Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica
Balkanica

New Series Vol.11, 1997, Fasc. 3-4

Evaluation of Fractional Integrals Involving Hypergeometric Functions

L. Galué *, *A. Al Zamel* **, *S. Kalla* **

Presented by V. Kiryakova

In this paper we establish that

$$\int_p^\infty (u-p)^{\alpha-1} u^n F(u)\, {}_rF_{s+1}(\alpha_1,\dots,\alpha_r;\,\alpha,\beta_1,\dots,\beta_s;\,-w(u-p))\,du$$
$$= \Gamma(\alpha)\int_0^\infty t^{-\alpha} e^{-pt} G^{(n)}(t)\, {}_rF_s\left(\alpha_1,\dots,\alpha_r;\,\beta_1,\dots,\beta_s;\,-\frac{w}{t}\right)dt,$$

where $G(t)$ is the inverse Laplace transform of $F(x)$; $p>0$, $Re(\alpha)>0$, if $r<s+1$; $Re(w)>0$ if $r=s+1$; $Re(1-\alpha+\min(\alpha_j))>0$ $(j=1,\dots,r)$ and $G(0)=G'(0)=\dots=G^{(n-1)}(0)=0$.

Further we apply this results to get a number of integrals, some of which seem to be new. Various particular cases are obtained.

1. Introduction

Recently, M. L. Glasser and V. Kowalenko [2] have established a method for evaluating fractional integrals. They demonstrated the effect of their techniques on fractional integrals of the form

$$I_\mu(p) = \int_p^\infty \frac{F(x)}{(x-p)^\mu}dx, \qquad Re(\mu) < 1. \tag{1}$$

They also extended the application of their techniques to many other classes of integrals not of fractional form.

In this paper, we give a natural generalization to their main theorem and produce many results as special cases. Some of these results were given in [2] as new results.

We also give a wide range of applications of our results to evaluate many fractional integrals.

2. Integrals involving hypergeometric functions

Theorem. *Let the inverse Laplace transform L^{-1} of $F(x)$ exist and be denoted by $G(t)$. Further, let $G(t)$ be n-times differentiable, $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$ and the Laplace transform of $G^{(n)}(t)$ exist for $x = p$. Then*

$$\int_p^\infty (u-p)^{\alpha-1} u^n F(u)\, {}_rF_{s+1}\left(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-w(u-p)\right) du = \Gamma(\alpha)\int_0^\infty t^{-\alpha} e^{-pt} G^{(n)}(t)\, {}_sF_s\left(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;-\frac{w}{t}\right) dt, \tag{2}$$

where $p > 0; Re(\alpha) > 0$ if $r < s+1$; $Re(w) > 0$ if $r = s+1$; $Re(1-\alpha+\min(\alpha_j)) > 0\,(j = 1,\ldots,r)$.

Proof. As

$$L[G(t);x] = F(x), \tag{3}$$

then

$$L[G^{(n)}(t);x] = x^n F(x), \tag{4}$$

where L is the Laplace transform $Lf = \int_0^\infty e^{-pt} f(t)dt$ and further, we have

$$\begin{aligned}
&\int_p^\infty (u-p)^{\alpha-1}\, u^n F(u)\, {}_rF_{s+1}\left(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-w(u-p)\right) du \\
&= \int_p^\infty (u-p)^{\alpha-1}\, {}_rF_{s+1}\left(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-w(u-p)\right) \\
&\times \left\{\int_0^\infty e^{-ut} G^{(n)}(t)dt\right\} du = \int_0^\infty G^{(n)}(t) \\
&\times \left\{\int_p^\infty e^{-ut}(u-p)^{\alpha-1}\, {}_rF_{s+1}\left(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-w(u-p)\right) du\right\} dt \\
&= \int_0^\infty e^{-pt} G^{(n)}(t) \left\{\int_0^\infty e^{-xt} x^{\alpha-1}\, {}_rF_{s+1}\left(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wx\right)dx\right\} dt.
\end{aligned}$$

Now, evaluating the x-integral of the above [4, p.851 (8)], we obtain

$$\int_p^\infty (u-p)^{\alpha-1}\, u^n F(u)\, {}_rF_{s+1}\left(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-w(u-p)\right) du = \Gamma(\alpha)\int_0^\infty t^{-\alpha}\, e^{-pt} G^{(n)}(t)\, {}_rF_s\left(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;-\tfrac{w}{t}\right) dt.$$

