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Holomorphic Fixed Point Theorem on Riemann Surfaces

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It is said that a Riemann surface S is hyperbolic, if its universal covering surface S is conformally isomorphic to the upper half plane H. The Riemann surface S can be realized as the orbit space H/Γ and the canonical mapping $p : H \mapsto H/\Gamma$ is a complex analytical projection. Here Γ is a discrete group of conformal automorphisms of H, such that every non-identity element of Γ acts without fixed points of H. The elements of Γ are called deck transformation and they can be characterized as holomorphic maps $A : H \mapsto H$ satisfying the identity $p \circ A = p$.

The metric $ds = |dw|/v$, $w = u + iv \in H$, is called the Poincaré metric on the upper half-plane H. The Poincaré distance $\rho(z_1, z_2) = \rho_H(z_1, z_2)$ between two points of H is the minimum, over all paths γ joining z_1 to z_2, of the integral $\int_\gamma ds$. This is the unique Riemannian metric on the half-plane H, up to a multiplication by a constant, which is invariant under every conformal automorphism of H.

Every hyperbolic surface S has a unique Poincaré metric which is complete, with Gaussian curvature equal to -1. Since the Poincaré metric on H is invariant under action of Γ, there is only one metric (scalar product) on S so that projection $p : H \mapsto S$ ($S \cong H/\Gamma$) is a local isometry. Here, just as the above, there is an associated Poincaré distance function $\rho = \rho_S$.

Theorem A. *Let $f : S \mapsto W$ be a holomorphic map between hyperbolic surfaces. Then,*

$$\rho_W(f(z), f(z_1)) \leq \rho_S(z, z_1), \qquad z, z_1 \in S. \tag{1}$$

Furthermore, if an equality holds for some $z \neq z_1$, then $f : S \mapsto W$ is onto.

Proof. Let $p : H \mapsto S$ and $p_1 : H \mapsto W$ be canonical projections. The mapping f can be lifted to analytic function F of H into H which satisfies $f \circ p = p_1 \circ F$. Now by the classical Schwarz-Pick result we have

$$(2) \qquad \rho(F(w), F(w_1)) \leq \rho(w, w_1), \qquad w, w_1 \in H,$$

where ρ is Poincaré (hyperbolic) distance on H. Combining the above facts, we derive (1).

If an equality holds in (1) for some $z \neq z_1$, then we can show that equality holds in (2) for the corresponding $w \neq w_1$. Hence, by the classical Shwarz-Pick result on H, we conclude that F is a conformal automorphism on H. Thus $F : H \mapsto H$ is onto and therefore, $f : S \mapsto W$ is onto. ■

Theorem 1. *Let S be a hyperbolic Riemann surface, let K be compact subset of S and let $f : S \mapsto S$ be a holomorphic mapping.*

If f is not onto and $f(K) \subseteq K$, then there is a unique fixed point $z_0 = f(z_0) \in K$.

Proof. Let ρ denote Poincaré distance $\rho = \rho_S$ on S and let $h(z) = \rho(z, f(z))$, $z \in S$. Using Theorem A and the triangle inequality, we can derive

$$|h(z_1) - h(z_2)| \leq 2\rho(z_1, z_2),$$

if z_1 and z_2 belong to S. Thus h is a continuous function on S. Hence, there is at least one $z_0 \in K$ such that

$$h(z_0) = \min_{z \in K} h(z).$$

From this fact and Theorem A, we have

$$\rho(f(z_0), f^{02}(z_0)) \leq \rho(z_0), f(z_0)) \leq \rho(f(z_0), f^{02}(z_0)),$$

where $f^{02}(z_0) = f(f(z_0))$. Thus,

$$\rho(z_0, f(z_0)) = \rho(f(z_0), f^{02}(z_0)).$$

Now, if $z_0 \neq f(z_0)$, another application of Theorem A shows that f is a conformal isomorphism. Hence, since f is not onto, we conclude that $z_0 = f(z_0)$.

If we suppose that there is $z_1 \in S$ such that $z_1 \neq z_0$ and $z_1 = f(z_1)$ is another fixed point, then we have

$$\rho(f(z_1), f(z_0)) = \rho(z_1, z_0).$$

Hence, we conclude as above, that f is a conformal isomorphism and we have a contradiction. Thus $z_0 = f(z_0)$ is a unique fixed point. ■

The example $f(z) = -z$ on $D\{z : 0 < |z| < 1\}$ shows that a mapping which maps the compact ring $K = \{z : \varepsilon \leq |z| \leq r\}$ $(0 < \varepsilon \leq r < 1)$ into itself, does not need to have any fixed point.

The example $w \mapsto w + i$ on the upper half-plane shows that a map which is not onto, does not need to have any fixed point.

Now, let us consider the Riemann sphere $\bar{C} = C \cup \{\infty\}$ with the usual conformal structure. The continuous mapping $f : \bar{C} \mapsto \bar{C}$ defined by $f(z) = -(1/r)e^{iw}$ for $z = re^{iw} \in C \setminus \{0\}$, $f(0) = \infty$ and $f(\infty) = 0$, has no fixed point. However, the next result shows that if f is a holomorphic mapping from $\bar{C}$ into $\bar{C}$, then f has a fixed point.

Proposition 1. *Let $f : \bar{C} \mapsto \bar{C}$ be a holomorphic function. Then f has at least one fixed point on $\bar{C}$.*

Proof. If f did not have fixed point, then the function

$$h(z) = (f(z) - z)^{-1}$$

would be holomorphic function from $\bar{C}$ into C. Since $\bar{C}$ is compact set, the maximum modulus theorem shows that $h(z) \equiv A \neq 0$, $z \in C$. Hence, $f(z) = z + A^{-1}$, $z \in C$. Thus, $f(\infty) = \infty$, which is contradiction with our assumption that there not fixed points. ■

The next example shows that a holomorphic map on a torus does not need to not have any fixed points. Let $T = C/G$ be a torus, where the covering group G of C over a torus consists of transformation $z \mapsto z + nw_1 + nw$ and where w_1 and w_2 are complex numbers, $\mathrm{Im}(w_1/w_2) \neq 0$, and m and n run through all integers Z. Let $[z]$ denote the orbit of point z. Thus $[z] = \{A(z) : A \in G\}$. The holomorphic function $f : T \mapsto T$ defined by $f([z]) = [z + w_1/2]$ has no fixed point.

In the case of several variables, we have the following result concerning fixed points. The proof is given in a next paper.

Theorem B *Let $D \subset C^n$ be a domain for which Carathéodory pseudodistance is a distance and let $f : D \mapsto D$ be a holomorphic mapping. If $\overline{f(D)}$ is a compact subset of D, then f has a fixed point in D.*

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