldots, n$,*

$$\sum_{k=1}^{n}(B_k x_k, B_k x_k) + mM\sum_{k=1}^{n}(A_k x_k, A_k x_k) \leq (M+m)\sum_{k=1}^{n}(A_k x_k, B_k x_k). \tag{8}$$

Equality holds in (8) if and only if, for each k, $x_k = u_{1k} + u_{2k}$ with $u_{1k} \perp u_{2k}$, where $B_k u_{1k} = mA_k u_{1k}$ and $B_k u_{2k} = MA_k u_{2k}$.

Proof. Lemma 1 with $k = 1$ (or Theorem 2 from [1]) gives

$$(B_k x_k, B_k x_k) + mM(A_k x_k, A_k x_k) \leq (M+m)(A_k x_k, B_k x_k) \tag{9}$$

with equality if and only if x_k is as stated in Theorem 1.

Summing (9) over $k = 1, \ldots, n$, gives (8). ■

Theorem 2. *Let the conditions of Theorem 1 be satisfied with $0 < m < M$. Then,*

$$\left(\sum_{i=1}^{n}(A_i x_i, A_i x_i)\right)\left(\sum_{i=1}^{n}(B_i x_i, B_i x_i)\right)$$

$$\leq \frac{(M+m)^2}{4Mm}\left(\sum_{i=1}^{n}(A_i x_i, B_i x_i)\right)^2. \tag{10}$$

Equality holds in (10) if and only if, for each k, $x_k = u_{1k} + u_{2k}$ with $u_{1k} \perp u_{2k}$, where $B_k u_{1k} = mA_k u_{1k}$, $B_k u_{2k} = MA_k u_{2k}$, and

$$m\sum_{i=1}^{n}(A_i u_{1i}, A_i u_{1i}) = M\sum_{i=1}^{n}(A_i u_{2i}, A_i u_{2i}). \tag{11}$$

Proof. Inequality (10) follows directly from inequality (8) and the obvious inequality

$$0 \leq \left\{\left[\sum_{i=1}^{n}(B_i x_i, B_i x_i)\right]^{1/2} - \left[mM\sum_{i=1}^{n}(A_i x_i, A_i x_i)\right]^{1/2}\right\}^2. \tag{12}$$

For the equality case we have the equality conditions from Theorem 1. Also, equality holds in (12) if and only if

$$\sum_{i=1}^{n}(B_i x_i, B_i x_i) = mM \sum_{i=1}^{n}(A_i x_i, A_i x_i). \tag{13}$$

Moreover, this inequality can be given in the form of (11) if we use the equality condition from Theorem 1. ∎

Theorem 3. *Let the conditions of Theorem 2 be satisfied. Then,*

$$\frac{\sum_{i=1}^{n}(B_i x_i, B_i x_i)}{\sum_{i=1}^{n}(A_i x_i, B_i x_i)} - \frac{\sum_{i=1}^{n}(A_i x_i, B_i x_i)}{\sum_{i=1}^{n}(A_i x_i, A_i x_i)} \le (\sqrt{M} - \sqrt{m})^2. \tag{14}$$

Equality holds in (14) if and only if, for each k, $x_k = u_{1k} + u_{2k}$ with $u_{1k} \perp u_{2k}$, where $B_k u_{1k} = mA_k u_{1k}, B_k u_k = MA_k u_{2k}$ and

$$\sqrt{m}\sum_{i=1}^{n}(A_i u_{1i}, A_i u_{i1}) = \sqrt{M}\sum_{i=1}^{n}(A_i u_{2i}, A_i u_{2i}). \tag{15}$$

P r o o f. Inequality (14) follows directly from inequality (8) and the obvious inequality

$$0 \le \left\{\sqrt{Mm}\left(\frac{\sum_{i=1}^{n}(A_i x_i, A_i x_i)}{\sum_{i=1}^{n}(A_i x_i, B_i x_i)}\right)^{1/2} - \left(\frac{\sum_{i=1}^{n}(A_i x_i, B_i x_i)}{\sum_{i=1}^{n}(A_i x_i, A_i x_i)}\right)^{1/2}\right\}^2. \tag{16}$$

In the equality case, we have the equality conditions from Theorem 1. Also, equality holds in (16) if and only if

$$\sqrt{Mm}\sum_{i=1}^{n}(A_i x_i, A_i x_i) = \sum_{i=1}^{n}(A_i x_i, B_i x_i), \tag{17}$$

which is equivalent to (15) with respect to the equality conditions given in Theorem 1. ∎

Complementary inequalities similar to those in Theorems 2 and 3 can also be given for (4).

