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Numerical Computation of Eigenvalues of Hamiltonian Matrices [1]

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An effective algorithm for computing eigenvalues of real Hamiltonian matrices is proposed. The algorithm is based on similar orthogonal and simplectic transformations. It can be used also for solution of matrix algebraic Riccati equation.

In this paper we consider square $2n \times 2n$ matrices. Let I_n be a unit $n \times n$ matrix and J be a block matrix $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

A real matrix H is called Hamiltonian (J-antisymmetric), if $J^T H J = -H^T$ [1, 5, 6]. The real Hamiltonian matrix H is of the type

$$H = H(A, B, D) = \begin{pmatrix} A & B \\ D & -A^T \end{pmatrix}, \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $B = B^T \in \mathbf{R}^{n\times n}$, $D = D^T \in \mathbf{R}^{n\times n}$, and $\mathbf{R}^{n\times n}$ is the set of real $n \times n$ matrices.

I k r a m o v [1] raises the question for construction an ortogonal and simplectic algorithm reducing the Hamiltonian matrix H to matrix $\tilde{H}$ of the type

$$\tilde{H} = H(\tilde{A}, \tilde{B}, 0) = \begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & -\tilde{A}^T \end{pmatrix},$$

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where $\tilde{A}$ is an upper Hessenberg matrix, $\tilde{B} = \tilde{B}^T$. We can use this algorithm for solving matrix algebraic Riccati equation

$$L(X) = XBX - XA - A^T X - D = 0, \tag{2}$$

where $A \in \mathbf{R}^{n\times n}$, $B = B^T \in \mathbf{R}^{n\times n}$, $D = D^T \in \mathbf{R}^{n\times n}$, B is a positive definite matrix and D is a positive semidefinite matrix.

The matrix Riccati equation has many apllications. In control theory, in research of mechanics systems we are to compute a solution of (2). It is well known that this equation has a symmetric positive definite solution P [2]. The computation of solution of (2) is equivalent to the solving of the spectral problem of the real square Hamiltonian matrix $H = H(A, -B, -D)$.

In this paper an algorithm for computing eigenvalues of H is presented. Let us consider

$$Hz = \lambda z,$$

where H is a Hamiltonian $2n \times 2n$ real matrix.

Our algorithm generalizes the algorithms from [3, 4, 6] and computes eigenvalues and eigenvectors of the matrix H.

In the algorithm we construct the sequence

$$H_{k+1} = U_k^{-1} H_k U_k = (h_{ij}^{(k)}),\ H_1 = H, k = 1, 2, 3, \ldots, \tag{3}$$

where $U_k = U_{p_k q_k}(\varphi_k)$ is a suitable matrix.

Since the computations in each step are similar, we consider the k-th step of the algorithm. We introduce the notations

$$H_k = H(A_k, B_k, D_k) = (h_{ij}^{(k)}),$$

$$A_k = (a_{\beta\gamma}^{(k)}),\ B = (b_{\beta\gamma}^{(k)}),\ D_k = (d_{\beta\gamma}^{(k)}).$$

and compute the matrices $H_k + H_k^T$ and C_k, for which we obtain:

$$H_k + H_k^T = H(A_k + A_k^T, B_k + D_k, D_k + B_k),$$

$$C_k = C(H_k) = (c_{ij}^{(k)}) = H_k H_k^T - H_k^T H_k = H(F_k, E_k, E_k),$$

where

$$F_k = F_k^T = A_k A_k^T + B_k B_k - A_k^T A_k - D_k D_k = (f_{\beta\gamma}^{(k)}),$$

$$E_k = E_k^T = A_k D_k - B_k A_k - A_k^T B_k + D_k A_k^T = (e_{\beta\gamma}^{(k)}).$$

We compute also the numbers

$$h^{(k)} = \max_{i \neq j} |h_{ij}^{(k)} + h_{ji}^{(k)}|, \quad c^{(k)} = \max_{i \neq j} |c_{ij}^{(k)}|^{\frac{1}{2}}.$$

The choice of the matrix U_k depends on the bigger of the numbers $h^{(k)}$ and $c^{(k)}$. The similar transformation (3) preserves the structure of H_k, i.e. $H_k = H(A_k, B_k, D_k)$.