■

As special cases of (2) we have:

i) If $n = 0$, then

$$\int_p^\infty (u-p)^{\alpha-1}\, F(u)\, {}_rF_{s+1}\left(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-w(u-p)\right) du = \Gamma(\alpha)\int_0^\infty t^{-\alpha}\, e^{-pt} G(t)\, {}_rF_s\left(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;-\tfrac{w}{t}\right) dt,$$

which corresponds to the result (4) obtained by Glasser and Kowalenko [2].

ii) If $n = 0$ and $\alpha_1 = 0$, then (2) reduces to the main result of [2], that is

$$\int_p^\infty (u-p)^{\alpha-1} F(u)du = \Gamma(\alpha) \int_0^\infty t^{-\alpha} e^{-pt} G(t)dt.$$

Further, we consider some special cases of the main theorem. By a change of variable, (2) can be expressed in the form:

$$\begin{aligned} &\int_0^\infty y^{\alpha-1}(y+p)^n F(y+p)\, {}_rF_{s+1}\left(\alpha_1,\ldots,\alpha_r;\, \alpha,\beta_1,\ldots,\beta_s;\, -wy\right) dy \\ &= \Gamma(\alpha) \int_0^\infty t^{-\alpha} e^{-pt} G^{(n)}(t)\, {}_rF_s\left(\alpha_1,\ldots,\alpha_r;\, \beta_1,\ldots,\beta_s;\, -\frac{w}{t}\right) dt, \end{aligned} \tag{5}$$

$p > 0, Re(\alpha) > 0$ if $r < s+1$; $Re(w) > 0$ if $r = s+1$; $Re(1-\alpha+\min(\alpha_j)) > 0\,(j = 1,\ldots,r)$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

By analytic continuation it follows (5) is valid also for $Re(p) > 0$.

Set $r = 1$ and $s = 0$ in (5) to obtain

$$\begin{aligned} &\int_0^\infty y^{\alpha-1}(y+p)^n F(y+p)\, {}_1F_1\left(\alpha_1;\alpha;-wy\right) dy \\ &= \Gamma(\alpha) \int_0^\infty t^{\alpha_1-\alpha} e^{-pt} G^{(n)}(t)(t+w)^{-\alpha_1} dt,. \end{aligned} \tag{6}$$

$Re(p) > 0, Re(w) > 0, Re(1-\alpha+\alpha_1) > 0$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$. Particular cases of (6) are obtained by using the results tabulated in [6, Section 7.11.1]:

i) The case $\alpha_1 = \alpha = a$ gives

$$\begin{aligned} &\int_0^\infty y^{\alpha-1}(y+p)^n F(y+p) e^{-wy} dy \\ &= \Gamma(a) \int_0^\infty e^{-pt}(t+w)^{-a} G^{(n)}(t) dt, \end{aligned} \tag{7}$$

$Re(p) > 0, Re(w) > 0$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

ii) The case $\alpha_1 = a, \alpha = 2a$ gives

$$\begin{aligned} &\int_0^\infty y^{a-1/2}(y+p)^n F(y+p)\, e^{-wy/2} I_{a-1/2}\left(-\frac{wy}{2}\right) dy \\ &= \frac{2^{1-2a}(-w)^{a-1/2}\Gamma(2a)}{\Gamma(a+1/2)} \int_0^\infty t^{-a}(t+w)^{-a} e^{-pt} G^{(n)}(t) dt, \end{aligned} \tag{8}$$

$Re(p) > 0, Re(w) > 0, Re(a) < 1$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

iii) The case $\alpha_1 = a, \alpha = a - m$ gives

$$\begin{aligned} &\int_0^\infty y^{a-m-1}(y+p)^n F(y+p) e^{-wy} L_m^{a-m-1}(wy)\, dy \\ &= \frac{(-1)^m \Gamma(a-m)(1-a)_m}{m!} \int_0^\infty t^m e^{-pt}(t+w)^{-a} e^{-pt} G^{(n)}(t) dt, \end{aligned} \tag{9}$$

$Re(p) > 0, Re(w) > 0$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

If we set $r = 2$ and $s = 1$ in equation (5), we obtain

$$\begin{aligned} &\int_0^\infty y^{\alpha-1}(y+p)^n F(y+p)\, {}_2F_2\,(\alpha_1,\alpha_2;\alpha,\beta_1;-wy)\,dy \\ &= \Gamma(\alpha)\int_0^\infty t^{-\alpha}e^{-pt}G^{(n)}(t)\, {}_2F_1\left(\alpha_1,\alpha_2;\beta_1;-\frac{w}{t}\right)dt, \end{aligned} \tag{10}$$

