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Global Existence for Semilinear Massless Dirac Equation with Small Data [1]

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We prove existence of global solutions to a semilinear massless Dirac equation. We construct solutions in the classical Sobolev spaces. Our approach is based on using the conservation law of charge together with an weighted estimate for L^∞ norm of the spinor field.

1. Introduction

In this work we consider the Cauchy problem:

$$\begin{aligned} \mathcal{D}\psi &= F(\psi), \\ \psi(0,x) &= \eta(x), \end{aligned} \tag{1}$$

where $\mathcal{D} \equiv i\gamma^\mu\partial_\mu$ (with the usual summation convention) is the Dirac operator, $\partial_0 = \partial_t, \partial_j = \partial_{x_j}, 1 \leq j \leq 3, F(\psi) = O(|\psi|^n), \psi = (\psi_1, ..., \psi_4)^t$ is the spinor field and γ^μ are the Dirac matrices. We use the following representation for these matrices:

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

$$1 \leq j \leq 3,$$

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where σ^μ are the Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We are interested for which n the initial value problem (1) is well-posed in classical Sobolev space H^s. The well-posedness means that if $\eta \in H^s$ is sufficiently small then $\psi(t,.) \in H^s$.We are also interested in decay properties of the solution ψ. Our approach is based on using energy method combining with an estimate for weighted L^∞ norm of the solution of (1).

We have the following relation:

$$\mathcal{D} \circ \mathcal{D} = (\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2)I_4 \tag{2}$$

where I_4 is the unit 4×4 matrix.

There are a lot of estimates for the wave equation one may try to use for the Dirac equation. By means of using a suitable $L^\infty - L^\infty$ estimate F.John [4] has obtained that for $p > 1 + \sqrt{2}$ the semilinear wave equation $(\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2)u = |u|^{p-1}u$ has a global solution with small initial data. Moreover, for $p \leq 1 + \sqrt{2}$ the solution might blow-up in finite time. One is not able to use F.John's estimate for the Dirac equation in its pure form because of impossibility to stop the losses of derivatives. In order to overcome this difficulty it is natural for one to use estimates of energy type. Combining F.John's estimate with the Sobolev embedding and energy estimate one obtains solution of (1) in H^3 for $n > 2 + \sqrt{2}$. In this work we replace Sobolev inequality (which always costs losses of derivatives) with a more refined approach based on ideas similar to these of W.v.Wahl [10]. For $n > 3$ we obtain a solution in H^2. In the case $n = 3$ we derive a solution in H^3(which is also done in Georgiev and Kovachev [3], as an improvement of Bachelot's [1] result).

The plan of work is the following. In Section 2 we prove an $L^\infty - L^\infty$ weighted estimate for the homogeneous problem. In Section 3 we prove an estimate for the wave equation which has an interest of its own. In Section 4 we define a Banach subspace of H^2 where the solutions are expected to belong. Decay properties of the solutions could be seen from the definition of this Banach space. In Section 5 we study low-regularity solutions with Sobolev exponent

$s = 2$ and for $n > 3$. In Section 6 we study well-posedness of (1) in H^s. If $n > 3$ is an integer then, (1) is well-posed in H^s for $s \geq 2$, providing the initial data are sufficiently small. To obtain this we also use an estimate from [9]. If n is not integer then (1) has global solution in H^s only when $n > s + 1$, and $s \geq 2$ is an integer.

2. $L^\infty - L^\infty$ weighted estimate for the homogeneous equation

We shall study the homogeneous Cauchy problem:

$$\begin{aligned} \mathcal{D}\psi &= 0, \\ \psi(0,x) &= \psi_0(x). \end{aligned} \tag{3}$$

Taking into account (2) we see that for every solution of (3) we have:

$$\begin{aligned} (\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2) I_4 \psi &= 0, \\ \psi(0,x) &= \psi_0(x), \\ \psi_t(0,x) = -\sum_{j=1}^{3} \gamma^0 \gamma^j \partial_j \psi_0 &:= \psi_1(x). \end{aligned} \tag{4}$$

We are going to use the following relation:

Lemma 1. (see [8]) *If f is a continuous function and $r := |x|$, then:*

$$\int_{|y-x|=t} f(|y|) dS_y = 2\pi \frac{t}{r} \int_{|r-t|}^{r+t} \lambda f(\lambda) d\lambda.$$

We shall also use the following estimate:

Lemma 2. *If $n \geq 2$, then:*

$$\int_a^b \frac{ds}{(c+s)^n} \leq \frac{b-a}{(c+a)^{n-1}(c+b)},$$

where $b \geq a \geq 0$, and $c \geq 0$.

