Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica
Balkanica

New Series Vol.11, 1997, Fasc. 3-4

Expressions of Legendre Polynomials Through Euler Polynomials [1]

Vu Kim Tuan and Nguyen Thi Tinh***

Presented by V. Kiryakova

The discrete orthogonality of the modified Lommel polynomials is applied to establish a formula of finite summation that is used to construct a relation between Legendre polynomials and Euler polynomials. Consequently, explicit coefficients of expansions of Legendre polynomials through Euler polynomials are obtained.

1. Introduction

The main aim of this work is to obtain an expansion formula of Legendre polynomials though Euler polynomials.

Legendre polynomials $P_n(x)$ are defined by

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n(x^2-1)^n}{dx^n}, \qquad n \geq 0,$$

and satisfy the following relation of orthogonality

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}, \tag{1}$$

or equivalently,

$$\int_0^1 P_n(2x-1)\,P_m(2x-1)\,dx = \frac{1}{2n+1}\delta_{mn}. \tag{2}$$

[1]The work of the first author was supported by the Kuwait University research grant SM 112.

([3], Chapter 3, 3.12.8 and 3.12.10), where

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

is the "Kronecker delta".

Euler polynomials $E_n(x)$ and Bernoulli numbers B_n, $n \geq 0$, are defined by

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}, \quad |z| < \pi, \tag{3}$$

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi, \tag{4}$$

(see [3], Chapter 1). Two of the properties of Euler polynomials $E_n(x)$ and Bernoulli numbers B_n used later are

$$B_{2k+1} = 0, \qquad k > 0; \tag{5}$$

$$\int_0^1 E_m(x) E_n(x)\, dx = 4(-1)^n \left(2^{m+n+2} - 1\right) \frac{m!n!}{(m+n+2)!} B_{m+n+2} \tag{6}$$

(see [5], Chapter 2).

Let $(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \nu(\nu+1)\ldots(\nu+n-1)$ be the Pochhammer symbol, ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ be a generalized hypergeometric function ([3], Chapter 4). Let $\{j_{\nu,n}\}$, $n = \pm1, \pm2, ...$, be the nonvanishing zeros of the Bessel function $J_\nu(x)$ ([4], Chapter 7) ordered by

$$\cdots < j_{\nu,-2} < j_{\nu,-1} < 0 < j_{\nu,1} < j_{\nu,2} < \cdots \quad .$$

Then

$$h_{n,\nu}(x) = (\nu)_n (2x)^n \; {}_2F_3\left(-n/2, (1-n)/2; \nu, -n, 1-\nu-n; -1/x^2\right) \tag{7}$$

are modified Lommel polynomials ([2]) satisfying the following discrete orthogonality

$$\sum_{k=-\infty}^{\infty}{}' h_{n,\nu}\left(\frac{1}{j_{\nu-1,k}}\right) h_{m,\nu}\left(\frac{1}{j_{\nu-1,k}}\right) \frac{1}{j^2_{\nu-1,k}} = \frac{\delta_{mn}}{2(\nu+n)}, \tag{8}$$

where the dash in the sum indicates that the term with index $k = 0$ is omitted. Formula (8) first appeared in [2] with the incorrect right-hand side.

2. Formula of finite summation

The following formula of finite summation will play an important role in the next section. It was given incorrectly in [5], Chapter 5, 5.1.1.7. For we could not find another reference, we shall establish it here by using the orthogonality relation (8) of the modified Lommel polynomials.

Lemma . *For $\sigma = 0$ or 1 the formula*

$$\sum_{l=0}^{n} \frac{(2n+2l+2\sigma)!\left(1-2^{2m+2l+2\sigma+2}\right) B_{2m+2l+2\sigma+2}}{(2l)!\,(2n-2l)!\,(2m+2l+2\sigma+2)!}$$

(9) $$= \frac{(-1)^{\sigma+1}\,(2n+\sigma)!}{4\,(4n+2\sigma+1)!}\,\delta_{mn}, \quad 0 \le m \le n,$$

is valid.