$Re(p) > 0, Re(w) > 0, Re(1-\alpha+\min(\alpha_j)) > 0\,(j = 1,2)$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

By letting $\alpha_1 = a, \alpha_2 = b, \beta_1 = a+b-1/2$ in (10) we get

$$\begin{aligned} &\int_0^\infty y^{\alpha-1}(y+p)^n F(y+p)\, {}_2F_2\,(a,b,\alpha,a+b-1/2;-wy)\,dy \\ &= 2^{a+b-3/2}\Gamma(\alpha)\Gamma(a+b-1/2)(-w)^{(3-2a-2b)/4} \\ &\times \int_0^\infty t^{a/2+b/2-\alpha-1/4}e^{-pt}(t+w)^{-1/2}P_{b-a-1/2}^{3/2-a-b}\left(\sqrt{1+\frac{w}{t}}\right)G^{(n)}(t)dt, \end{aligned} \tag{11}$$

$Re(p) > 0, Re(w) > 0, Re(1-\alpha+\min(a,b)) > 0$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

For $r = s = 1$ (5) reduces to

$$\begin{aligned} &\int_0^\infty y^{\alpha-1}(y+p)^n F(y+p)\, {}_1F_2\,(\alpha_1,;\alpha,\beta_1;-wy)\,dy \\ &= \Gamma(\alpha)\int_0^\infty t^{-\alpha}e^{-pt}G^{(n)}(t)\, {}_1F_1\left(\alpha_1;\beta_1;-\frac{w}{t}\right)dt, \end{aligned} \tag{12}$$

$Re(p) > 0, Re(1+\alpha_1) > Re(\alpha) > 0$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

Another special case of (12) can be obtained by letting $\alpha_1 = a, \beta_1 = 2a$ in the following form

$$\begin{aligned} &\int_0^\infty y^{\alpha-1}(y+p)^n F(y+p)\, {}_1F_2\,(a,\alpha,2a;-wy)\,dy \\ &= 2^{2a-1}\Gamma(\alpha)\Gamma(a+1/2)(-w)^{1/2-a} \\ &\times \int_0^\infty t^{a-\alpha-1/2}e^{-(pt+w/2t)}\,I_{a-1/2}\left(-\frac{w}{2t}\right)G^{(n)}(t)dt, \end{aligned} \tag{13}$$

$Re(p) > 0, Re(1+a) > Re(\alpha) > 0$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

If $r = s = 2, \alpha_1 = 1, \alpha_2 = 3/2, \beta_1 = 2, \beta_2 = b$ and $\alpha = 3-b$ in (5) we get, by virtue of the results [6, No.7.12.1.5 and 7.15.1.3] and [4, No.5.7.6],

$$\begin{aligned} &\int_0^\infty y^{3/2-b}(y+p)^n F(y+p)\, J_{b-1}\left(\sqrt{wy}\right) J_{2-b}\left(\sqrt{wy}\right)dy \\ &= \frac{b}{3\Gamma(b)}\, w^{-1/2}\int_0^\infty t^{b-2}e^{-pt}\left[1-{}_1F_1\left(\frac{3}{2};b;-\frac{w}{t}\right)\right]G^{(n)}(t)dt, \end{aligned} \tag{14}$$

$Re(p) > 0, 0 < Re(\alpha) < 2$ and $G(0) = G'(0) = \ldots = G^{(n-1)}(0) = 0$.

We can also produce many other results as special cases when taking particular values of r and s.

3. Applications

In this section we consider specific expressions for $F(x)$ and $G^{(n)}(t)$ in (5) in order to evaluate some fractional integrals.

From [1, No.5.4.10],

$$L^{-1}\left[p^{-2\lambda}(p^2+a^2)^{-nu}\right] = \frac{t^{2\lambda+2\nu-1}}{\Gamma(2\lambda+2\nu)}\,{}_1F_2(\nu;\lambda+\nu,\lambda+\nu+\frac{1}{2};-\frac{1}{4}a^2t^2)$$

with $Re(\lambda+\nu) > 0$.

Setting $\lambda = -\frac{n}{2}$ in this result and from (4),

$$G^{(n)}(t) = \frac{t^{2\nu-n-1}}{\Gamma(2\nu-n)}\,{}_1F_2(\nu;\nu-\frac{n}{2},\nu-\frac{n}{2}+\frac{1}{2};-\frac{1}{4}a^2t^2)$$

and $F(x) = (x^2+a^2)^{-\nu}$, $\nu > n/2$.

