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# Representation of Right Cyclically Ordered Groups as Groups of Automorphisms of a Cyclically Ordered Set

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It is proved by P.M. Chon, P.F. Conrad and M.I. Zaytzeva that the group $G$ is a right orderable group iff $G$ is 0-isomorphic onto a subgroup of the automorphism's group of an appropriate linearly ordered set. The analogous result is proved for lattice ordered groups by W.Ch. Holland. In this paper an analogous result will be proved for right cyclically ordered groups.

## 1. Basic notions

The notions of a cyclically ordered set (CO - set) and a cyclically ordered group (CO - group) are introduced by L.S. Rieger [4]. The notions of a right cyclically ordered group (RCO - group) and a partially cyclically ordered group (PCO - group) are introduced by S.D. Zheleva in [6], where some properties of these groups are investigated.

For the sake of completeness we recall some basic notions conserning cyclic orders, which will be used in this paper.

Let $M$ be a set with card $M \geq 3$ and let $C$ be a ternary relation on the set $M$. The relation $C$ is called a partial cyclic order on $M$ iff it is:

1) asymmetric, i.e. $(a,b,c) \in C \ \& \ a \neq b \neq c \neq a \Longrightarrow (a,c,b) \overline{\in} C$;
2) cyclic, i.e. $(a,b,c) \in C \Longrightarrow (b,c,a) \in C$;
3) transitive, i.e. $(a,b,c) \in C \ \& \ (a,c,d) \in C \Longrightarrow (a,b,d) \in C$.

The pair $(M,C)$ is called a partially cyclically ordered set iff the relation $C$ is a partial cyclic order on the set $M$.

The PCO - set $(M, C)$ is a linearly cyclically ordered set iff just one of the possibilities $(a, b, c) \in C$ or $(a, c, b) \in C$ holds for each triple of distinct elements $a, b, c \in M$.

The algebraic system $(G, \bullet, C)$ is called a right cyclically ordered group iff the following conditions are valid:

I. $(G, \bullet)$ is a group;

II. $(G, C)$ is a CO - set;

III. $(a, b, c) \in C \Longrightarrow (ax, bx, cx) \in C$ for each $x \in G$.

The mapping $f$ of the PCO - set $(M_1, C_1)$ into the PCO - set $(M_2 C_2)$ is called a CO - homomorphism iff $(a, b, c) \in C_1$ implies $(af, bf, cf) \in C_2$.

Any bijective CO - homomorphism $f$ of $(M_1, C_1)$ onto $(M_2, C_2)$ such that $f^{-1}$ is a CO - homomorphism of $(M_2, C_2)$ onto $(M_1, C_1)$ is called a CO - isomorphism.

Any CO - isomorphism of the PCO - set $(M, C)$ onto itself is called a CO - automorphism of the set $M$.

## 2. The right cyclically ordered automorphism's group of a cyclically ordered set

Let $(M, C)$ be a CO - set. We denote the group of CO - automorphisms of the set $(M, C)$ by $(\mu(M), \circ)$.

**Theorem 1.** *The group of CO - automorphisms of a cyclically ordered set is a right cyclically orderable group.*

Proof. Let $(M, C)$ be a CO - set and let $a$ be a fixed element on the set $M$. For any $x, y, \in M$ we put

$$(1) \qquad a <_a x <_a y \text{ iff } a \neq x \neq y \neq a \text{ and } (a, x, y) \in C.$$

The binary relation $<_a$ is a linear order on the set $M$ with the least element $a$. Let $S(a) = \{f \in \mu(M) / \ af = a\}$. It is easy to verify that $S(a)$ is the group of all o - automorphisms of the ordered set $(M, <_a)$. The group $(S(a), \circ)$ is right orderable group by Chon's - Conrad's theorem [1],[2].

If $C_\mu$ is the ternary relation on the group $\mu(M)$, defined by

$$(2) \quad (f, g, h) \in C_\mu \text{ iff } \begin{cases} (af, ag, ah) \in C, & \text{if } af \neq ag \neq ah \neq af; \\ gf^{-1} > e \text{ on } S(a), & \text{if } af = ag \neq ah; \\ hg^{-1} > e \text{ on } S(a), & \text{if } af \neq ag = ah; \\ fh^{-1} > e \text{ on } S(a), & \text{if } ah = af \neq ag; \\ gf^{-1} > e \ \& \ hg^{-1} > e, & \text{or } gh^{-1} > e \& fh^{-1} > e, \text{ or} \\ fh^{-1} > e \ \& \ gf^{-1} > e & \text{on } S(a), \text{ if } af = ag = ah, \end{cases}$$

then $(\mu(M), \circ, C_\mu)$ is a RCO - group.

In [6, Theorem 4] it is proved that any RCO - group has a representation as a group of automorphisms of a cyclically ordered set. ∎

**Theorem 2.** *Any RCO - group $(G, \bullet, C)$ is CO - isomorphic onto a subgroup of the right cycilically ordered automorphism's group of the CO - set $(G, C)$.*

Proof. Let $(G, \bullet, C)$ be RCO - group and $C_\mu$ be the right cyclic order on the group $\mu(G)$, defined by formulas (2), where $a = e$ is the unit of the group $G$. The mapping $f_g : x \longrightarrow xg$ for each $x \in G$, where $g$ is a fixed element of $G$, is a CO - automorphism of the CO-set $(G, C)$. The mapping $f$ of $G$ into $\mu(G)$, defined by $gf = f_g$ for each $g \in G$, is a CO - isomorphism of the RCO-group $(G, \circ, C)$ onto $(f(G), \circ, C'_\mu)$, where $C'_\mu$ is the restriction of the right cyclic order $C_\mu$ on the subgroup $f(G)$. Therefore the theorem is proved. ∎

Let $(M, C)$ be a CO - set and let $(\mu(M), \circ, C_\mu)$ be a RCO - group of automorphisms of the set $(M, C)$. If $(G, \bullet, C_G)$ is a group with a ternary relation $C_G$ such that $(G, \bullet, C_G)$ is isomorphic onto the group $(H, \circ, C_H)$, where $H \leq \mu(M)$ and $C_H \leq C_\mu$, then $(G, \bullet, C_G)$ is a RCO - group.

The truthfulness of the main result follows from Theorem 1, Theorem 2 and the above fact.

**Main Theorem.** *The group $G$ is RCO - group iff $G$ is CO - isomorphic onto a subgroup of the automorphism's group of an appropriate CO - set.*

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