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Invariants of Commutative Group Algebras [1]

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Let K be an algebraically closed field of characteristic p and let G be an abelian group. Then the structure of the group algebra KG as K-algebra is being investigated, when G is a divisible group, when G is a direct sum (i.e. a restricted, bounded direct product) of cyclic groups and when G is a countable algebraically compact group or a torsion algebraically compact group. We shall show that in general, a complete set of invariants for the structure of the commutative algebra KG consists of the isomorphism class of the group G_p, the isomorphism class of the factor-group G/G_0 and the cardinality of the factor-group G_0/G_p, where G_0 is a torsion part (the maximal torsion subgroup) of G and G_p is its p-primary component. Thus the algebra KG over a field K can be determined up to isomorphism from the parts G_p, G/G_0 and $|G_0/G_p|$.

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1. Introduction

Let K be a field of characteristic, $\mathrm{char}K = p \neq 0$, and suppose KG is a group algebra of the abelian group G over the field K. Let again, G_p be a p-component of the torsion subgroup G_0 of a group G.

In the paper [2], when G is a divisible group or G is an arbitrary direct sum of cyclic groups, and G_0 is a p-torsion group (i.e. $G_0 = G_p$), the K-isomorphic class of KG is being depends on G. This class determines the isomorphic class of G, i.e. the K-isomorphism $KG \cong KH$ for any group H holds if and only if $G \cong H$.

Besides, also in [2] is coments the *isomorphism problem* for some classes of abelian p-groups and p-mixed abelian groups (i.e. abelian groups for which G_0 is p-primary), i.e. actually to a problem, when $KG \cong KH$ as K-algebras

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implies $G \cong H$. There is being discussed, too. the *isomorphism problem* of group algebras of abelian groups over every field (over all fields of prime characteristic), as this problem has a positive solution for the important classes of simply presented torsion groups, of an arbitrary direct sums of cyclic groups and, of direct products of divisible groups and bounded group, respectively. Moreover, a survey of most of the results we have had so far is also to be found there, for a isomorphism of commutative modular group algebras.

In his research [5], W. May obtain a full system of invariants for the semisimple commutative group algebra KG over an algebraic closed field K, whose characteristic not divide the orders of the torsion elements in G (i.e. G having no element whose order is equal to the characteristic of K).

In the present paper, an invariant set (i.e. a complete invariant system) of the K-algebra KG over an algebraic closed field K with $\mathrm{char} K = p > 0$ is being computed in the terms of a group G, for the cases when

(*) G is a direct sum of cyclic groups of arbitrary power (cardinality).

(**) G is a direct product of a divisible group and of a bounded group.

(***) G is a torsion simply presented group.

The proofs of this results are based on some theoretic-group statements from a classical group-theory and from the unit groups of commutative modular group rings, on the preceding basic May's result and on some others propositions of May for commutative modular group algebras (see [4]).

2. Group lemmas

Lemma 1 . *Let G be an abelian group. Therefore*

(1) *If $H \subseteq G_0$, then $(G/H)_0 = G_0/H$.*

(2) *If $H \subseteq G_p$, then $(G/H)_p = G_p/H$.*

Proof. Take $x \in (G/H)_0 \subseteq G/H$, hence $x = gH$, where $g \in G$ and $x^n = H$ for any positive integer n. Thus $g^n H = H$, i.e. $g^n \in H \subseteq G_0$. So, $g^n \in G_0$, i.e. $g \in G_0$ is a torsion element. Finally $x \in G_0/H$ and $(G/H)_0 \subseteq G_0/H$. Further, it is evident that, G_0/H is a torsion group and $G_0/H \subseteq G/H$. Consequently $G_0/H \subseteq (G/H)_0$ and thus we complete the proof. (2) can be proved anallogously to (1). The lemma is true.