So, after substituting in (5) we get,

$$\int_0^\infty y^{\alpha-1}[(y+p)^2+a^2]^{-\nu}\,{}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy$$
$$= \frac{\Gamma(\alpha)}{\Gamma(2\nu-n)}\int_0^\infty t^{-\alpha+2\nu-n-1}\,e^{-pt}\,{}_1F_2(\nu;\nu-\tfrac{n}{2},\nu-\tfrac{n}{2}+\tfrac{1}{2};-\tfrac{1}{4}a^2t^2)$$
$$\times\,{}_rF_s\left(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;-\frac{w}{t}\right)dt,\quad Re(\nu-\tfrac{n}{2}) > 0.$$

In view of the results [6, No.8.4.51.1, No.8.2.2.14],

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;-x) = \Gamma\left[\begin{array}{c} b_1,\ldots,b_q \\ a_1,\ldots,a_p\end{array}\right] G^{1,p}_{p,q+1}\left[x\left|\begin{array}{c} 1-a_1,\ldots,1-a_p \\ 0,1-b_1,\ldots,1-b_q\end{array}\right.\right],$$

$$G^{m,n}_{p,q}\left[z\left|\begin{array}{c}(a_p)\\(b_q)\end{array}\right.\right] = G^{n,m}_{q,p}\left[\frac{1}{z}\left|\begin{array}{c}(1-b_q)\\(1-a_p)\end{array}\right.\right],\quad \arg(1/z) = -\arg z,$$

using the extension in series from ${}_1F_2(\nu;\nu-\frac{n}{2},\nu-\frac{n}{2}+\frac{1}{2};-\frac{1}{4}a^2t^2)$ and interchanging the order of the integral and the sum,

$$\int_0^\infty y^{\alpha-1}(y+p)^n[(y+p)^2+a^2]^{-\nu}\,{}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy$$

$$= \frac{\Gamma(\alpha)}{\Gamma(2\nu-n)}\,\Gamma\left[\begin{array}{c}\beta_1,\ldots,\beta_s\\ \alpha_1,\ldots,\alpha_r\end{array}\right]\sum_{k=0}^{+\infty}\frac{(\nu)_k\left(-\frac{a^2}{4}\right)^k}{\left(\nu-\frac{n}{2}\right)_k\left(\nu-\frac{n}{2}+\frac{1}{2}\right)_k k!}\,I_1,$$

where

$$I_1 = \int_0^\infty t^{2k-\alpha+2\nu-n-1}\, e^{-pt}\, G^{r,1}_{s+1,r}\left[\frac{t}{w}\,\middle|\, \begin{matrix} 1,\beta_1,\ldots,\beta_s \\ \alpha_1,\ldots,\alpha_r \end{matrix}\right] dt.$$

Using the result [5, No.3.3.6a],

$$\int_0^\infty x^{\sigma-1} e^{-wx}\, G^{m,n}_{p,q}\left[\eta x^{k/\rho}\,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right] dx = w^{-\sigma}(2\pi)^{(1-k)/2+c^*(1-\rho)} k^{\sigma-1/2} \rho^U$$
$$\times\, G^{\rho m,\rho n+k}_{\rho p+k,\rho q}\left[\frac{k^k \eta^\rho \rho^{\rho(p-q)}}{w^k}\,\middle|\, \begin{matrix} \Delta(k,1-\sigma),\Delta(\rho,a_p) \\ \Delta(\rho,b_q) \end{matrix}\right],$$

provided $Re(\sigma+\frac{k}{\rho}\min b_j)>0\,(j=1,\ldots,m), Re(w)>0, c^*=m+n-\frac{p}{2}-\frac{q}{2}>0, |\arg\eta|<c^*\pi, U=\sum_{j=1}^q b_j-\sum_{j=1}^p a_j+\frac{p}{2}-\frac{q}{2}+1$, where the symbol $\Delta(r,a)$ represents the set of r parameters $\frac{a}{r},\frac{a+1}{r},\ldots,\frac{a+r-1}{r}$, we get

$$I_1 = p^{-2k+\alpha-2\nu+n}\, G^{r,2}_{s+2,r}\left[\frac{1}{pw}\,\middle|\, \begin{matrix} 1-2k+\alpha-2\nu+n,1,\beta_1,\ldots,\beta_s \\ \alpha_1,\ldots,\alpha_r \end{matrix}\right],$$

$Re(p)>0, Re(2\nu-n+\min(\alpha_j))>Re(\alpha)$ for $j=1,\ldots,r;\ r>s-1, |\arg(\frac{1}{w})|<(r-s+1)\frac{\pi}{2}$.