Theorem 4. *Let $A_1, \dots, A_n$ and $B_1, \dots, B_n$ be permutable self-adjoint operators such that (5) holds with $0 < m < M$. Then the following operator inequality holds:*

$$\sum_{k=1}^{n} A_k^2 \sum_{k=1}^{n} B_k^2 \le \frac{(M+m)^2}{4mM}\left(\sum_{k=1}^{n} A_k B_k\right)^2. \tag{18}$$

Equality holds in (18) if and only if $(B_kA_k^{-1} - mE)(ME - B_kA_k^{-1})$ *is the zero operator for each* $k = 1, \ldots, n$ *and if*

$$\sum_{k=1}^{n} B_k^2 = mM \sum_{k=1}^{n} A_k^2. \tag{19}$$

Proof. Inequality (18) follows directly from inequality (6) and the obvious inequality

$$0 \le \left\{ \left(\sum_{i=1}^{n} B_i^2 \right)^{1/2} - \left(Mm \sum_{i=1}^{n} A_i^2 \right)^{1/2} \right\}^2. \tag{20}$$

Equality holds in (20) if and only if (19) holds which together with the equality conditions of Lemma 1 gives the equality conditions of Theorem 4. ∎

Theorem 5. *Let the conditions of Theorem 4 be satisfied. Then the following operator inequality holds:*

$$\left(\sum_{i=1}^{n} B_i^2 \right) / \left(\sum_{i=1}^{n} A_iB_i \right) - \left(\sum_{i=1}^{n} A_iB_i \right) / \left(\sum_{i=1}^{n} A_i^2 \right) \le (\sqrt{M} - \sqrt{m})^2. \tag{21}$$

Equality holds if and only if $(B_kA_k^{-1} - mE)(ME - B_kA_k^{-1})$ *is the zero operator for each* $k = 1, \ldots, n$ *and if*

$$\sqrt{Mm} \sum_{i=1}^{n} A_i^2 = \sum_{i=1}^{n} A_iB_i. \tag{22}$$

Proof. Inequality (21) follows directly from inequality (6) and the obvious inequality

(23)

$$0 \le \left\{ \sqrt{Mm} \left(\sum_{i=1}^{n} A_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} A_iB_i \right)^{-1/2} - \left(\sum_{i=1}^{n} A_i^2 \right)^{-1/2} \left(\sum_{i=1}^{n} A_iB_i \right)^{1/2} \right\}^2.$$

Equality holds in (23) if and only if (22) holds, which, together with the equality conditions from Lemma 1, gives the equality conditions of Theorem 5. ∎

Remark. For analogous results for Hölder and Minkowski inequalities, see [2].

References

[1] J.B. D i a z, F.T. M e t c a l f, Complementary inequalities, III: Inequalities complementary to Schwarz's inequality in Hilbert space. *Math. Annalen* **162** (1965), 120-139.

[2] B. M o n d, J.E. P e č a r i ć, Hölder and Minkowski inequalities in Hilbert space. *Math. Balkanica* **8**, No 1 (1994), 45-49.

[3] F. R i e s z, B. S z.- N a g y, *Functional Analysis* (Translation from the 2nd French Ed. by Leo F. Boron). Frederick Ungar Pub. Co., 1955.

* *School of Mathematics, La Trobe University*
Bundoora, Victoria, 3083, AUSTRALIA

Received: 03.07.1995

** *Faculty of Textile Technology, University of Zagreb*
Zagreb, CROATIA
&
School of Mathematics, La Trobe University
Bundoora, Victoria, 3083, AUSTRALIA