Lemma 2 . *Let G and H be abelian groups and let $\varphi: G \to H$ be an epimorphism with kernel $\ker\varphi \subseteq G_p$. Then the restriction map $\varphi_{|G_p} = \varphi_p: G_p \to H_p$ is an epimorphism with kernel $\ker\varphi_p = \ker\varphi$.*

Proof. Trivially, if $g_p \in G_p$ then $\varphi(g_p) \in H_p$, because φ is a group homomorphism and $\varphi(1_G) = 1_H$, where $1_G \in G_p$ is the identity of G (similarly for 1_H).

Now let give $h_p \in H_p$. Furthermore, there exists an element $g \in G$ such that $\varphi(g) = h_p$. But h_p is a p-element, say $h_p^{p^i} = 1$ for some natural i. Hence $\varphi(g^{p^i}) = 1$, say $g^{p^i} \in \ker\varphi \subseteq G_p$. This is equivalent to $g \in G_p$. Finally φ_p is a surjection with kernel, $\ker\varphi_p = \ker\varphi \cap G_p = \ker\varphi$. Thus, the proof is over.

3. Unit groups of commutative modular group rings

Denote in this section, K a commutative ring with identity of prime characteristic p and G an abelian group. Throughout this paper $U(KG)$ is a multiplicative (unit) group of the ring KG (i.e. the group of units in KG), and $U_p(KG)$ is its p-primary component, respectively.

The following statement is valid.

Lemma 3 . *Let K does not have nilpotent elements. Then $U_p(KG) = 1$ if and only if $G_p = 1$.*

Proof. If $U_p(KG) = 1$, then $1 = G_p \subseteq U_p(KG)$.

Let us now $G_p = 1$. Since $\mathrm{char}K = p$ and K is a ring without nilpotent elements, i.e. K possesses a trivially nilradical, then $U_p(K) = 1$. Start with $u \in U_p(KG)$, hence $u = \sum_1^k \alpha_j g_j$, where $\alpha_j \in K$, $\sum_1^k \alpha_j = 1$, $g_j \in G$ and $u^{p^i} = 1$ for any natural i. Thus $\sum_1^k \alpha_j^{p^i} g_j^{p^i} = 1$. But this is a canonic element, i.e. $g_j^{p^i} \neq g_{j+1}^{p^i}$ $(1 \leq j \leq k,\ g_{k+1} = g_1)$, because $G_p = 1$ and $g_j \neq g_{j+1}$ for each $1 \leq j \leq k$, k – natural. Clearly $\alpha_1^{p^i} = 1$, $g_1^{p^i} = 1$ and $\alpha_2^{p^i} = \cdots = \alpha_k^{p^i} = 0$. Elementary $(\alpha_1 - 1)^{p^i} = 0$, $g_1 \in G_p = 1$ and $\alpha_j = 0$, $2 \leq j \leq k$. Finally $\alpha_1 = 1$, $g_1 = 1$ and $\alpha_j = 0$ for $2 \leq j \leq k$. Further obviously, $u = 1$ and $U_p(KG) = 1$. So, this completes the proof.

Suppose H is a subgroup of G and $I(KG; H)$ is a relative fundamental ideal of the ring (algebra) KG by (with respect to) the group H, respectively, i.e. the ideal of KG, generated by the elements $1 - h$, when h varies in H. Certainly, if $v \in I(KG; H)$ then $v = \sum_1^k x_j(1 - h_j)$, where $x_j \in KG$ and $h_j \in H$. Thus, assuming that H is a p-torsion group, therefore $I(KG; H)$ is a $(p$-)nil-ideal (i.e. $v^{p^i} = 0$ for any $i \in N$) and consequently $1 + I(KG; H)$ is a multiplicative abelian p-group in a group ring KG.

