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Results Concerning Common Fixed Point in Random Normed Spaces

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In this paper we consider pairs of self mappings in a convex and complete subspace of a random normed space and prove some results concerning the approximations of fixed points under a new contraction condition.

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The concept of random normed space was introduced by Serstnev [8]. Fixed point theorems in random normed spaces have been proved by Bocsan [2], Hadzic [3] and a number of other mathematicians. In the present paper we prove common fixed point theorems for pairs of self mappings satisfying a contraction condition in a convex and complete subspace of a random normed space.

Definition 1. Let L denote the set of all distribution functions. A triplet $(X, \mathcal{J}, t)$ of a real or complex linear space X, a mapping $\mathcal{J} : X \to L$ and T - norm t is called a random normed space, if and only if it satisfies the following conditions in which F_u denotes the distribution function $\mathcal{J}(u)$:

(a) $$F_u(0) = 0 \text{ for all } u \text{ in } X.$$

(b) $$F_u(x) = H(x) \text{ iff } u = 0, \text{ where } H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(c) If q is non-zero scalar, then

$$F_{qu}(x) = F_u(x/|q|) \quad \text{for all } u \text{ in } X \text{ and } x \text{ in } R.$$

(d) $$F_{u+v}(x+y) \geq t[F_u(x),\ F_v(y)] \quad \text{for all } u, v \text{ in } X,$$

$$x \geq 0, \qquad y \geq 0.$$

(e) $$t[x,y] \geq \max[x+y-1, 0] \quad \text{for all } x, y \text{ in } [0,1].$$

For topological preliminaries Schweizer and Sklar [7] is an excellent reference.

Theorem 1. *Let $(X, \mathcal{J}, t)$ be a random normed space with $t = \min[x,y]$ for every x, y in $[0,1]$ and $C(X)$ be a convex and complete subspace of X. T_1, T_2 be two mappings of $C(X)$ into itself. The sequence $\{u_n\}$ in $C(X)$ be defined as*

(1) $$u_{2n+1} = (1-s)u_{2n} + sT_1u_{2n}$$

$$u_{2n+2} = (1-s)u_{2n+1} + sT_2u_{2n+1}$$

for some s in $(0,1)$ and $n = 0, 1, 2, \ldots$

Then T_1 and T_2 have a unique common fixed point if for k in $(0,1)$

(2) $$F_{T_1u-T_2v}(kx) \geq \min[F_{u-v}(x), F_{u-T_1u}(x/s), F_{v-T_2v}(x/s)$$

$$F_{u-T_2v}((1+s)x/s), F_{v-T_1u}((1+s)x/s)]$$

holds for all u, v in $C(X)$ and $x \geq 0$.

In the proof of above theorem following lemma is employed.

Lemma 1. *Let $\{Y_n\}$ be a sequence in subspace $C(X)$ of random normed space $(X, \mathcal{J}, t)$ where t is continuous and satisfies $t[x,x] \geq x$ for every $x \in [0,1]$. If there exists a constant $k \in (0,1)$ such that*

(3) $$F_{Y_n-Y_{n+1}}(kx) \geq F_{Y_{n-1}-Y_n}(x)$$

for all n, then $\{Y_n\}$ is a Cauchy sequence.

The proof is similar to the lemma for Menger space proved by Bharucha-Reid and Sehgal [1] so we omit the details.

P r o o f o f T h e o r e m 1. Let $u_0 \in C(X)$ be arbitrary and for $s \in (0,1)$ consider a sequence $\{u_n\}$ in $C(X)$ as defined by (1). Then for $k' \in (0,1)$ such that $k' + s > 1$ and $x \geq 0$

$$F_{u_{2n+1}-u_{2n+2}}(k'x) - F_{(1-s)(u_{2n}-u_{2n+1})+s(T_1u_{2n}-T_2u_{2n+1})}((1-s)x + (k'+s-1)x)$$

$$\geq t[F_{(1-s)(u_{2n}-u_{2n+1})}((1-s)x),\ F_{s(T_1u_{2n}-T_2u_{2n+1})}(k'+s-1)x\ ] \quad (\text{ by (d)})$$

$$= t[F_{u_{2n}-u_{2n+1}}(x), F_{T_1u_{2n}-T_2u_{2n+1}}(k'+s-1)x/s] \quad (\text{ by (c)})$$

(4) $$= \min[F_{u_{2n}-u_{2n+1}}(x), F_{T_1u_{2n}-T_2u_{2n+1}}(k'+s-1)x/s].$$

But $0 < k' < 1$ and $0 < s < 1 \Longrightarrow k' + s < 1 + s$. This gives $k' + s - 1 < s$. Also $k' + s > 1$ gives $0 < k' + s - 1$. Thus, $0 < k' + s - 1 < s$ and hence $0 < (k' + s - 1)/s < 1$.