Proof. First we consider the case $\sigma = 0$. From (7) it is easy to see that the modified Lommel polynomials $h_{n,\nu}(x)$ is an even or an odd polynomial according as n is even or odd, i.e. $h_{n,\nu}(-x) = (-1)^n\, h_{n,\nu}(x)$. Therefore,

(10) $$v_n(x) = h_{2n,1/2}\left(\frac{2\sqrt{x}}{\pi}\right), \quad n = 0, 1, \ldots,$$

is a polynomial of precise degree n of variable x. From (7) and (10) we have

$$\begin{aligned} v_n(x) &= (1/2)_{2n}\,\frac{4^{2n}x^n}{\pi^{2n}}\;{}_2F_3\left(-n, 1/2-n; 1/2, -2n, 1/2-2n; -\frac{\pi^2}{4x}\right) \\ &= \frac{(4n-1)!!2^{2n}x^n}{\pi^{2n}} \sum_{k=0}^{n} \frac{(-n)_k\,(1/2-n)_k}{(1)_k\,(1/2)_k\,(1/2-2n)_k\,(-2n)_k}\left(-\frac{\pi^2}{4x}\right)^k, \end{aligned}$$

where $k!! = k\,(k-2)\,(k-4)\ldots\left(k-2[\frac{k}{2}]+2\right)$. Using the formula

$$4^k\,(a)_k\,(a+1/2)_k = (2a)_{2k},$$

we obtain

$$\begin{aligned} v_n(x) &= \frac{(4n-1)!!2^{2n}x^n}{\pi^{2n}} \sum_{k=0}^{n} \frac{(-2n)_{2k}}{(1)_{2k}\,(-4n)_{2k}}\left(-\frac{\pi^2}{x}\right)^k \\ &= \frac{(4n-1)!!2^{2n}x^n}{\pi^{2n}} \sum_{k=0}^{n} \frac{(2n-2k+1)_{2k}}{(2k)!\,(4n-2k+1)_{2k}}\left(-\frac{\pi^2}{x}\right)^k. \end{aligned}$$

Changing now $n-k$ by k we get

$$v_n(x) = (-1)^n 2^{2n}(4n-1)!!\sum_{k=0}^{n}\frac{(2k+1)_{2n-2k}}{(2n-2k)!\,(2n+2k+1)_{2n-2k}}\left(-\frac{x}{\pi^2}\right)^k$$

$$(11)\qquad = (-1)^n\sum_{k=0}^{n}\frac{(2n+2k)!}{(2k)!\,(2n-2k)!}\left(-\frac{x}{\pi^2}\right)^k.$$

Since ([4], Chapter 7)

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x,$$

then

$$j_{-1/2,k} = (2k+1)\pi/2, \qquad k = 0, \pm 1, \pm 2, \dots .$$

Therefore, the discrete orthogonal relation (8) for the sequence of polynomials $\{v_n(x)\}$ becomes

$$(12)\qquad \sum_{k=0}^{\infty} v_n\left(\frac{1}{(2k+1)^2}\right) v_m\left(\frac{1}{(2k+1)^2}\right)\cdot\frac{1}{(2k+1)^2} = \frac{\pi^2}{8(4n+1)}\delta_{mn}.$$

For the coefficient of x^n in $v_n(x)$ is $\frac{(4n)!}{(2n)!\pi^{2n}}$ the orthogonal relation (12) can be rewritten in the form

$$(13)\qquad \sum_{k=0}^{\infty} v_n\left(\frac{1}{(2k+1)^2}\right)\cdot\frac{1}{(2k+1)^{2m+2}} = \frac{(2n)!\pi^{2n+2}}{8(4n+1)!}\delta_{mn}, \quad 0 \le m \le n.$$

Putting the explicit representation (11) of $v_n(x)$ into (13) we get

$$\sum_{k=0}^{\infty}\left[\sum_{l=0}^{n}\frac{(-1)^{n+l}(2n+2l)!}{(2l)!\,(2n-2l)!\pi^{2l}}\left(\frac{1}{(2k+1)^2}\right)^l\right]\frac{1}{(2k+1)^{2m+2}}$$

$$(14)\qquad = \frac{(2n)!\pi^{2n+2}}{8(4n+1)!}\delta_{mn}, \quad 0 \le m \le n.$$

Changing the order of summation in (14) we obtain

$$\sum_{l=0}^{n}\frac{(-1)^{n+l}(2n+2l)!}{(2l)!\,(2n-2l)!\pi^{2l}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2m+2l+2}} = \frac{(2n)!\pi^{2n+2}}{8(4n+1)!}\delta_{mn},$$

$$0 \le m \le n.$$

Applying the formula ([3], Chapter 1, $ 1.13)

$$(15)\qquad \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2m+2l+2}} = \frac{(-1)^{m+l}\pi^{2m+2l+2}\left(2^{2m+2l+2}-1\right)}{2(2m+2l+2)!}B_{2m+2l+2},$$

we get

$$(16)\ \sum_{l=0}^{n} \frac{(2n+2l)\left(1-2^{2n+2l+2}\right)! B_{2m+2l+2}}{(2l)!\,(2n-2l)!\,(2m+2l+2)!} = -\frac{(2n)!}{(4n+1)!4}\delta_{mn}, 0 \le m \le n.$$

Hence formula (9) is proved in case $\sigma = 0$.