Then there are five possible cases:

A.1. Let $|f_{pq}^{(k)}|^{\frac{1}{2}} = c^{(k)} \geq h^{(k)}$, $1 \leq p = p_k < q = q_k \leq n$, $\varphi = \varphi_k$. In this case the matrix $U_k = U_{pq}(\varphi)$ is of the type

$$U_k = U_{pq}(\varphi) = \begin{pmatrix} V & 0 \\ 0 & V^{-1} \end{pmatrix}, \tag{4}$$

where $V \in \mathbf{R}^{n \times n}$ and

$$V = (v_{\beta\gamma}) = \begin{cases} v_{pp} = v_{qq} = \mathrm{ch}\varphi \\ v_{pq} = v_{qp} = \mathrm{sh}\varphi \\ v_{\beta\gamma} = 0, \quad (\beta\gamma) \notin \{(p,p),(p,q),(q,p),(q,q)\}. \end{cases} \tag{5}$$

In the matrix (5) we define φ by

$$\mathrm{th}\varphi = \frac{2f_{pq}^{(k)}}{G + 2L}, \tag{6}$$

where

$$\begin{aligned} G &= 2\sum_{i \neq p,q} ((a_{ip}^{(k)})^2 + (a_{iq}^{(k)})^2 + (a_{pi}^{(k)})^2 + (a_{qi}^{(k)})^2) \\ &+ \sum_{i \neq p,q} ((b_{ip}^{(k)})^2 + (b_{iq}^{(k)})^2 + (b_{pi}^{(k)})^2 + (b_{qi}^{(k)})^2) \\ &+ \sum_{i \neq p,q} ((d_{ip}^{(k)})^2 + (d_{iq}^{(k)})^2 + (d_{pi}^{(k)})^2 + (d_{qi}^{(k)})^2), \\ L &= 2T_A^2 + 2\xi_A^2 + T_B^2 + \xi_B^2 + T_D^2 + \xi_D^2, \\ T_A &= a_{pp}^{(k)} - a_{qq}^{(k)},\ T_B = b_{pp}^{(k)} + b_{qq}^{(k)},\ T_D = d_{pp}^{(k)} + d_{qq}^{(k)}, \end{aligned}$$

$$\xi_A = a_{pq}^{(k)} - a_{qp}^{(k)},\ \xi_B = b_{pq}^{(k)} + b_{qp}^{(k)},\ \xi_D = d_{pq}^{(k)} + d_{qp}^{(k)}.$$

A.2. Let $|e_{pq}^{(k)}|^{\frac{1}{2}} = c^{(k)} \geq h^{(k)}$, $1 \leq p = p_k < q = q_k \leq n$, $\varphi = \varphi_k$. In this case we choose the matrix $U_k = U_{pq}(\varphi)$ of the form

$$U_k = U_{pq}(\varphi) = \begin{pmatrix} V & Z \\ Z & V \end{pmatrix}, \tag{7}$$

where $V \in \mathbf{R}^{n\times n}$, $Z \in \mathbf{R}^{n\times n}$ are of the form

$$\begin{cases} V = \mathrm{diag}[I_{p-1}, \mathrm{ch}\varphi, I_{q-p-1}, \mathrm{ch}\varphi, I_{n-q}] \\ Z = (z_{\beta\gamma}) = \begin{cases} z_{pq} = z_{qp} = \mathrm{sh}\varphi \\ z_{\beta\gamma} = 0, \quad (\beta,\gamma) \notin \{(p,q),(q,p)\}. \end{cases} \end{cases} \tag{8}$$

In the matrices (8) we define φ by

$$\mathrm{th}\varphi = \frac{2e_{pq}^{(k)}}{G+2L}, \tag{9}$$

where

$$\begin{aligned} L \;=\; & 2(a_{pp}^{(k)} + a_{qq}^{(k)})^2 + 2(b_{pq}^{(k)} - d_{pq}^{(k)})^2 + (a_{pq}^{(k)} + a_{pq}^{(k)})^2 \\ + \; & (a_{qp}^{(k)} + a_{qp}^{(k)})^2 + (b_{pp}^{(k)} - d_{qq}^{(k)})^2 + (b_{qq}^{(k)} - d_{pp}^{(k)})^2. \end{aligned}$$

When $p = q$ we choose V and Z in the matrix $U_{pq}(\varphi)$ from (7) of the type

$$\begin{cases} V = \mathrm{diag}[I_{p-1}, \mathrm{ch}\varphi, I_{n-p}] \\ Z = (z_{\beta\gamma}) = \begin{cases} z_{pp} = \mathrm{sh}\varphi \\ z_{\beta\gamma} = 0, \quad (\beta,\gamma) \neq (p,p). \end{cases} \end{cases} \tag{10}$$