Proof. We have that

$$\int_a^b \frac{ds}{(c+s)^n} \leq \frac{1}{(c+a)^{n-2}} \int_a^b \frac{ds}{(c+s)^2} = \frac{b-a}{(c+a)^{n-1}(c+b)}.$$

This completes the proof of the Lemma 2. ∎

Now we are going to prove an $L^\infty - L^\infty$ weighted estimate for the solution of (3).

Theorem 1. (see [8]) *If $\psi(t,x)$ is a solution of (3), then:*

$$|\tau_+\tau_-^{k-2}\psi|_{L^\infty(R_+\times R^3)}$$
$$\le \text{const}\ (|(1+|x|)^{k-1}\psi_0(x)|_{L^\infty} + |(1+|x|)^k\nabla^1\psi_0(x)|_{L^\infty}),$$

where $k>2$ and $\tau_\pm(t,x) = 1 + |t\pm|x||$.

Remark. By $\nabla^k\psi$ we shall denote the following vector: $(\partial^\alpha\psi)_{0\le|\alpha|\le k}$.

Proof of Theorem 1. We shall use the following representation formula for the solution of (4):

$$\psi(t,x) = \frac{1}{4\pi t}\int_{|y-x|=t}\psi_1(y)dS_y$$

$$(5)\qquad +\frac{1}{4\pi t^2}\int_{|y-x|=t}\psi_0(y)dS_y + \frac{t}{4\pi}\int_{|\omega|=1} < \nabla\psi_0(x+t\omega),\omega > d\omega.$$

We shall estimate each term in the right hand-side of (5).

1) Using Lemma 1 and Lemma 2, we get:

$$|\frac{1}{4\pi t}\int_{|y-x|=t}\psi_1(y)dS_y| \le \frac{c}{t}\int_{|y-x|=t}|\nabla^1\psi_0(y)|dS_y$$
$$\le c|(1+|x|)^k\nabla^1\psi_0(x)|_{L^\infty}\cdot\frac{1}{t}\cdot\int_{|y-x|=t}\frac{dS_y}{(1+|y|)^k}$$
$$= c|(1+|x|)^k\nabla^1\psi_0(x)|_{L^\infty}\cdot\frac{1}{r}\cdot\int_{|r-t|}^{r+t}\frac{\lambda d\lambda}{(1+\lambda)^k}$$
$$\le c|(1+|x|)^k\nabla^1\psi_0(x)|_{L^\infty}\cdot\frac{1}{r}\cdot\frac{r+t-|r-t|}{(1+|r-t|)^{k-2}(1+r+t)}$$
$$\le \frac{c|(1+x)^k\nabla^1\psi_0(x)|_{L^\infty}}{(1+t+r)(1+|r-t|)^{k-2}},$$

since $r+t-|r-t|\le 2r$.

2) As in 1), we get:

$$|\frac{t}{4\pi}\int_{|\omega|=1} < \nabla\psi_0(x+t\omega),\omega > d\omega|$$
$$\le |\frac{t}{4\pi}\cdot\frac{1}{t^2}\cdot\int_{|y-x|=t}|\nabla^1\psi_0(y)|dS_y|$$
$$\le c|(1+x)^k\nabla^1\psi_0(x)|_{L^\infty}\cdot\frac{1}{t}\cdot\int_{|y-x|=t}\frac{dS_y}{(1+|y|)^k}$$
$$\le \frac{|(1+|x|)^k\nabla^1\psi_0(x)|_{L^\infty}}{(1+t+r)(1+|t-r|)^{k-2}}.$$

3) Using Lemma 1 and Lemma 2, we get:

$$
\begin{aligned}
&|\frac{1}{4\pi t^2}\int_{|y-x|=t}\psi_0(y)dS_y| \\
&\le c|(1+|x|)^{k-1}\psi_0(x)|_{L^\infty}\cdot\frac{1}{t^2}\cdot\int_{|y-x|=t}\frac{dS_y}{(1+|y|)^{k-1}} \\
&= c|(1+|x|)^{k-1}\psi_0(x)|_{L^\infty}\cdot\frac{1}{tr}\cdot\int_{|r-t|}^{r+t}\frac{\lambda d\lambda}{(1+\lambda)^{k-1}}.
\end{aligned}
$$