We consider now the case $\sigma = 1$. Let

$$(17)\qquad t_n(x) = x^{-1}[v_{n+1}(x) + v_n(x)].$$

Since $v_n(0) = (-1)^n$, it is easy to see that $t_n(x)$ is a polynomial of precise degree n. After replacing $v_n(x)$ and $v_{n+1}(x)$ in (17) by their explicit representation formulas (11) we obtain

$$(18)\qquad t_n(x) = \frac{2(-1)^n(4n+3)}{\pi^2}\sum_{k=0}^{n}\frac{(2n+2k+2)!}{(2k+1)!\,(2n-2k)!}\left(\frac{-x}{\pi^2}\right)^k.$$

Using formula (12) twice we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} t_n\left(\frac{1}{(2k+1)^2}\right)\frac{1}{(2k+1)^{2m+4}} = \sum_{k=0}^{\infty} v_{n+1}\left(\frac{1}{(2k+1)^2}\right)\frac{1}{(2k+1)^{2m+2}}\\
&+ \sum_{k=0}^{\infty} v_n\left(\frac{1}{(2k+1)^2}\right)\frac{1}{(2k+1)^{2m+2}} = \sum_{k=0}^{\infty} v_n\left(\frac{1}{(2k+1)^2}\right)\frac{1}{(2k+1)^{2m+2}}\\
(19)\qquad &= \frac{(2n)!\pi^{2n+2}}{8(4n+1)!}\delta_{mn}, \quad 0 \le m \le n.
\end{aligned}$$

Putting expression (18) into (19) we get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\sum_{l=0}^{n}\frac{(-1)^{n+l}(2n+2l+2)!}{(2l+1)!\,(2n-2l)!\pi^{2l}}\left(\frac{1}{(2k+1)^2}\right)^l\right]\frac{1}{(2k+1)^{2m+4}}\\
(20)\qquad &= \frac{(2n+1)!\pi^{2n+4}}{8(4n+3)!}\delta_{mn}, \quad 0 \le m \le n.
\end{aligned}$$

Changing the order of summation in (20) we have

$$\begin{aligned}
&\sum_{l=0}^{n}\frac{(-1)^{n+l}(2n+2l+2)!}{(2l+1)!\,(2n-2l)!\,(2n-2l+1)!\pi^{2l}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2m+2l+4}}\\
(21)\qquad &= \frac{(2n+1)!\pi^{2n+4}}{8(4n+3)!}, \quad 0 \le m \le n.
\end{aligned}$$

Replacing the infinite sum in (21) by the formula (15) we obtain

$$\sum_{l=0}^{n} \frac{(2n+2l+2)!\left(1-2^{2m+2l+4}\right) B_{2m+2l+4}}{(2l)!\,(2n-2l)!\,(2m+2l+4)!}$$

$$= \frac{(2n+1)!}{4\,(4n+3)!}\delta_{mn}, \quad 0 \le m \le n. \tag{22}$$

Hence, the formula (9) is proved for the case $\sigma = 1$. ■

3. The relation between Legendre and Euler polynomials

Now we shall use the formula of finite summation (9) to prove the following expansion theorem.

Theorem. *Let $\{P_n(x)\}$ be Legendre polynomial sequence and $\{E_n(x)\}$ be Euler polynomial sequence. The following equality holds*

$$P_{2n+\sigma}(2x-1) = \sum_{l=0}^{n} \frac{(2n+2l+2\sigma)!}{(2l)!\,(2l+\sigma)!\,(2n-2l)!} E_{2l+\sigma}(x), \tag{23}$$

$$(n \ge 0; \quad \sigma = 0 \text{ or } 1).$$

P r o o f. First, we apply the lemma in case $\sigma = 0$. In formula (16) replacing $B_{2m+2l+2}$ by formula (6) we get

$$\int_0^1 \frac{E_{2m}(x)}{(2m)!} \sum_{l=0}^{n} \frac{(2n+2l)!E_{2l}(x)}{[(2l)!]^2\,(2n-2l)!} = \frac{(2n)!\delta_{mn}}{(4n+1)!}, \quad 0 \le m \le n. \tag{24}$$

If we set

$$\sum_{l=0}^{n} \frac{(2n+2l)!E_{2l}(x)}{[(2l)!]^2\,(2n-2l)!} = P_{2n}^{*}(x), \tag{25}$$

then $P_{2n}^{*}(x)$ is a polynomial of degree $2n$, and we obtain

$$\int_0^1 E_{2m}(x)\,P_{2n}^{*}(x)\,dx = \frac{\{(2n)!\}^2}{(4n+1)!}\delta_{mn}, \quad 0 \le m \le n. \tag{26}$$