Let G and $\overline{G}$ be abelian groups and $f: G \to \overline{G}$ is a group homomorphism. Hence $\overline{f}: KG \to K\overline{G}$ is a homomorphism of K-algebras (K-homomorphism), define as follows by $\overline{f}(\sum_1^k r_j g_j) = \sum_1^k r_j f(g_j)$. Well, if f is a monomorphism (respectively, epimorphism) then $\overline{f}$ is also monomorphism (respectively, epimorphism). Finally, if f is an isomorphism, then $\overline{f}$ is an isomorphism, too, which extends f, say $G \cong \overline{G}$ implies that $KG \cong K\overline{G}$ as K-algebras.

Of course, the natural map $G \xrightarrow{\varphi} G/H$ induced the K-algebras epimorphism $KG \xrightarrow{\overline{\varphi}} K(G/H)$ extending φ, with kernel $I(KG;H)$, and the group homomorphism $U(KG) \to U(K(G/H))$ by restriction onto $\overline{\varphi}$, with kernel $U(KG) \cap (1 + I(KG;H))$.

Suppose $V(KG)$ is a normed unit subgroup of $U(KG)$, i.e. the group of normalized units in $U(KG)$, and $V_p(KG)$ is its p-component. Furthermore we get a homomorphism $V(KG) \to V(K(G/H))$ with kernel $V(KG) \cap (1 + I(KG;H))$. If H is a p-group, then this homomorphism is a surjection. Indeed, start with $x \in V(K(G/H))$, hence immediately there exists an element $y \in KG$, such that $\overline{\varphi}(y) = x$. Besides does exist $x' \in V(K(G/H))$, say $xx' = 1$ and then $\overline{\varphi}(yy') = 1$ for some $y' \in KG$, i.e. $\overline{\varphi}(1 - yy') = 0$ and consequently $1 - yy' \in \ker \overline{\varphi} = I(KG;H)$. Thus $(1 - yy')^{p^i} = 0$ for any $i \in N$, hence $y^{p^i} y'^{p^i} = 1$, i.e. $y \in U(KG)$. Evidently, y is a normed unit, since $x = \sum_1^k \alpha_j \varphi(g_j)$, where $\alpha_j \in K$, $g_j \in G$, $\sum_1^k \alpha_j = 1$, put $y = \sum_1^k \alpha_j g_j$. Finally $y \in V(KG)$ and the map is an epimorphism, with kernel a p-group $1 + I(KG;H) \subseteq V(KG)$. Using Lemma 2, $V_p(KG) \to V_p(K(G/H))$ is a surjection with kernel $1 + I(KG;H) \subseteq V_p(KG)$, anallogicly $U_p(KG) \to U_p(K(G/H))$ is a surjection with the same kernel. Suppose K have a trivial nilradical, i.e. $U_p(K) = 1$. Thus $V_p(KG) = U_p(KG)$, $V_p(K(G/H)) = U_p(K(G/H))$ and so, $U_p(KG)/(1 + I(KG;H))$
$\cong U_p(K(G/H))$. For the next case $H = G_p$, we obtain that $U_p(KG)/(1 + I(KG;G_p)) \cong U_p(K(G/G_p)) = 1$ by Lemma 3, because from Lemma 1, $(G/G_p)_p = 1$. Finally, we have the following lemma.

Lemma 4. *If K is a ring without nilpotent elements, then $U_p(KG) = 1 + I(KG;G_p)$.*

Lemma 5. ([2,3]) *For every ordinal ε*

$$U_p^{p^\varepsilon}(KG) = U_p(K^{p^\varepsilon} G^{p^\varepsilon}) .$$

From Lemma 4 and Lemma 5, we deduce

Proposition 6 . *Let K be a ring without nilpotent elements. Then $U_p(KG)$ is a direct sum of cyclic groups if and only if G_p is a direct sum of cylic groups.*

Proof. However $G_p \subseteq U_p(KG)$ and hence G_p is a direct sum of cyclic groups, whenever $U_p(KG)$ is the same.