If

$$I(t,r)=\frac{1}{tr}\int_{|r-t|}^{r+t}\frac{\lambda d\lambda}{(1+\lambda)^{k-1}},$$

then:

Case 1. $t\le r\le 1$ or $r\le t\le 1$

We have that:

$$I(t,r)\le\frac{1}{tr}\int_{|r-t|}^{r+t}\lambda d\lambda = 4\le\frac{c}{(1+t+r)(1+|t-r|)^{k-2}}.$$

Case 2. $t\ge r$ and $t\ge 1$

We have that:

$$
\begin{aligned}
&I(t,r)\le\frac{1}{tr}\int_{t-r}^{t+r}\frac{d\lambda}{(1+\lambda)^{k-2}} \\
&\le\frac{c}{tr}\cdot\frac{2r}{(1+t+r)(1+t-r)^{k-3}}\le\frac{c}{(1+t+r)(1+|t-r|)^{k-2}},
\end{aligned}
$$

since $\frac{1}{t}\le\frac{c}{1+t-r}$, when $t\ge 1$.

Case 3. $r\ge t$ and $r\ge 1$

Case 3 is similar to Case 2. This completes the proof of Theorem 1. ■

One can obtain the following corollary:

Corollary 1. *For every $s\ge 0$ the following estimate holds:*

$$
\begin{aligned}
&|\tau_+\tau_-^{k-2}\nabla_x^s\psi|_{L^\infty(R_+\times R^3)} \\
&\le \text{const }(|(1+|x|)^k\nabla^{s+1}\psi_0(x)|_{L^\infty}+|(1+|x|)^{k-1}\nabla^s\psi_0(x)|_{L^\infty}),
\end{aligned}
$$

where $k>2$.

3. An estimate for the inhomogeneous wave equation

Let us consider the Cauchy problem:

$$(\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2)\psi = F(t,x),$$
$$\psi(0,x) = \psi_t(0,x) = 0. \tag{6}$$

We assume that the function $F(t,x)$ has the form $F(t,x) = F_1(t,x) \cdot F_2(t,x)$. The following estimate holds:

Theorem 2. *If $\psi(t,x)$ is solution of (6), then the following estimate holds:*

$$|(1+t+|x|)(1+|t-|x||)^{l-1-\frac{2}{q}}\psi(t,x)|$$
$$\leq \text{const} \sup_{0\leq s\leq t} \Big(\int_{|y-x|=t-s} |F_1(s,y)|^p dS_y\Big)^{\frac{1}{p}} \cdot |\tau_+^l \tau_-^k F_2|_{L^\infty(R_+\times R^3)},$$

where:

$$1 \leq q \leq 2, \frac{1}{p} + \frac{1}{q} = 1,$$
$$l \geq 1 + \frac{2}{q}, k \geq 0.$$

Proof. Using the representation formula for $\psi(t,x)$, we get:

$$\psi(t,x) = \frac{1}{4\pi}\int_0^t \frac{1}{t-s}\Big(\int_{|y-x|=t-s} F(s,y)dS_y\Big)ds$$

Using Hölder inequality one obtains:

$$|\psi(t,x)|$$
$$\leq \text{const}\ \Big(\int_0^t \frac{1}{t-s}\Big(\int_{|y-x|=t-s} |F_1(s,y)|^p dS_y\Big)^{\frac{1}{p}}$$
$$\Big(\int_{|y-x|=t-s} \frac{dS_y}{(1+s+|y|)^{lq}(1+|s-|y||)^{kq}}\Big)^{\frac{1}{q}} ds\Big) |\tau_+^l \tau_-^k F_2|_{L^\infty(R_+\times R^3)}.$$

By using Lemma 1 one obtains:

$$|\psi(t,x)| \leq \text{const} \cdot r^{-\frac{1}{q}} I(t,r)$$
$$\times \sup_{0\leq s\leq t} \Big(\int_{|y-x|=t-s} |F_1(s,y)|^p dS_y\Big)^{\frac{1}{p}} |\tau_+^l \tau_-^k F_2|_{L^\infty(R_+\times R^3)}, \tag{7}$$

where due to Lemma 1:

$$I(t,r)=\int_0^t \frac{1}{(t-s)^{1-\frac{1}{q}}}\Big(\int_{|r-t+s|}^{r+t-s}\frac{\lambda d\lambda}{(1+s+\lambda)^{lq}(1+|s-\lambda|)^{kq}}\Big)^{\frac{1}{q}}.$$