Let, $(k' + s - 1)/s = k$.

Using (2) we have

$$F_{T_1u_{2n}-T_2u_{2n+1}}(kx) \geq \min[F_{u_{2n}-u_{2n+1}}(x),\ F_{u_{2n}-T_1u_{2n}}(x/s),$$

$$F_{u_{2n+1}-T_2u_{2n+1}}(x/s),\ F_{u_{2n}-T_2u_{2n+1}}((1+s)x/s),\ F_{u_{2n+1}-T_1u_{2n}}((1+s)x/s)].$$

Since using (1) we get

(*i*) $$u_{2n} - T_1u_{2n} = (1/s)(u_{2n} - u_{2n+1})$$

(*ii*) $$u_{2n+1} - T_2u_{2n+1} = (1/s)(u_{2n+1} - u_{2n+2})$$

(*iii*) $$u_{2n} - T_2u_{2n+1} = (u_{2n} - u_{2n+1}) + (1/s)(u_{2n+1} - u_{2n+2})$$

(*iv*) $$u_{2n+1} - T_1u_{2n} = ((1-s)/s)(u_{2n} - u_{2n+1})$$

$$F_{u_{2n+1}-u_{2n+2}}(kx) \geq \min[F_{u_{2n}-u_{2n+1}}(x), F_{(1/s)(u_{2n}-u_{2n+1})}(x/s),$$

$$F_{(1/s)(u_{2n+1}-u_{2n+2})}(x/s),\ F_{(u_{2n}-u_{2n+1})+(1/s)(u_{2n+1}-u_{2n+2})}((1+s)x/s),$$

$$F_{((1-s)/s)(u_{2n}-u_{2n+1})}((1+s)x/s).$$

$$= \min[F_{u_{2n}-u_{2n+1}}(x), F_{u_{2n}-u_{2n+1}}(x), F_{u_{2n+1}-u_{2n+2}}(x),$$

$$F_{(u_{2n}-u_{2n+1})+1/s(u_{2n+1}-u_{2n+2})}((1+s)x/s), F_{u_{2n}-u_{2n+1}}((1+s)x/(1-s))].$$

$$\geq \min[F_{u_{2n}-u_{2n+1}}(x), F_{u_{2n}-u_{2n+1}}(x), F_{u_{2n+1}-u_{2n+2}}(x),$$

(5) $$F_{u_{2n}-u_{2n+1}}(x), F_{u_{2n+1}-u_{2n+2}}(x), F_{u_{2n}-u_{2n+1}}((1+s)x/(1-s))].$$

Using (4) and (5)

$$F_{u_{2n+1}-u_{2n+2}}(k'x) \geq \min[F_{u_{2n}-u_{2n+1}}(x), F_{u_{2n+1}-u_{2n+2}}(x)].$$

(Since $F_{u_{2n}-u_{2n+1}}((1+s)x/(1-s)) > F_{u_{2n}-u_{2n+1}}(x)$).

Now observing that $F_{u_{2n+1}-u_{2n+2}}(k'x) < F_{u_{2n+1}-u_{2n+2}}(x)$ we have

$$F_{u_{2n+1}-u_{2n+2}}(k'x) \geq F_{u_{2n}-u_{2n+1}}(x).$$

In general $F_{u_{n+1}-u_{n+2}}(k'x) \geq F_{u_n-u_{n+1}}(x)$.

Therefore by Lemma 1 $\{u_n\}$ is a Cauchy sequence in $C(X)$. $C(X)$ being complete subspace of $X, \{u_n\}$ converges to a point (say) z in $C(X)$.