Therefore,

$$\int_0^\infty y^{\alpha-1}(y+p)^n[(y+p)^2+a^2]^{-\nu}\, {}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy$$
$$= \frac{\Gamma(\alpha)}{\Gamma(2\nu-n)}\Gamma\left[\begin{matrix} \beta_1,\ldots,\beta_s \\ \alpha_1,\ldots,\alpha_r \end{matrix}\right] p^{\alpha-2\nu+n} \sum_{k=0}^{+\infty} \frac{(\nu)_k\left(-\frac{a^2}{4p^2}\right)^k}{\left(\nu-\frac{n}{2}\right)_k\left(\nu-\frac{n}{2}+\frac{1}{2}\right)_k k!}$$
$$\times\, G^{2,r}_{r,s+2}\left[pw\,\middle|\, \begin{matrix} 1-\alpha_1,\ldots,1-\alpha_r \\ 2k-\alpha+2\nu-n,0,1-\beta_1,\ldots,1-\beta_s \end{matrix}\right], \tag{15}$$

$Re(p)>0;\ Re(\alpha)>0$ if $r<s+1;\ Re(w)>0$ if $r=s+1;\ Re(\nu-n/2)>0, Re(2\nu-n+\min(\alpha_j))>Re(\alpha)$ for $j=1,\ldots,r,\ r>s-1, |\arg(\frac{1}{w})|<(r-s+1)\frac{\pi}{2}$.

Analogously, we obtain some other results, listed below, with the references used.

$$\int_0^\infty y^{\alpha-1}(y+p)^n(y+p-\beta)^{-2\nu}\, {}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy$$
$$= \frac{\Gamma(\alpha)}{\Gamma(2\nu-n)}\Gamma\left[\begin{matrix} \beta_1,\ldots,\beta_s \\ \alpha_1,\ldots,\alpha_r \end{matrix}\right] p^{\alpha-2\nu+n} \sum_{k=0}^{+\infty} \frac{(2\nu)_k}{(2\nu-n)_k k!}\left(\frac{\beta}{p}\right)^k$$
$$\times\, G^{2,r}_{r,s+2}\left[pw\,\middle|\, \begin{matrix} 1-\alpha_1,\ldots,1-\alpha_r \\ k-\alpha+2\nu-n,0,1-\beta_1,\ldots,1-\beta_s \end{matrix}\right], \tag{16}$$

$Re(p) > 0, Re(\alpha) > 0$ if $r < s+1$; $Re(w) > 0$ if $r = s+1$; $Re(\nu - n/2) > 0, Re(2\nu - n + \min(\alpha_j)) > Re(\alpha)$ for $j = 1,\ldots,r$, $r > s-1, |\arg(\frac{1}{w})| < (r-s+1)\frac{\pi}{2}$. ([1, No.5.4.8]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n e^{a/(y+p)}\,\gamma(n+1,\tfrac{a}{y+p})$$
$$\times\, {}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy = \frac{\Gamma(\alpha)}{(n+1)}\Gamma\left[\begin{array}{c}\beta_1,\ldots,\beta_s\\ \alpha_1,\ldots,\alpha_r\end{array}\right] a^{n+1}p^{\alpha-1}$$
$$\sum_{k=0}^{+\infty}\frac{\left(\frac{a}{p}\right)^k}{(n+2)_k k!}\,G^{2,r}_{r,s+2}\left[pw\left|\begin{array}{c}1-\alpha_1,\ldots,1-\alpha_r\\ 1-\alpha+k,0,1-\beta_1,\ldots,1-\beta_s\end{array}\right.\right], \tag{17}$$

$Re(p) > 0$; $Re(\alpha) > 0$ if $r < s+1$; $Re(w) > 0$ if $r = s+1$; $Re(1+\min(\alpha_j)) > Re(\alpha)$ for $j = 1,\ldots,r$, $r > s-1, |\arg(\frac{1}{w})| < (r-s+1)\frac{\pi}{2}$. ([1, No.5.11.26], [6, No.7.13.1.1]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n\, Q_{2\nu}(\sqrt{y+p})\,{}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy$$
$$= \frac{\sqrt{\pi}\Gamma(\alpha)\Gamma(2\nu+1)}{2^{2\nu+1}\Gamma(2\nu+3/2)\Gamma(\nu-n+1/2)}\Gamma\left[\begin{array}{c}\beta_1,\ldots,\beta_s\\ \alpha_1,\ldots,\alpha_r\end{array}\right] p^{n+\alpha-\nu-1/2}$$
$$\times\sum_{k=0}^{+\infty}\frac{\left(\nu+\frac{1}{2}\right)_k(\nu+1)_k p^{-k}}{\left(2\nu+\frac{3}{2}\right)_k\left(\nu-n+\frac{1}{2}\right)_k k!} \tag{18}$$
$$\times\, G^{2,r}_{r,s+2}\left[pw\left|\begin{array}{c}1-\alpha_1,\ldots,1-\alpha_r\\ 1/2+k-\alpha+\nu-n,0,1-\beta_1,\ldots,1-\beta_s\end{array}\right.\right],$$