By changing variables $\mu=\lambda^2$ and using Lemma 2 one obtains:

$$I(t,r)\le c\int_0^t \frac{1}{(t-s)^{1-\frac{1}{q}}}\Big(\int_{(r-t+s)^2}^{(r+t-s)^2}\frac{d\mu}{(1+s^2+\mu)^{\frac{lq}{2}}}\Big)^{\frac{1}{q}}$$

$$\le c\int_0^t \frac{1}{(t-s)^{1-\frac{1}{q}}}\Big(\frac{(r+t-s)^2-(r-t+s)^2}{(1+s^2+(r-t+s)^2)^{\frac{lq}{2}-1}(1+s^2+(r+t-s)^2)}\Big)^{\frac{1}{q}}ds$$

$$\le c\,\frac{r^{1/q}}{(1+r+t)^{2/q}}\int_0^t \frac{(t-s)^{\frac{2}{q}-1}}{(1+s+|r-t+s|)^{l-\frac{2}{q}}}ds$$

$$\le c\,\frac{r^{\frac{1}{q}}}{1+t+r}\int_0^t \frac{ds}{(1+s+|r-t+s|)^{l-\frac{2}{q}}}=c\,\frac{r^{\frac{1}{q}}}{1+t+r}J(t,r),$$

where

$$J(t,r)=\int_0^t \frac{ds}{(1+s+|r-t+s|)^{l-\frac{2}{q}}}.$$

Now we shall estimate $J(t,r)$.

Case 1. $r\ge t$
By using Lemma 2 one obtains:

$$J(t,r)=\int_0^t \frac{ds}{(1+r-t+2s)^{l-\frac{2}{q}}}$$

$$\le \frac{t}{(1+r+t)(1+r-t)^{l-1-\frac{2}{q}}}\le \frac{1}{(1+r-t)^{l-1-\frac{2}{q}}}.$$

Case 2. $r\le t$
We shall divide the integral into two parts. When $s\in[0,t-r]$ the integral can be majorised by:

$$\int_0^{t-r}\frac{ds}{(1+t-r)^{l-\frac{2}{q}}}=\frac{t-r}{(1+t-r)^{l-\frac{2}{q}}}\le \frac{1}{(1+t-r)^{l-1-\frac{2}{q}}}.$$

When $s \in [t-r, t]$, we have that

$$\int_{t-r}^{t} \frac{ds}{(1+r-t+2s)^{l-\frac{2}{q}}}$$

$$\leq \frac{r}{(1+t+r)(1+t-r)^{l-1-\frac{2}{q}}} \leq \frac{1}{(1+t-r)^{l-1-\frac{2}{q}}}.$$

Therefore, Case 2 is completed.

We derive the following:

$$J(t,r) \leq \frac{c}{(1+|r-t|)^{l-1-\frac{2}{q}}}.$$

Moreover,

$$I(t,r) \leq \frac{c r^{\frac{1}{q}}}{(1+t+r)(1+|t-r|)^{l-1-\frac{2}{q}}}.$$

Using the above estimate in (7) one completes the prove of Theorem 2. ∎

4. Basic estimate

We introduce the following Banach spaces:

$$V_1 = \{\psi \in C(R_+ \times R^3) : \sup_{t \in R_+, x \in R^3} |(1+t+|x|)(1+|t-|x||)^{n-3}\psi(t,x)| < \infty\},$$

$$V_2 = \{\psi : \sup_{t \in R_+} |\nabla^1 \nabla_x^1 \psi(t,.)|_{L^2} < \infty\}$$

V_1 and V_2 are Banach spaces with the following norms:

$$|\psi|_{V_1} = \sup_{t \in R_+, x \in R^3} |(1+t+|x|)(1+|t-|x||)^{n-3}\psi(t,x)|,$$

$$|\psi|_{V_2} = \sup_{t \in R_+} |\nabla^1 \nabla_x^1 \psi(t,.)|_{L^2}$$

Let us denote $V = V_1 \bigcap V_2$. V is a Banach space with the following norm:

$$|\psi|_V = |\psi|_{V_1} + |\psi|_{V_2}$$

V is such a space where the solutions are expected to belong.