Conversely, now let G_p be a direct sum of cyclic groups. Write down $G_p = \bigcup_1^\infty M_j$, put $M_j \subseteq M_{j+1}$ and $M_j \cap G_p^{p^j} = 1$. By Lemma 4, $U_p(KG) =$

$1+I(KG;G_p)$, therefore $U_p(KG)=\bigcup_1^\infty(1+I(KG;M_j))$ and $1+I(KG;M_j)\subseteq 1+I(KG;M_{j+1})$. Indeed, $1+I(KG;M_j)\subseteq 1+I(KG;G_p)$ for every j, hence $\bigcup_1^\infty(1+I(KG;M_j))\subseteq U_p(KG)$. Assuming that $w\in U_p(KG)$, then $w=1+\sum_1^k x_l(1-a_l)$, where $x_l\in KG$ and $a_l\in G_p=\bigcup_1^\infty M_j$, thus $a_l\in M_{jl}$ for each $1\leq l\leq k$, $k\in N$. Further obviously $a_l\in M_{j'}$ for any fixed natural j' and for all $1\leq l\leq k$, since $(M_j)_1^\infty$ is ascending chain. Finally $w\in 1+I(KG;M_{j'})$ and so $U_p(KG)\subseteq\bigcup_1^\infty(1+I(KG;M_j))$, which proves the equality.

Moreover using Lemma 5, for every j we calculated that (cf [2,3]), $[1+I(KG;M_j)]\cap U_p^{p^j}(KG)=[1+I(KG;M_j)]\cap U_p(K^{p^j}G^{p^j})=1+I(K^{p^j}G^{p^j};G^{p^j}\cap M_j)=1$, because $M_j\cap G^{p^j}=M_j\cap G_p^{p^j}=1$, when $M_j\subseteq G_p$. Indeed, if choose $c\in[1+I(KG;M_j)]\cap U_p(K^{p^j}G^{p^j})$, then $c=\sum_1^k r_lg_l$ and $\sum_1^k r_l=\sum\limits_{g_l\in G^{p^j}} r_l=1$, where $r_l\in K^{p^j}$, $g_l\in G^{p^j}$ and

$$\sum_{g_l\in gM_j} r_l=\begin{cases}0, & g\notin M_j\\ 1, & g\in M_j\end{cases}$$

for each $g\in G$ (whence and for every $g_{(j)}\in G^{p^j}$). Well, immediately $c=\sum_1^k r_lg_l$ and

$$\sum_{g_l\in g_{(j)}(M_j\cap G^{p^j})} r_l=\begin{cases}0, & g_{(j)}\notin M_j\cap G^{p^j}\\ 1, & g_{(j)}\in M_j\cap G^{p^j}\end{cases}$$

for each $g_{(j)}\in G^{p^j}$. Finally if $g_{(jt)}\in G^{p^j}$, then $c=1+\sum\limits_{t\in N}\sum\limits_{g_l\in g_{(jt)}(M_j\cap G^{p^j})} r_lg_l(1-g_l^{-1}.g_{(jt)})\in 1+I(K^{p^j}G^{p^j};M_j\cap G^{p^j})$, as desired.

Consequently by the fundamental criterion of Kulikov (see [1], p.106, Theorem 17.1), $U_p(KG)$ is a direct sum of cyclic groups and the proposition is completely proved.

4. Invariants of commutative modular group algebras

Now we formulate the main results for isomorphism of commutative group algebras, in three sections.

A. Invariants of modular group algebras of direct sums of cyclic groups

Theorem A. (Invariants) *Let K be an algebraically closed field of characteristic $p>0$ and let G be a direct sum of cyclic groups. Then $KH\cong KG$ as K-algebras for some group H if and only if*

(1) *H is abelian.*

(2) $H_p \cong G_p$.
(3) $H/H_0 \cong G/G_0$.
(4) $|H_0/H_p| = |G_0/G_p|$.