In this case we define the parameter φ from (10) by formula

$$\mathrm{th}\varphi = \frac{2e_{pp}^{(k)}}{G+2(D^2+\xi^2)}, \tag{11}$$

$$G = 2\sum_{i\neq p,p+n} (h_{ip}^{(k)})^2 + (h_{pi}^{(k)})^2 + (h_{ip+n}^{(k)})^2 + (h_{p+ni}^{(k)})^2,$$

$$D = h_{pp}^{(k)} - h_{p+np+n}^{(k)}, \quad \xi = h_{pp+n}^{(k)} - h_{p+np}^{(k)}.$$

Remark. In the cases **A.1** and **A.2**, when we choose U_k from (4) or (8) or (10) and we choose φ from (6), or (9), or (11) respectively, we have

$$||H_k||^2 - ||H_{k+1}||^2 \geq \frac{1}{3}\frac{(c^{(k)})^4}{||H_k||^2}. \tag{12}$$

A.3. Let $|a_{pq}^{(k)} + a_{qp}^{(k)}| = h^{(k)} > c^{(k)}$, $1 \leq p = p_k < q = q_k \leq n$, $\varphi = \varphi_k$. In this case we choose the matrix $U_k = U_{pq}(\varphi)$ of the type

$$U_{pq}(\varphi) = \text{diag}[T_{pq}(\varphi), T_{pq}(\varphi)], \tag{13}$$

where $T_{pq}(\varphi) \in \mathbf{R}^{n\times n}$ is a matrix of the form

$$T_{pq}(\varphi) = (t_{\beta\gamma}) = \begin{cases} t_{pp} = t_{qq} = \cos\ \varphi \\ t_{pq} = -t_{qp} = -\sin\ \varphi \\ t_{\beta\gamma} = \delta_{\beta\gamma}\ \ (\beta,\gamma) \notin \{(p,p),(p,q),(q,p),(q,q)\}. \end{cases}$$

For φ in the matrix (13) we have

$$\text{tg}2\varphi = \frac{a_{pq}^{(k)} + a_{qp}^{(k)}}{a_{pp}^{(k)} - a_{qq}^{(k)}}. \tag{14}$$

A.4. Let $|b_{pq}^{(k)} + b_{qp}^{(k)}| = h^{(k)} > c^{(k)}$, $1 \leq p = p_k < q = q_k \leq n$, $\varphi = \varphi_k$. In this case we choose the matrix $U_k = U_{pq}(\varphi)$ of the type

$$U_{pq}(\varphi) = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}, \tag{15}$$

where $C,\ S \in \mathbf{R}^{n\times n}$ are determined by

$$\begin{aligned} C &= \text{diag}[I_{p-1}, \cos\varphi, I_{q-p}, \cos\varphi, I_{m-q}], \\ S &= (s_{\beta\gamma}) = \begin{cases} s_{pq} = s_{qp} = \sin\varphi \\ s_{\beta\gamma} = 0,\ \ (\beta,\gamma) \notin \{(p,q),(q,p)\}. \end{cases} \end{aligned}$$

For φ in the matrix (15) we have

$$\text{tg}2\varphi = \frac{b_{pq}^{(k)} + d_{qp}^{(k)}}{a_{pp}^{(k)} + a_{qq}^{(k)}}. \tag{16}$$

A.5. Let $|b_{pp}^{(k)} + b_{pp}^{(k)}| = h^{(k)} > c^{(k)}$, $1 \leq p = p_k = q = q_k \leq n$, $\varphi = \varphi_k$. In this case we choose the matrix $U_k = U_{pq}(\varphi)$ of the type $(1 \leq p = q \leq n)$

$$U_{pq}(\varphi) = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}, \tag{17}$$

where $C,\ S \in \mathbf{R}^{n\times n}$ are

$$\begin{aligned} C &= \text{diag}[I_{p-1}, \cos\varphi, I_{m-p}], \\ S &= (s_{\beta\gamma}) = \begin{cases} s_{pp} = \sin\varphi \\ s_{\beta\gamma} = 0,\ \ (\beta,\gamma) \notin \{(p,p)\}. \end{cases} \end{aligned}$$

For φ in the matrix (17) we have

$$\mathrm{tg}2\varphi = \frac{b_{pp}^{(k)} + d_{pp}^{(k)}}{a_{pp}^{(k)} + a_{pp}^{(k)}}.$$