Taking into account (2) one can see that the integral equation corresponding to (1) is:

$$\psi(t,x) = \phi(t,x) + \frac{1}{4\pi}\int_0^t \frac{1}{t-s}\Big(\int_{|y-x|=t-s} \mathcal{D}F(\psi)dS_y\Big)ds. \tag{8}$$

Here $\phi(t,x)$ is the solution of the homogeneous problem with initial data $\eta(x)$.

Let us define L as:

$$LF(\psi) = \frac{1}{4\pi}\int_0^t \frac{1}{t-s}\Big(\int_{|y-x|=t-s} \mathcal{D}F(\psi)dS_y\Big)ds$$

The following a priori estimate holds:

Theorem 3. *If $\psi \in V$, then $LF(\psi) \in V$ and*

$$|LF(\psi)|_V \le \text{const } |\psi|_V^n,$$

where $n > 3$.

To prove Theorem 3 we need the next technical estimate which enables one to evaluate surface integrals over the light cone with usual L^2-norms.

Lemma 3. *If $\psi : R^3 \to C^4$ is such that $|\nabla^1\psi| \in L^2(R^3)$, then the following inequality holds*

$$\int_{|y-x|=t} |\psi(y)|^2 dS_y < \text{const} \int_{R^3} |\nabla^1\psi(y)|^2 dy.$$

P r o o f. First, if $|\nabla^1 f| \in L^1(R^3)$, then via Gauss formula one has:

$$\Big|\int_{|x-y|=t} f(y)dS_y\Big|$$
$$= \Big|\sum_{j=1}^{4}\frac{1}{t}\int_{|x-y|\le t} \partial_j((x_j-y_j)f(y))dS_y\Big|$$
$$\le c\,\Big(\int_{|y-x|\le t} |\nabla^1 f(y)|dy + \int_{|y-x|\le t}\frac{|f(y)|}{|y-x|}dy\Big).$$

Further, via Cauchy inequality one has:

$$\int_{|y-x|=t} |\psi(y)|^2 dS_y$$
$$\le c\,\Big(\int_{|y-x|\le t} |\nabla^1|\psi(y)|^2|dy + \int_{R^3}\frac{1}{|x-y|}|\psi(y)|^2dy\Big)$$
$$\le c\,\Big(\int_{R^3} |\nabla^1\psi(y)|^2dy + \Big(\int_{R^3}|\psi(y)|^2dy\Big)^{1/2}\Big(\int_{R^3}\frac{1}{|x-y|^2}|\psi(y)|^2dy\Big)^{1/2}\Big).$$

Now we are in position to apply Hardy inequality:

$$\int_{R^3} \frac{1}{|x|^2}|f(x)|^2dx \le \int_{R^3} |\nabla^1 f(x)|^2dx,$$

which completes the proof of Lemma 3. ∎

Proof of Theorem 3. Using Lemma 3 and Theorem 2 for

$$p = q = 2, l = n-1, k = (n-1)(n-3), F_1 = |\nabla^1\psi|, F_2 = |\psi|^{n-1},$$

one obtains:

$$\begin{aligned} &|(1+t+|x|)(1+|t-|x|)^{n-3}LF(\psi)| \\ \le \text{const } \tau_+\tau_-^{n-3}\psi|^{n-1}_{L^\infty(R_+\times R^3)} \cdot \sup_{0\le s\le t} &\Big(\int_{|y-x|=t-s} |\nabla^1\psi|^2dS_y\Big)^{\frac{1}{2}} \\ &\le \text{const } |\psi|^{n-1}_{V_1}\cdot|\psi|_{V_2}. \end{aligned}$$

Hence we obtain that

$$|LF(\psi)|_{V_1} \le \text{const } |\psi|^n_V.$$

It remains to estimate $|LF(\psi)|_{V_2}$. Using the energy estimate for the wave equation and the Sobolev embedding one can obtain:

$$\begin{aligned} &|\nabla^1\nabla^1_x LF(\psi(t,.))|_{L^2} \\ &\le c\int_0^t |\nabla^1_x \mathcal{D}F(\psi(\tau,.))|_{L^2}d\tau \\ &\le c\int_0^t (|\nabla^1_x F'(\psi(\tau,.))\psi_t(\tau,.)|_{L^2} + |\nabla^2_x F(\psi(\tau,.))|_{L^2})d\tau \\ &\le c\int_0^t (|\nabla^1_x\psi(\tau,.)|_{L^4}|\psi_t(\tau,.)|_{L^4}|\psi|^{n-2}_{V_1}(1+\tau)^{-n+2} \\ &\quad +|\nabla^1_x\psi_t(\tau,.)|_{L^2}|\psi|^{n-1}_{V_1}(1+\tau)^{-n+1})d\tau \\ &\quad +c\int_0^t (|\nabla^2_x\psi(\tau,.)|_{L^2}|\psi|^{n-1}_{V_1}(1+\tau)^{-n+1} \\ &\quad +|\nabla^1_x\psi(\tau,.)|^2_{L^2}|\psi|^{n-2}_{V_1}(1+\tau)^{-n+2})d\tau \\ &\le c\int_0^t ((1+\tau)^{-n+2}|\psi|^2_{V_2}|\psi|^{n-2}_{V_1} + (1+\tau)^{-n+1}|\psi|_{V2}|\psi|^{n-1}_{V_1})d\tau \le \text{const } |\psi|^n_V. \end{aligned}$$

Hence:

$$|LF(\psi)|_{V_2} \le \text{const } |\psi|^n_V.$$

This completes the proof of Theorem 3. ■

Remark. In a similar way one can obtain that for $s \geq 0$ we have:

$$|\nabla^s LF(\psi)|_{V_1} \leq \text{const } |\nabla^s \psi|_V^n.$$

5. Low-regularity solutions

In this section we obtain low-regularity solutions of (1) in H^2 providing the initial data sufficiently small. Our main tool is the basic estimate of the previous section.

Theorem 4. *Suppose that the initial data $\eta \in H^2$ of (1) satisfy the following asymptotic decay property*

$$|(1+|x|)^{n-1}\nabla^1\eta(x)|_{L^\infty} + |(1+|x|)^{n-2}\eta(x)|_{L^\infty} + |\eta|_{H^2} < \epsilon,$$

where $n > 3$ and $\epsilon > 0$ is sufficiently small. Then there exists a global solution

$$\psi \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{2-j}).$$

Proof. We consider the usual iteration:

$$\psi_{m+1} = \phi + LF(\psi_m), \tag{9}$$

where ϕ is the solution of the homogeneous equation with initial data η, and $\psi_0 = \phi$. Using Theorem 1, the basic estimate and the energy estimate one obtains:

$$|\psi_{m+1}|_V \leq C(\epsilon + |\psi_m|_V^n).$$

Further we set:

$$Y_\delta = (\psi : |\psi|_V \leq \delta).$$

It is easy to see that $\psi_{m+1} \in Y_\delta$, when $\psi_m \in Y_\delta$,providing ϵ and δ such that $C(\epsilon + \delta^n) < \delta$. Similarly to the proof of the basic estimate, one obtains:

$$|\psi_{m+1} - \psi_m|_V \leq c(|\psi_m|_V^{n-1} + |\psi_{m-1}|_V^{n-1})|\psi_m - \psi_{m-1}|_V.$$

Therefore the iteration (9) is contraction in Y_δ for small δ. By the usual argument we obtain that ψ_m converges in Y_δ to ψ which is the desired low-regularity solution. This completes the proof of Theorem 4. ■

6. Higher regularity solutions

In this section we try to construct solutions of (1) in H^s, for $s \geq 2$. By the Sobolev embedding we see that for large s solutions are also classical (i.e. belonging to C^1). First let us consider the case when n is an integer. The following theorem holds:

Theorem 5. *Suppose the initial data $\eta \in H^s, s \geq 2$, is sufficiently small. Then the initial value problem (1) has a global solution*

$$\psi \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{s-j}),$$

for $n = 4, 5, 6...$

To prove Theorem 5 we need the following lemmas:

Lemma 4. (see[6]) *If $\psi_1, \psi_2 \in H^s(R^3; C^4)$, then the following estimate holds*

$$|\nabla^s \psi_1 \psi_2| \leq \text{const } (|\nabla^s \psi_1|_{L^{p_1}} |\psi_2|_{L^{q_1}} + |\psi_1|_{L^{p_2}} |\nabla^s \psi_2|_{L^{q_2}}),$$

where $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{2}$.

The following obvious corollary holds.