Now we prove that $T_2z = z$.

Let $T_2z(\epsilon,\lambda)$ be an (ϵ,λ) neighbour hood of T_2z. Since $u_n \to z$ for $(1-k)\epsilon/2k > 0$, $\lambda > 0$ there exists an integer $N_1((1-k)\epsilon/2k,\ \lambda)$ such that $F_{u_{2n}-u_{2n+1}}((1-k)\epsilon/2k) > 1-\lambda$ and $F_{z-u_{2n+1}}((1-k)\epsilon/2k) > 1-\lambda$ for all $n \geq N_1$.

And for $(1-s)\epsilon/2s > 0$ and $\lambda > 0$ there exists an integer $N_2((1-s)\epsilon/2s,\lambda)$ such that $F_{u_{2n}-u_{2n+1}}((1-s)\epsilon/2s) > 1-\lambda$ and $F_{z-u_{2n+1}}((1-s)\epsilon/2s) > 1-\lambda$ for all $n \geq N_2$.

Taking $N = \max[N_1, N_2]$ we have

$$(6)\quad \begin{cases} F_{u_{2n}-u_{2n+1}}((1-k)\epsilon/2k) > 1-\lambda, & F_{z-u_{2n+1}}((1-k)\epsilon/2k) > 1-\lambda. \\ F_{u_{2n}-u_{2n+1}}((1-s)\epsilon/2s) > 1-\lambda, & F_{z-u_{2n+1}}((1-s)\epsilon/2s) > 1-\lambda. \end{cases}$$

For all $n \geq N$.

By (1) and (2)

$$\begin{aligned} F_{(1/s)(u_{2n+1}-u_{2n})+u_{2n}-T_2z}(\epsilon) &= F_{T_1u_{2n}-T_2z}(\epsilon) \\ \geq \min[F_{u_{2n}-z}(\epsilon/k), &F_{u_{2n}-T_1u_{2n}}(\epsilon/ks), F_{z-T_2z}(\epsilon/ks), \\ &F_{u_{2n}-T_2z}((1+s)\epsilon/sk), F_{z-T_1u_{2n}}((1+s)\epsilon/sk)] \end{aligned}$$

Since,

$$F_{u_{2n}-z}(\epsilon/k) = F_{u_{2n}-u_{2n+1}+u_{2n+1}-z}(((1+k)\epsilon/2k) + ((1-k)\epsilon/2k)))$$

$$(v)\qquad \geq \min[F_{u_{2n}-u_{2n+1}}((1+k)\epsilon/2k), F_{u_{2n+1}-z}((1-k)\epsilon/2k)].$$

$$(vi)\qquad \begin{aligned} F_{u_{2n}-T_1u_{2n}}(\epsilon/ks) &= F_{u_{2n}+((1-s/s)u_{2n}-(1/s)u_{2n+1}}(\epsilon/ks) \\ &= F_{(1/s)(u_{2n}-u_{2n+1})}(\epsilon/ks) \\ &= F_{u_{2n}-u_{2n+1}}(\epsilon/k), \end{aligned}$$

$$\begin{aligned} F_{z-T_2z}(\epsilon/ks) &= F_{z-T_2z}((1-k)\epsilon/2ks + (1-k)\epsilon/2ks + (1+s)\epsilon/2s) + (1-s)\epsilon/2s) \\ &\geq \min[F_{z-u_{2n}+(1/s)(u_{2n}-u_{2n+1})}((1-k)\epsilon/2ks + (1-k)\epsilon/2ks + (1-s)\epsilon/2s), \end{aligned}$$

$$F_{(1/s)(u_{2n+1}-u_{2n})+u_{2n}-T_2z}((1+s)\epsilon/2s)]$$
$$\geq \min[F_{z-u_{2n+1}}((1-s)\epsilon/2s), F_{u_{2n+1}-u_{2n}}((1-k)\epsilon/2ks),$$
$$F_{u_{2n}-u_{2n+1}}((1-k)\epsilon/2k), F_{(1/s)(u_{2n+1}-u_{2n})+u_{2n}-T_2z}((1+s)\epsilon/2s)]$$
$$\geq \min[F_{z-u_{2n+1}}((1-s)\epsilon/2s), F_{u_{2n+1}-u_{2n}}((1-k)\epsilon/2k), F_{u_{2n}-u_{2n+1}}((1-k)\epsilon/2k)$$