Using (6) again and noticing that $B_{2m+1} = 0$ for $m > 0$, we have

$$\int_0^1 E_{2m+1}(x)\,P_{2n}^{*}(x)\,dx = \int_0^1 E_{2m+1}(x) \sum_{l=0}^{n} \frac{(2n+2l)!E_{2l}(x)}{[(2l)!]^2\,(2n-2l)!}dx$$

$$(27) \quad = \sum_{l=0}^{n} \frac{(2n+2l)!}{[(2l)!]^2 (2n-2l)!} \int_0^1 E_{2m+1}(x) E_{2l}(x)\, dx$$

$$= \sum_{l=0}^{n} \frac{4(2n+2l)!(2m+1)!\left(2^{2m+2l+3}-1\right)}{(2l)!(2n-2l)!(2m+2l+3)!} B_{2m+2l+3} = 0.$$

From (26) and (27) we can conclude that the polynomial $P_{2n}^{*}(x)$ is orthogonal to the system $\{E_0(x), E_1(x), ..., E_{2n-1}(x)\}$ with respect to the weight 1 on the interval $[0,1]$. Since the system of Legendre polynomials $P_n(2x-1)$ is also orthogonal with respect to the weight 1 on $[0,1]$, then there exists a sequence of scalars $\{\alpha_n\}$ such that

$$(28) \qquad P_{2n}^{*}(x) = \alpha_n P_{2n}(2x-1), \qquad n \geq 0.$$

The coefficients α_n can be found exactly. Indeed, identifying the coefficients of x^{2n} in two polynomials in (28) we get $\alpha_n = 1$ and the proof of the theorem in case $\sigma = 0$ is finished.

For the case $\sigma = 1$ we shall apply the lemma for the corresponding σ. The method used here is similar to the previous one.

Again replacing $B_{2m+2l+4}$ in (22) by formula (6) we get

$$\int_0^1 \frac{E_{2m+1}(x)}{(2m+1)!} \sum_{l=0}^{n} \frac{(2n+2l+2)! E_{2l+1}(x)}{(2l)!(2l+1)!(2n-2l)!} dx$$

$$(29) \qquad = \frac{(2n+1)!}{(4n+3)!} \delta_{mn}, \quad 0 \leq m \leq n.$$

Setting

$$(30) \qquad \sum_{l=0}^{n} \frac{(2n+2l+2)! E_{2l+1}(x)}{(2l)!(2l+1)!(2n-2l)!} = P_{2n+1}^{*}(x),$$

then $P_{2n+1}^{*}(x)$ is a polynomial of precise degree $2n+1$, and we have

$$(31) \qquad \int_0^1 E_{2m+1}(x) P_{2n+1}^{*}(x)\, dx = \frac{\{(2n+1)!\}^2 \delta_{mn}}{(4n+3)!}, \quad 0 \leq m \leq n.$$

Completely similar to the corresponding step in the proof of the theorem in case $\sigma = 0$ we also obtain

$$(32) \qquad \int_0^1 E_{2m}(x) P_{2n+1}^{*}(x)\, dx = 0, \qquad 0 \leq m \leq n.$$

Now, from (31), (32) and reasoning the same as before we can conclude that there exists a sequence of scalars $\{\beta_n\}$ with

$$(33) \qquad P_{2n+1}^{*}(x) = \beta_n P_{2n+1}(2x-1), \qquad n \geq 0.$$

Comparing the coefficients of x^{2n+1} we can find out the value of β_n to be 1, and the proof of the theorem is finished. ∎

References

[1] T. S. Chihara, *An Introduction to Orthogonal Polynomials.* Gordon and Breach, New York-London-Paris, 1978.

[2] D. Dickinson, On Lommel and Bessel polynomials. *Proc. Amer. Math. Soc.* **5** (1954), 946–956.

[3] A. Erdelyi et al., *Higher Transcendental Functions*, Vol. **1**. McGraw-Hill, New York-Toronto-London,1952.

[4] A. Erdelyi et al. *Higher Transcendental Functions*, Vol. **2**. McGraw-Hill, New York-Toronto-London, 1953.

[5] A. P. Prudnikov, Yu. Brychkov, O. I. Marichev, *Integrals and Series*, Vol. **3**: *More Special Functions.* Gordon and Breach, Philadelphia-Reading-Paris-Montreux-Tokyo-Melbourne, 1990.

* *Department of Mathematics and Informatics*
Faculty of Science, Kuwait University
P.O. Box 5969, Safat 13060
KUWAIT

** *Hanoi Teacher's Training College N°1*
VIETNAM

Received: 12.12.1995