Lemma. *Let α and β be integers and $1 \leq \alpha \neq \beta \leq 2n$. Let $\tilde{H} = U^{-1}HU$, where U is chosen from (4) or (7) and φ is chosen from (6) or (9) respectively, depending on α and β. Then*

$$|\tilde{h}_{ij} - h_{ij}| \leq \sqrt{\frac{3}{2}}|c_{\alpha\beta}|^{\frac{1}{2}} \quad \text{for all} \quad i, j,$$

where $H = H(A,B,D) = (h_{ij})$, $\tilde{H} = H(\tilde{A},\tilde{B},\tilde{D}) = (\tilde{h}_{ij})$, $C(H) = H(F,E,E) = (c_{ij})$.

This lemma is proved by analogy with Lemma 1 [4].

Theorem. *For the sequence (3) we have*

I. $C(H_k) \to 0$, $k \to \infty$.

II. The symmetric matrix $\frac{1}{2}(H_k + H_k^T)$ tends to the diagonal matrix $\frac{1}{2}(H_0 + H_0^T)$, where $H_0 = H(A_0, B_0, D_0) = (h_{ij}^{(0)})$,

$$\frac{1}{2}(H_0 + H_0^T) = diag[h_{11}^{(0)}, \ldots, h_{2n2n}^{(0)}],$$

where $h_{ii}^{(0)}$ are real parts of eigenvalues of H.

III. Let for $1 \leq p \neq q \leq 2n$ we have $h_{pp}^{(0)} \neq h_{qq}^{(0)}$. Then

$$h_{pq}^{(k)} \to 0, \quad k \to \infty.$$

IV. Let for $1 \leq p \neq q \leq 2n$ we have $h_{pp}^{(0)} = h_{qq}^{(0)}$ and for each $t \neq p, q$ and $(1 \leq t \leq 2n)$ we have $h_{tt}^{(0)} \neq h_{pp}^{(0)}$. Then

$$h_{pq}^{(k)} \to h_{pq}^{(0)}, \quad k \to \infty,$$

where $h_{pq}^{(0)}$ is the imaginary part of eigenvalues with a real part $h_{pp}^{(0)}$.

The theorem is proved by analogy with Theorem from [4].

Our algorithm for computing of eigenvalues and eigenvectors of real Hamiltonian matrix can be used successfully for solving the matrix equation (2). One method [2] for computing solution P of (2) follows. We reduce the matrix $H = H(A,-B,-D)$ with QR-transformations in form of Shur

$$\tilde{H} = U^T H U, \quad U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}.$$

For the solution P of equation (2) we have $P = U_{21}U_{11}^{-1}$.

This method can be changed as follows. We compute eigenvalues and eigenvectors of the matrix $H = H(A, -B, -D)$ with our algorithm. Let U be the matrix of eigenvectors of H and $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. We compute the solution P.

We make numerical experiments for computing of solution P of algebraic Riccati equation (2). The first method, who uses QR-transformations we denote by $W1$ and the second method, who uses our algorithm for computing eigenvalues of Hamiltonian matix H, we denote by $W2$. The method $W1$ is compared with the method $W2$ for computing the solution of Riccati equation.

Our algorithm uses less memory. The iterative process (3) computes maximum element on each step. In the program of our algorithm a cyclic choice on the pivot indices (p,q) is used. The computations were made on computer Pentium. The elements of the blocks of H are chosen with function for random numbers. The experiments are accomplished for different n. For each trial we compute the time for obtaining a solution P of (2) and the accuracy of the solution P – $||L(P)||_1$. The results from experiments show that the time for computing P with algorithm $W2$ is bigger.

There exist examples for which the accuracy for solution P, obtained from $W2$ is less than accuracy from $W1$. Such kind of example for matrix $H = H(A, B, D)$ is the following:

$$A = \begin{cases} ij & i = j \\ i + j & i \neq j \end{cases}$$

$$B = \begin{cases} ij & i = j \\ 0 & i \neq j \end{cases} \qquad D = \begin{cases} i & i = j \\ 0 & i \neq j \end{cases}.$$

Table 1 shows the values of $||L(P)||_1$ for different values of n, which are obtained from algorithms $W1$ and $W2$.

	W1	W2	W1	W2
n	10	10	20	20
$\|\|L(P)\|\|_1$	$3.6755 \cdot 10^{-4}$	$1.0343 \cdot 10^{-7}$	$4.2971 \cdot 10^{-3}$	$1.5521 \cdot 10^{-6}$

Table 1.

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