$Re(p) > 0$; $Re(\alpha) > 0$ if $r < s+1$; $Re(w) > 0$ if $r = s+1$; $Re(\nu-n) > -\frac{1}{2}$, $Re(\nu-n+\frac{1}{2}+\min(\alpha_j)) > Re(\alpha)$ for $j = 1,\ldots,r$, $r > s-1, |\arg(\frac{1}{w})| < (r-s+1)\frac{\pi}{2}$. ([1, No.5.13.10]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n K_{2n}(2\sqrt{a(y+p)})\,{}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy$$
$$= 2^{-2\alpha-4n}\sqrt{\pi}\Gamma(\alpha)a^{-\alpha-n}e^{-2\sqrt{ap}}$$
$$\times\sum_{k=0}^{+\infty}\frac{\prod_{j=1}^r(\alpha_j)_k}{\prod_{j=1}^s(\beta_j)_k k!}\left(-\frac{w}{4a}\right)^k\Psi\left(-\alpha-k-2n+\tfrac{1}{2},-2\alpha-2k-4n+1;4\sqrt{ap}\right), \tag{19}$$

$Re(p) > 0$; $Re(\alpha) > 0$ if $r < s+1$; $Re(w) > 0$ if $r = s+1$, $Re(a) > 0$.
([1, No.5.16.40], [3, No.3.471.9], [4,, No.9.13.15]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n e^{a/2(y+p)}\, M_{n,\mu}\left(\frac{a}{y+p}\right)$$
$$\times\, {}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy = \frac{\Gamma(\alpha)}{\Gamma(\mu-n+1/2)}$$
$$\times\,\Gamma\left[\begin{array}{c}\beta_1,\ldots,\beta_s\\ \alpha_1,\ldots,\alpha_r\end{array}\right] a^{\mu+1/2}p^{\alpha+n-\mu-1/2}\sum_{k=0}^{+\infty}\frac{\left(\frac{a}{p}\right)^k}{(2\mu+1)_k k!} \tag{20}$$
$$\times\, G^{2,r}_{r,s+2}\left[pw\left|\begin{array}{c}1-\alpha_1,\ldots,1-\alpha_r\\ 1/2+\mu+k-\alpha-n,0,1-\beta_1,\ldots,1-\beta_s\end{array}\right.\right],$$

$Re(p)>0, Re(\alpha)>0$ if $r<s+1$; $Re(w)>0$ if $r=s+1$; $Re(n-\mu)<\frac{1}{2}, Re(\mu-n+\frac{1}{2}+\min(\alpha_j))>Re(\alpha)$ for $j=1,\ldots,r$, $r>s-1, |\arg(\frac{1}{w})|<(r-s+1)\frac{\pi}{2}$. ([1, No.5.20.2]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n e^{(y+p)/2a} W_{\kappa,\mu}\left(\frac{y+p}{a}\right)$$
$$\times\, {}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy = \frac{\Gamma(\alpha)}{\Gamma(-n-\kappa)}\,\Gamma\left[\begin{array}{c}\beta_1,\ldots,\beta_s\\ \alpha_1,\ldots,\alpha_r\end{array}\right]$$
$$a^{-\kappa}p^{\alpha+n+\kappa}\sum_{k=0}^{+\infty}\frac{\left(\frac{1}{2}-\kappa+\mu\right)_k\left(\frac{1}{2}-\kappa-\mu\right)_k}{(-n-\kappa)_k k!}\left(-\frac{a}{p}\right)^k$$
$$\times\, G^{2,r}_{r,s+2}\left[pw\left|\begin{array}{c}1-\alpha_1,\ldots,1-\alpha_r\\ k-\alpha-n-\kappa,0,1-\beta_1,\ldots,1-\beta_s\end{array}\right.\right], \tag{21}$$