Proof. It is evident that KH is a commutative algebra, i.e. H is an abelian group. Therefore from [4], $G/G_0 \cong H/H_0$ and $K(G/G_p) \cong K(H/H_p)$ as K-algebras. Since the factor-group G/G_p have not p-elements (similarly for H/H_p), then by [5], $|(G/G_p)_0| = |(H/H_p)_0|$, i.e. Lemma 1 implies $|G_0/G_p| = |H_0/H_p|$. Moreover $G_p \subseteq G$ is a direct sum of cyclic groups and using Proposition 6, $U_p(KG) \cong U_p(KH)$ is a direct sum of cyclic groups, i.e. H_p is the same. But hence, $G_p \cong H_p$ by the fact that G_p and H_p have equal functions of Ulm-Kaplansky (see [4]). So, the first part is proved.

Now, let the conditions (1-4) hold. The group G is a direct sum of cyclic groups, hence $G/G_0 \cong H/H_0$ are direct sums of cyclic groups and G and H splits ([1], p.91, Theorem 14.4). Thus $G \cong G_0 \times G/G_0 \cong G_p \times G/G_p$ and analogicaly $H \cong H_0 \times H/H_0 \cong H_p \times H/H_p$, because G_p is a direct factor of G_0 (similarly for H_p). Further certainly $G/G_0 \cong G/G_p/G_0/G_p = G/G_p/(G/G_p)_0$ by Lemma 1 and, respectively $H/H_0 \cong H/H_p/(H/H_p)_0$. That is why, from [5] we deduce that $K(G/G_p) \cong K(H/H_p)$. Besides $G_p \cong H_p$ implies however $KG_p \cong KH_p$. Finally $KG \cong KG_p \otimes_K K(G/G_p) \cong KH_p \otimes_K K(H/H_p) \cong KH$, i.e. $KG \cong KH$ as K-algebras. This completes the proof of the theorem.

Proposition A . *Let K be a field with* char$K = p \neq 0$ *and G a direct sum of cyclic groups. If H is a group such that there exists a K-isomorphism $KH \cong KG$, then the following are valid:*

(1) *H is abelian.*
(2) $H_p \cong G_p$.
(3) $H/H_0 \cong G/G_0$.
(4) $|H_0/H_p| = |G_0/G_p|$.

Proof. Suppose again, $\overline{K}$ is an algebraic closure of K, hence exists an injection $K \to \overline{K}$. Thus $\overline{K}G \cong KG \otimes_K \overline{K}$, $\overline{K}H \cong KH \otimes_K \overline{K}$ and the two isomorphisms with an isomorphism $KG \cong KH$ does imply $\overline{K}G \cong \overline{K}H$. Then the proposition follows from the central theorem, immediately.

Corollary A. (Isomorphism) *Let K be a field,* char$K = p \neq 0$ *and G be a direct sum of cyclic groups, whose torsion part G_0 is a p-group. Then $KH \cong KG$ as K-algebras for some group H if and only if $H \cong G$.*

Proof. As we shall see, $G \cong G_0 \times G/G_0$ and $H \cong H_0 \times H/H_0$. A further application of the proposition shows that, $G_0 \cong H_0$ and $G/G_0 \cong H/H_0$. Thus, $G \cong H$. The proof is over.

B. Invariants of modular group algebras of algebraically compact abelian groups.

Lemma B. *Let K be a field with $\mathrm{char} K = p > 0$ and G be an abelian group such that G_p be algebraically compact. Then for any group H, $KH \cong KG$ as K-algebras implies $H_p \cong G_p$.*

Proof. Write $G_p = (G_p)_d \times (G_p)_r$, where $(G_p)_d$ is a maximal divisible subgroup of G_p and $(G_p)_r$ is a reduced part (subgroup) of G_p. From ([1], p.189, Corollary 38.3 and p.199, Corollary 40.3) follows that $(G_p)_r$ is bounded, i.e. $G_p^{p^i}$ is divisible for any $i \in N$. Write $H_p = (H_p)_d \times (H_p)_r$. We will prove that, $H_p^{p^i}$ is divisible, i.e. $(H_p)_r$ is bounded.