(*vii*) $$F_{(1/s)(u_{2n+1}-u_{2n})+u_{2n}-T_2z}((1+s)\epsilon/2s)],$$

$$F_{u_{2n}-T_2z}((1+s)\epsilon/ks) = F_{u_{2n}-((1/s)(u_{2n+1}-u_{2n})+u_{2n})+(1/s)(u_{2n+1}-u_{2n})+u_{2n}-T_2z}(\epsilon/ks+\epsilon/k).$$

(*viii*) $$\geq \min[F_{(u_{2n}-u_{2n+1})}(\epsilon/k), F_{(1/s)(u_{2n+1}-u_{2n})+u_{2n}-T_2z)}(\epsilon/k)],$$

and

$$F_{z-T_1u_{2n}}((1+s)\epsilon/ks) = F_{z+((1-s)/s)u_{2n}-(1/s)(u_{2n+1})}(\epsilon/k+\epsilon/ks)$$
$$\geq \min[F_{z-u_{2n}}(\epsilon/k), F_{(1/s)(u_{2n}-u_{2n+1})}(\epsilon/ks)]$$
$$\geq \min[F_{z-u_{2n+1}}((1-k)\epsilon/2k), F_{u_{2n+1}-u_{2n}}((1+k)\epsilon/2k),$$

(*ix*) $$F_{u_{2n}-u_{2n+1}}(\epsilon/k)].$$

We have,

$$F_{(1/s)(u_{2n+1}-u_{2n})+u_{2n}-T_2z}(\epsilon) \geq \min[F_{u_{2n}-u_{2n+1}}((1-k)\epsilon/2k),$$
$$F_{z-u_{2n+1}}((1-k)\epsilon/2k), F_{z-u_{2n+1}}((1-s)\epsilon/2s)]$$
$$\geq (1-\lambda) \quad \text{for all } : n \geq N \qquad (\text{ by } (6))$$

Since $\{u_n\}$ converges to $z, T_2z = z$

Similarly $T_1z = z$.

To prove the uniqueness of z as common fixed point of T_1 and T_2, let y be another common fixed point. They by (2) for some $x > 0$, we have

$$F_{z-y}(kx) \geq \min[F_{z-y}(x), f_{z-z}(x/s), F_{y-y}(x/s),$$
$$F_{z-y}((1+s)\,x/s), F_{y-z}((1+s)x/s)]$$
$$= F_{z-y}(x) \geq \ \dots \geq F_{z-y}(x/k^n) \to 1 \quad \text{as } n \to \infty.$$

This proves $y = z$. ∎

Letting $T_1 = T_2 = T$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let $C(X)$ be convex and complete subspace of a random normed space X and T be a mapping on $C(X)$ into itself satisfying,*

$$F_{Tu-Tv}(kx) \geq \min[F_{u-v}(x),\ F_{u-Tu}(x/s),\ F_{v-Tv}(x/s),$$

$$F_{u-Tv}((1+s)\ x/s),\ F_{v-Tu}((1+s)x/s)]$$

for all $u, v \in C(X), k, s \in (0,1)$ and $x \geq 0$. Then T has a unique fixed point in $C(X)$. Deterministic analogue of Theorem 1 is the following.

Corollary 2. *Let $C(X)$ be a convex and complete subspace of a normed space X. T_1 and T_2 be self mappings on $C(X)$. Then T_1 and T_2 have a common fixed point if*

$$||T_1u - T_2v|| \leq k \max[||u - v||, s||u - T_1u||,$$

$$s||v - T_2v||, (s/1+s)||u - T_2v||, (s/1+s)||v - T_1u||]$$

for all $u, v \in C(X)$ and $k, s \in (0,1)$.

If $C(X)$ is complete (but not convex) subspace of X, then letting $s = 1$ we get following well known probabilistic contraction condition.