$Re(p)>0$; $Re(\alpha)>0$ if $r<s+1$; $Re(w)>0$ if $r=s+1$; $Re(n+\kappa)<0, Re(-n-\kappa+\min(\alpha_j))>Re(\alpha)$ for $j=1,\ldots,r$, $|\arg(a)|<\pi$. ([1, No.5.20.9]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n e^{a/(y+p)}\gamma\left(\nu,\frac{a}{y+p}\right)\, {}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy$$
$$=\frac{\Gamma(\alpha)}{\nu\Gamma(\nu-n)}\,\Gamma\left[\begin{array}{c}\beta_1,\ldots,\beta_s\\ \alpha_1,\ldots,\alpha_r\end{array}\right] a^\nu p^{\alpha+n-\nu}\sum_{k=0}^{+\infty}\frac{(1)_k\left(\frac{a^k}{p}\right)}{(\nu+1)_k(\nu-n)_k k!}$$
$$\times\, G^{2,}_{r,r,s+2}\left[pw\left|\begin{array}{c}1-\alpha_1,\ldots,1-\alpha_r\\ k+\nu-n-\alpha,0,1-\beta_1,dots,1-\beta_s\end{array}\right.\right], \tag{22}$$

$Re(p)>0$; $Re(\alpha)>0$ if $r<s+1$; $Re(w)>0$ if $r=s+1$; $Re(\nu)>0, Re(\nu-n+\min(\alpha_j))>Re(\alpha)$ for $j=1,\ldots,r$, $r>s-1, |\arg(\frac{1}{w})|<(r-s+1)\pi$. ([1, No.5.11.29]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n J_\nu\left(\frac{4a}{y+p}\right) {}_rF_{s+1}(\alpha_1,\dots,\alpha_r;\alpha,\beta_1,\dots,\beta_s;-wy)dy$$
$$= \frac{\Gamma(\alpha)}{\Gamma(\nu+1)\Gamma(\nu-n)}\Gamma\left[\begin{array}{c}\beta_1,\dots,\beta_s\\ \alpha_1,\dots,\alpha_r\end{array}\right](2a)^\nu p^{n+\alpha-\nu}$$
$$\times \sum_{k=0}^{+\infty}\frac{\left(-\frac{a^2}{p^2}\right)^k}{(\nu+1)_k\left(\frac{\nu}{2}-\frac{n}{2}\right)_k\left(\frac{\nu}{2}-\frac{n}{2}+\frac{1}{2}\right)_k k!} \tag{23}$$
$$\times G^{2,r}_{r,s+2}\left[pw\left|\begin{array}{c}1-\alpha_1,\dots,1-\alpha_r\\ 2k+\nu-n-\alpha,0,1-\beta_1,\dots,1-\beta_s\end{array}\right.\right],$$

$Re(p)>0; Re(\alpha)>0$ if $r<s+1; Re(w)>0$ if $r=s+1; Re(\nu-n)>0, Re(\nu-n+\min(\alpha_j))>Re(\alpha)$ for $j=1,\dots,r$, $r>s-1, |\arg(\frac{1}{w})|<(r-s+1)\pi$. ([1, No.5.14.12]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n e^{a/(y+p)} I_\nu\left(\tfrac{a}{y+p}\right) {}_rF_{s+1}(\alpha_1,\dots,\alpha_r;\alpha,\beta_1,\dots,\beta_s;-wy)dy$$
$$= \frac{\Gamma(\alpha)}{\Gamma(\nu+1)\Gamma(\nu-n)}\Gamma\left[\begin{array}{c}\beta_1,\dots,\beta_s\\ \alpha_1,\dots,\alpha_r\end{array}\right]\frac{a^\nu}{2^\nu}p^{\alpha+n-\nu}\sum_{k=0}^{+\infty}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{2a}{p}\right)^k}{(2\nu+1)_k(\nu-n)_k k!}$$
$$\times G^{2,r}_{r,s+2}\left[pw\left|\begin{array}{c}1-\alpha-1,\dots,1-\alpha_r\\ k+\nu-n-\alpha,0,1-\beta_1,\dots,1-\beta_s\end{array}\right.\right], \tag{24}$$