We may assume that analogously to Proposition A, K is algebraically closed, hence perfect. Besides $KG = KH$, where H is a normalized group basis of KG, $H \leq V(KG)$. Consequently $KG^{p^i} = KH^{p^i}$ for this $i \in N$, and $1 + I(KG^{p^i}; G_p^{p^i}) = U_p(KG^{p^i}) = U_p(KH^{p^i}) = 1 + I(KH^{p^i}; H_p^{p^i})$, i.e. $I(KG^{p^i}; G_p^{p^i}) = I(KH^{p^i}; H_p^{p^i})$, see Lemma 4. Thus $I^p(KG^{p^i}; G_p^{p^i}) = I(KG^{p^{i+1}}; G_p^{p^i}) = I(KH^{p^{i+1}}; H_p^{p^{i+1}}) = I^p(KH^{p^i}; H_p^{p^i})$, i.e. $I(KG^{p^{i+1}}; G_p^{p^i}) = I(KH^{p^{i+1}}; H_p^{p^{i+1}})$. Hence $I(KG^{p^i}; G_p^{p^i}) = I(KH^{p^i}; H_p^{p^{i+1}})$ and $I(KH^{p^i}; H_p^{p^i}) = I(KH^{p^i}; H_p^{p^{i+1}})$, i.e. $H_p^{p^i} = H_p^{p^{i+1}} = (H_p^{p^i})^p$. Finally $H_p^{p^i}$ is divisible, i.e. $(H_p)_r$ is bounded, as desired. Moreover $(G_p)_d \cong (H_p)_d$ and the Ulm-Kaplansky invariants of $(G_p)_r$ and $H_p)_r$ are equal [4]. Thus, $(G_p)_r \cong (H_p)_r$ and $G_p \cong H_p$. The proof is complete.

Theorem B. (Invariants) *Let K be an algebraically closed field of characteristic $p \neq 0$ and let G be an abelian group whose p-component G_p is algebraically compact. Then $KH \cong KG$ as K-algebras for some group H if and only if*

(1) *H is abelian.*

(2) *$H_p \cong G_p$.*

(3) *$H/H_0 \cong G/G_0$.*

(4) *$|H_0/H_p| = |G_0/G_p|$.*

Proof. Using the lemma, analogously to Theorem A, $KG \cong KH$ implies that, (1-4) are true.

We claim (1-4) are valid. The group G_p is algebraically compact and pure in G. Hence $G \cong G_p \times G/G_p$. Similarly for $H \cong H_p \times H/H_p$. Again from (3) and (4) by [5], $K(G/G_p) \cong K(H/H_p)$. This isomorphism will be reduced to $KG \cong KH$, since $KG \cong KG_p \otimes_K K(G/G_p)$, $KH \cong KH_p \otimes_K K(H/H_p)$ and $KG_p \cong KH_p$. The theorem is completely proved.

Proposition B1. *Let K be an algebraically closed field of* $\mathrm{char} K = p > 0$ *and let G be an abelian groups, such that the torsion part G_0 is algebraically compact. Then KH and KG are K-isomorphic for any group H if and only if, H is abelian, H_p and G_p are isomorphic, H/H_0 and G/G_0 are isomorphic and H_0/H_p and G_0/G_p have equal cardinalities.*

Proof. The component G_p is algebraically compact as a direct factor in G_0 (see [1]), hence the theorem implies the statement. This proves the proposition.

Remark. If $G_0 = G_p$, then is valid the Karpilovsky statement, that $G \cong H$ (see [2]).

Proposition B2. *Let K be an algebraically closed field with* $\mathrm{char} K = p \neq 0$ *and let G be a direct product of a divisible group and of a bounded group. Then KH and KG are K-isomorphic for some group H if and olnly if, H is abelian, H_p and G_p are isomorphic, H/H_0 and G/G_0 are isomorphic and H_0/H_p and G_0/G_p have equal cardinalities.*

Proof. Follows immediately from the theorem, because G_p is a direct product of a divisible and a bounded group, hence she is algebraically compact (see [1]). The proposition is true.