Corollary 2. *If $\psi \in H^s$, then*

$$|F(\psi)|_{H^s} \leq const|\psi|_{H^s}|\psi|_{L^\infty}^{n-1},$$

where $F(\psi) = O(|\psi|^n)$.

There are two different approaches for proving Lemma 4. One of them is based on using Fourier analysis (see [2], [6]). The other one is based on using Gagliardo-Nirenberg inequality.

We shall also use the following estimate:

Lemma 5. (Conservation law of charge for the Dirac equation) *If $\psi(t,x)$ is a solution of*

$$\mathcal{D}\psi = F(t,x), \tag{10}$$

then the following estimate holds:

$$|\psi(t,.)|_{L^2} \leq \text{const } (|\psi(0,.)|_{L^2} + \int_0^t |F(s,.)|_{L^2} ds).$$

The proof of Lemma 5 is based on multiplying the equation (10) with $-i\gamma^0\psi$ and integration over R^3.

Commutating the Dirac equation (10) with ∂_j, $0 \le j \le 3$, we can obtain similar estimates for $|\nabla^k\psi(t,.)|(k \ge 0)$.

Proof of Theorem 5. Now using the corollary of Lemma 4, the conservation law of charge for the Dirac equation and Sobolev embedding one obtains for the solution of (1) and any $l \ge 0$:

$$
\begin{aligned}
|\nabla^1\nabla_x^{l+1}\psi(t,.)|_{L^2} &\le c\,(\epsilon + \int_0^t |\nabla_x^{l+1}F'(\psi(\tau,.))\psi_t(\tau,.)|_{L^2}d\tau) \\
&\quad + \int_0^t |\nabla_x^{l+2}F(\psi(\tau,.))|_{L^2}d\tau) \le c\,(\epsilon \\
&\quad + \int_0^t (|\psi_t(\tau,.)|_{L^4}|\nabla_x^{l+1}\psi(\tau,.)|_{L^4}|\psi(\tau,.)|_{L^\infty}^{n-2} \\
&\quad +|\nabla_x^{l+1}\psi_t(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-1} + |\nabla_x^{l+2}\psi(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-1})d\tau) \\
&\le c\,(\epsilon + \int_0^t (|\nabla_x^1\psi_t(\tau,.)|_{L^2}|\nabla_x^{l+2}\psi(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-2} \\
&\quad +|\nabla_x^{l+1}\psi_t(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-1} + |\nabla_x^{l+2}\psi(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-1})d\tau) \\
&\le c\,(\epsilon + \int_0^t (|\nabla_x^l\psi|_{V_2}^2|\psi|_{V_1}^{n-2}(1+\tau)^{-n+2} + |\nabla_x^l\psi|_{V2}|\psi|_{V_1}^{n-1}(1+\tau)^{-n+1})d\tau).
\end{aligned}
$$

Hence, if we set

$$\|\psi\|_l = |\nabla_x^l\psi|_{V_2} + |\tau_+\tau_-^{\frac{n-3}{}}\psi|_{L^\infty},$$

then we can obtain

$$\|\psi\|_l \le c(\epsilon + \|\psi\|_l^n).$$

And by usual argument one obtains solution $\psi \in \bigcap_{j=0}^1 C^j([0,\infty); H^{s-j}(R^3))$ $(s \ge 2)$, providing ϵ is sufficiently small (i.e. the initial data). This completes the proof of Theorem 5. ■

Remark. If we consider the case $n = 3$, then we also can obtain global solution in H^s, but for $s \ge 3$ (which is also done in [3]). Following the same way like in the proof of Theorem 5, we can achieve the desired result by setting:

$$|||\psi|||_l = |\nabla_x^l\psi|_{V_2} + |\tau_+\tau_-^{n-3}\nabla^1\psi|_{L^\infty},$$

but in this case $l \geq 1$.

Similarly to the proof of Theorem 5, one obtains:

$$|\nabla^1\nabla_x^{l+1}\psi(t,.)|_{L^2} \leq c\ (\epsilon + \int_0^t (|\psi_t(\tau,.)|_{L^\infty}|\nabla_x^{l+1}\psi(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-2}$$
$$+|\nabla_x^{l+1}\psi_t(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-1} + |\nabla_x^{l+2}\psi(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-1})d\tau)$$
$$\leq c\ (\epsilon + \int_0^t (1+\tau)^{-n+1}(|\psi_t|_{V_1}|\nabla_x^l\psi|_{V_2}|\psi|_{V_1}^{n-2} + |\nabla_x^l\psi|_{V_2}|\psi|_{V_1}^{n-1})d\tau).$$

Taking into account the remark after Theorem 3, we can go on like in the proof of Theorem 5, but in this case for the convergence of integrals it is necessary that $n > 2$ (i.e. we include the case $n = 3$).