$Re(p)>0; Re(\alpha)>0$ if $r<s+1; Re(w)>0$ if $r=s+1; Re(\nu-n)>0, Re(\nu-n+\min(\alpha_j))>Re(\alpha)$ for $j=1,\dots,r$, $r>s-1, |\arg(\frac{1}{w})|<(r-s+1)\pi$. ([1, No.5.16.18]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n L_\nu\left(\tfrac{2a}{y+p}\right) {}_rF_{s+1}(\alpha_1,\dots,\alpha_r;\alpha,\beta_1,\dots,\beta_s;-wy)dy$$
$$= \frac{2\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\nu+3/2)\Gamma(\nu-n+1)}\Gamma\left[\begin{array}{c}\beta_1,\dots,\beta+s\\ \alpha_1,\dots,\alpha_r\end{array}\right]a^{\nu+1}p^{\alpha+n-\nu-1}$$
$$\times \sum_{k=0}^{+\infty}\frac{(1)_k\left(\frac{a^2}{4p^2}\right)^k}{\left(\frac{3}{2}\right)_k\left(\nu+\frac{3}{2}\right)_k\left(\frac{\nu-n+1}{2}\right)_k\left(\frac{\nu-n}{2}+1\right)_k k!} \tag{25}$$
$$\times G^{2,r}_{r,s+2}\left[pw\left|\begin{array}{c}1-\alpha_1,\dots,1-\alpha_r\\ 1-\alpha+\nu-n+2k,0,1-\beta_1,\dots,1-\beta_s\end{array}\right.\right],$$

$Re(p)>0; Re(\alpha)>0$ if $r<s+1; Re(w)>0$ if $r=s+1; Re(\nu-n)>-1, Re(\nu-n+1+\min(\alpha_j))>Re(\alpha)$ for $j=1,\dots,r$, $r>s-1, |\arg(\frac{1}{w})|<(r-s+1)\pi$.

([1, No.5.17.22]).

$$\int_0^\infty y^{\alpha-1}(y+p)^n \left[(y+p)^3+a^3\right]^{-\nu} {}_rF_{s+1}(\alpha_1,\ldots,\alpha_r;\alpha,\beta_1,\ldots,\beta_s;-wy)dy$$
$$= \Gamma\left[\begin{array}{c}\beta_1,\ldots,\beta_s\\ \alpha_1,\ldots,\alpha_r\end{array}\right] \frac{\Gamma(\alpha)}{\Gamma(3\nu-n)}\, p^{\alpha+n-3\nu} \sum_{k=0}^{+\infty} \frac{(\nu)_k\left(-\frac{a^3}{27p^3}\right)^k}{\left(\nu-\frac{n}{3}\right)_k\left(\nu-\frac{n}{3}+\frac{1}{3}\right)_k\left(\nu-\frac{n}{3}+\frac{2}{3}\right)_k k!}$$
$$\times G^{2,r}_{r,s+2}\left[pw\left|\begin{array}{c}1-\alpha-1,\ldots,1-\alpha_r\\ 3k-\alpha-n+3\nu,0,1-\beta_1,\ldots,1-\beta_s\end{array}\right.\right], \tag{26}$$

$Re(p) > 0; Re(\alpha) > 0$ if $r < s+1; Re(w) > 0$ if $r = s+1; Re(\nu - n/3) > 0, Re(3\nu-n+\min(\alpha_j)) > Re(\alpha)$ for $j = 1,\ldots,r$, $r > s-1$, $|\arg(\frac{1}{w})| < (r-s+1)\pi$. ([1, No.5.4.11]).

Acknowledgement

L. Galué and S. L. Kalla would like to thank C.O.N.D.E.S. of "Universidad del Zulia" for financial support.

References

[1] A. E r d é l y i, H. B a t e m a n (ed-s), *Tables of Integral Transforms*, Vol. **1**. Mc Graw-Hill, New York, 1954.

[2] M. L. G l a s s e r, V. K o w a l e n k o, Extensions and results from a method for evaluating fractional integrals. *Integral Transforms & Special Functions* (1996). To appear.

[3] I. S. G r a d s h t e y n, I. M. R y z h i k, *Tables of Integrals, Series and Products*. Academic Press, New York, 1980.

[4] N. N. L e b e d e v, *Special Functions and Their Applications.* Prentice Hill, New York, 1965.

[5] A. M. M a t h a i, R. K. S a x e n a, *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences.* Springer-Verlag, New York, 1973.

[6] A. P. P r u d n i k o v, Yu. A. B r y c h k o v, I. O. M a r i c h e v, *Integrals and Series: More Special Functions*, Vol. **3**. Gordon and Breach, New York, 1986.

*) *Centro de Investigación de Matemática Aplicada,*
Facultad de Ingeniería, Universidad del Zulia,
Maracaibo, VENEZUELA

Received: 12.12.1995
Revised: 20.04.1996

**) *Department of Mathematics,*
Kuwait University, P.O. Box 5969, Safat 13060,
KUWAIT