Corollary B1. *Let K be an algebraically closed field of* $\mathrm{char} K = p > 0$ *and let G be a countable algebraically compact group. Then the system $\{G_p, G/G_0, |G_0/G_p|\}$ is a complete set of invariants of the K-algebra KG.*

Proof. It is well-known that, G is a direct product of a divisible and of a bounded group ([1], p.200, Exercise 3(a)), therefore by the last proposition the corollary is true.

Corollary B2. *Let K be an algebraically closed field with* $\mathrm{char} K = p \neq 0$ *and let G be a torsion algebraically compact group. Then the system $\{G_p, |G/G_p|\}$ is a complete set of invariants of the K-algebra KG.*

Proof. From [1], G is also a direct product of a divisible group and of a bounded group. Furthermore from the proposition follows that, the corollary is verified.

C. Invariants of modular group algebras of simply presented torsion abelian groups

Theorem C. (Invariants) *Let K be an algebraically closed field of characteristic $p > 0$ and let G be a torsion abelian group whose p-component G_p is simply presented. Then $KH \cong KG$ as K-algebras for some group H if and only if*

(1) *H is abelian torsion.*
(2) $H_p \cong G_p$.
(3) $|H/H_p| = |G/G_p|$.

Proof. From $KG \cong KH$ follows that, $G_p \cong H_p$ (see [2]) and $K(G/G_p) \cong K(H/H_p)$ (see [4]). Hence $|G/G_p| = |H/H_p|$. Besides H is torsion, since $1 = G/G_0 \cong H/H_0$. Finally (1), (2), (3) are true.

Now, if (1), (2) and (3) are valid, then as we well-know $G \cong G_p \times G/G_p$, $H \cong H_p \times H/H_p$, therefore $KG \cong KG_p \otimes_K K(G/G_p)$, $KH \cong KH_p \otimes_K K(H/H_p)$. But by [5], $K(G/G_p) \cong K(H/H_p)$, thus $KG \cong KH$, because $KG_p \cong KH_p$, as required. So, the theorem is verified.

Corollary C. *Let K be an algebraically closed field of characteristic $p \neq 0$ and let G be an abelian simply presented torsion group. Then $KH \cong KG$ as K-algebras for some group H is and only if*
(1) *H is abelian torsion.*
(2) $H_p \cong G_p$.
(3) $|H/H_p| = |G/G_p|$.

5. Problems for isomorphism of commutative group algebras

Now we consider some questions for isomorphism of group algebras and for generalizations of the preceding results.

First, we note that if the abelian group G belongs to the class of divisible groups or to the class of direct sums of cyclic groups or to their (for the two classes) direct product, respectively, then G is a splitting group. That is why in the first two casses, the full system invariants for KG over an algebraic closed field of prime characteristic p is $\{G_p, G/G_0, |G_0/G_p|\}$. When G is a direct product of a divisible group and of a direct sum of cyclic groups, then is the complete system the same? And for an algebraic compact group G with $|G| > \aleph_0$, what is the invariant set?

Second, we will demonstrate also another example.

Example. ([6], p.267, Exercise 28). Let $F = F_p$ be a simple field with characteristic p and K is a field, obtained from F by the addition of all primitive q-th root of unity for all primes $q \neq p$. Then K is an algebraically closed field, char$K = p$ and an isomorphism results in this research are valid for K. Besides F as a finite field is perfect, but is not algebraic closed; $FG \cong FH$ for every two groups G and H does imply $\overline{F}G \cong \overline{F}H$. Thus $\{G_p, G/G_0, |G_0/G_p|\}$ is a system of invariants for FG, but probably nocomplete however, since the special May construction in [5] for a semisimple case of FG over F, when $G_p = 1$, is not true.

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