Hence we have the following

Theorem 6. *If $\eta \in H^s (s \geq 3)$ is sufficiently small, then we have global solution of (1) $\psi \in \bigcap_{j=0}^1 C^j([0,\infty); H^{s-j}(R^3))$, for $n = 3, 4, 5, \ldots$*

Remark. This result seems to be sharp for the integer case for n, since if $n = 2$, then one may expect the solution of (1) to blow-up in finite time (see [5]).

Now we may turn to noninteger case for n. This case is more complicated because we are not able to use Lemma 4. However, the following generalised form of Theorem 3 holds:

Theorem 7. *If $n > s + 1$, $s \geq 2$ and s is an integer, then for every $\eta \in H^s$ which is sufficiently small the initial value problem (1) has global solution $\psi \in \bigcap_{j=0}^1 C^j([0,\infty); H^{s-j}(R^3))$.*

Proof. By the conservation law of charge for the Dirac equation we obtain for $n > l + 3$:

$$|\nabla^1\nabla_x^{l+1}\psi(t,.)|_{L^2} \leq c\ (\epsilon +$$
$$\int_0^t |\nabla_x^{l+1}F'(\psi(\tau,.))\psi_t(\tau,.)|_{L^2}d\tau) + \int_0^t |\nabla_x^{l+2}F(\psi(\tau,.))|_{L^2}d\tau). \tag{11}$$

Now by using Hölder inequality one obtains:

$$|\nabla_x^{l+2}F(\psi(\tau,.))|_{L^2} \leq c(|\nabla_x^{l+2}\psi(\tau,.)|_{L^2}|\psi(\tau,.)_{L^\infty}^{n-1}$$
$$+|\nabla_x^{l+1}\psi(\tau,.)|_{L^4}|\nabla_x^1\psi(\tau,.)|_{L^4}|\psi(\tau,.)|_{L^\infty}^{n-2}$$
$$+\sum_M |\nabla_x^{\alpha_1}\psi(\tau,.)|_{L^2}|\nabla_x^{\alpha_2}\psi(\tau,.)|_{L^\infty}\ldots|\nabla_x^{\alpha_k}\psi(\tau,.)|_{L^\infty}|\psi(\tau,.)|_{L^\infty}^{n-|\alpha|}, \tag{12}$$

where the sum is taken over M:

$$M := (\alpha = (\alpha_1, ..., \alpha_k) : |\alpha| \leq l + 2, l \geq \alpha_1 \geq ... \geq \alpha_k).$$

Further via the Sobolev embedding the left hand-side of (12) can be estimated from:

$$\begin{aligned}|\nabla_x^{l+2} F(\psi(\tau,.))|_{L^2} \leq c\ (|\nabla_x^{l+2}\psi(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-1} \\ +|\nabla_x^{l+2}\psi(\tau,.)|_{L^2}|\nabla_x^2\psi(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-2} \\ +\sum_M |\nabla_x^{\alpha_1}\psi(\tau,.)|_{L^2}|\nabla_x^{\alpha_2+2}\psi(\tau,.)|_{L^2}...|\nabla_x^{\alpha_k+2}\psi(\tau,.)|_{L^2}|\psi(\tau,.)|_{L^\infty}^{n-|\alpha|} \\ \leq c\ (1+\tau)^{n-l-2}\|\psi\|_l^n.\end{aligned} \tag{13}$$

In a similar way one can obtain:

$$|\nabla_x^{l+1} F'(\psi(\tau,.))\psi_t(\tau,.)|_{L^2} \leq c(1+\tau)^{n-l-2}\|\psi\|_l^n. \tag{14}$$

Hence if we use (13) and (14) in (11), then we obtain:

$$\|\nabla_x^l \psi\|_{V_2} \leq c(\epsilon + \|\psi\|_l^n).$$

Moreover one has:

$$\|\psi\|_l \leq c(\epsilon + \|\psi\|_l^n).$$

Now it remains to use the usual argument to complete the proof of Theorem 7. ■

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