Corollary 3. *Let $C(X)$ be complete subspace of random normed space X and T_1, T_2 be self mappings on $C(X)$. Then T_1, T_2 have a common fixed point if,*

$$F_{T_1u-T_2v}(kx) \geq \min[F_{u-v}(x),\ F_{u-T_1u}(x),\ F_{v-T_2v}(x)$$

$$F_{u-T_1v}(2x),\ F_{v-T_1u}(2x)] \text{ for all } u, v \in C(X),$$

$k \in (0,1)$ and $x \geq 0$.

Letting $k = s$ Theorem 1 gives following Corollary.

Corollary 4. *Let $C(X)$ be convex and complete subspace of random normed space and T_1, T_2 be self mappings on $C(X)$, then T_1, T_2 have a common fixed point if.*

$$F_{T_1u-T_2v}(kx) \geq \min[F_{u-v}(x),\ F_{u-T_1u}(x/k),\ F_{v-T_2v}(x/k),$$

$$F_{u-T_2v}((1+k)x/k), F_{v-T_1u}((1+k)x/k)]$$

for all u, v in $C(X)$, $k \in (0,1)$, $x \geq 0$.

A slight generalization of Theorem 1 is as follows.

Theorem 2. *Let $C(X)$ be a convex and complete subspace of a random normed space X and let T_1 and T_2 be two self mappings of $C(X)$ into itself such that there exist positive integers p and q satisfying,*

$$F_{T_1^p u-T_2^q v}(kx) \geq \min[F_{u-v}(x),\ F_{u-T_1^p u}(x/s), \tag{7}$$

$$F_{v-T_2^q v}(x/s),\ F_{u-T_2^q v}((1+s)x/s),\ F_{v-T_1^p u}((1+s)x/s)]$$

for all $u, v \in C(X)$, $s, k \in (0,1)$, $x \geq 0$.

Then T_1 *and* T_2 *have a unique common fixed point.*

Proof. Let $S_1 = T_1^p$ and $S_2 = T_2^q$, then by Theorem 1, S_1 and S_2 have a unique common fixed point say u. Thus $S_1 u = T_1^p u = u$ and $S_2 u = T_2^q u = u$. Then $T_1(T_1^p u) = T_1^p(T_1 u) = T_1 u$ and $T_2(T_2^q u) = T_2^q(T_2 u) = T_2 u$. We show that $T_1 u = T_2 u$ is also a common fixed point of S_1 and S_2.

Suppose $T_1 u \neq T_2 u$, then from (7), we get

$$\begin{aligned}
F_{T_1u-T_2u}(kx) &= F_{S_1(T_1u)-S_2(T_2u)}(kx)\\
&= F_{T_1^p(T_1u)-T_2^q(T_2u)}(kx)\\
&\geq \min[F_{T_1u-T_2u}(x),\ F_{T_1u-T_1^p(T_1u)}(x/s),\\
&\quad F_{T_2u-T_2^q(T_2u)}(x/s),\ F_{T_1u-T_2^q(T_2u)}((1+s)x/s),\\
&\quad F_{T_2u-T_1^p(T_1u)}((1+s)x/s)].\\
&= \min[F_{T_1u-T_2u}(x), F_{T_1u-T_1u)}(x/s), F_{T_2u-T_2u)}(x/s),\\
&\quad F_{T_1u-T_2u}((1+s)x/s),\ F_{T_2u-T_1u}((1+s)x/s)].\\
&= F_{T_1u-T_2u}(x).
\end{aligned}$$

Which is a contradiction and hence $T_1 u = T_2 u$ is a common fixed point of S_1 and S_2. By uniqueness of u, if follows that $T_1 u = u = T_2 u$. This completes the proof. ■

Letting $T_1 = T_2 = T$ in the above Theorem 2, we obtain the following corollary.

Corollary 5. *Let* $C(X)$ *be a convex and complete subspace of a random normed space* X *and let* $T: C(X) \to C(X)$ *be a mapping such that there exist positive integers* p *and* q *satisfying,*

$$F_{T^p u-T^q v}(kx) \geq \min[F_{u-v}(x),\ F_{u-T^p u}(x/s)$$

$$F_{v-T^q v}(x/s),\ F_{u-T^q v}((1+s)x/s),\ F_{v-T^p u}((1+s)x/s)]$$

for all $u, v \in C(X), s, k \in (0,1)$ *and* $x \geq 0$. *Then* T *has a unique fixed point.*

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