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Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New Series Vol. 12, 1998, Fasc. 1-2

Lattice Cyclically Ordered Groups

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Presented by P.Kenderov

The notion of cyclically ordered group (CO-group) is introduced by L. Rieger. Some properties of right cyclically ordered groups (RCO-groups) and of partially cyclically ordered groups (PCO-groups) are investigated by S. Zheleva. It is proved that the group of automorphisms of a cycilically ordered set is a RCO-orderable group.

In this paper the notion of a lattice cyclically ordered group will be introduced. It will be proved that the group of automorphisms of a cyclically ordered set is a lattice cyclically orderable group.

AMS Subj. Classification: 06AXX, 06BXX, 06F15

Key Words: lattices, cyclically ordered groups, automorphisms

1. Basic notions

In this section we introduce the notion of a lattice cyclically ordered group.

Let M be a set with card $M \geq 3$ and a, b, c elements on this set. Let (M, C) be a partially cycilically ordered set (PCO-set). We denote the fact $(a, b, c) \in C$ only by (a, b, c).

Definition 1.1. The elements a and b are cyclically comparable elements on the PCO-set (M, C) iff an element c exists such that $c \in M$ and (a, b, c) or (a, c, b) holds.

Definition 1.2. Let C_a be a cycle (CO-set) of the PCO-set (M, C), containing the element a. The elements a, b, c are incomparable elements on the PCO-set (M, C) iff there is no cycle C_a such that $b \in C_a$ and $c \in C_a$.

Definition 1.3. Let a, b, u, v be elements on the PCO-set (M, C).

The element u will be called a maximal left cyclic limit of the elements a and b iff cycles C_a and C_b exist such that $a \overline{\in} C_b$, $b \overline{\in} C_a$, $u \in C_a \cap C_b$ and (x, u, a) & (x, u, b) for each $x \in C_a \cap C_b$, $x \neq u$.

The element v will be called a minimal right cyclic limit of the elements a and b iff cycles C_a and C_b exist such that $a\overline{\in}C_b$, $b\overline{\in}C_a$, $v \in C_a \cap C_b$ and (a, v, y) & (b, v, y) for each $y \in C_a \cap C_b$, $y \neq v$.

We denote by $a \wedge_c b$ and $a \vee_c b$ any maximal left cyclic limit and any minimal right cyclic limit of the elements a and b.

Definition 1.4. The pair (M, C) will be called a lattice cyclically ordered set (lc-set) iff the following conditions are valid:

I. (M, C) is a PCO-set;

II. For every two different elements a and b on the set M one of the possibilities holds:

1) Every two cycles C_a and C_b, containing respectively the elements a and b, have no common elements;

2) The elements a and b are comparable elements or they have a maximal left cyclic limit and a minimal right cyclic limit.

If (M, C) is a lc-set, then the relation C will be called a lattice cyclic order (lc-order).

Definition 1.5. The algebraic system $(G, \bullet, C)$ will be called a lattice cyclically ordered group (lc-group) iff:

1) $(G, \bullet, C)$ is a PCO-group;

2) (G, C) is a lc-set.

The group G is a lattice cyclically orderable (lc-orderable) group iff at least one lc-order C exists such that $C \leq G^3$.

2. Examples for lc-groups

In this section some examples for lc-groups are given.

Example 2.1. Every CO-group is a lc-group.

Example 2.2. Every l-group is a lc-orderable group.

Example 2.3. Let $(I, \prec)$ be a well ordered set and let $(G_i, \bullet, C_i)$ be CO-groups with card $G_i \geq 3$ for each $i \in I$. In the product $G = \prod_{i \in I} G_i$ we survey the ternary relation C, defined as: $(a, b, c) \in C$ iff $(a_\alpha, b_\alpha, c_\alpha) \in C_\alpha$, where $a_\alpha \neq b_\alpha \neq c_\alpha \neq a_\alpha$ and $a_\beta = b_\beta = c_\beta$ for each $\beta \prec \alpha$.

The PCO-group $(G, \bullet, C)$ is called a lexicographic product of CO-groups. If CO-groups $(G_i, \bullet, C_i)$ have a nontrivial cyclic order for each $i \in I$, then the lexicographic product $(G, \bullet, C)$ is a lc-group, in which every two different elements are cyclically comparable.

Example 2.4. Let $(\mathbb{C}, +, C_1)$ and $(\mathbb{P}, +, C_2)$ be lc-groups, where C_1 and C_2 are the cyclic orders, induced respectively by the l-order $P(\mathbb{C}) = \{(a_1, a_2) \in$

$\mathbb{C}/a_1 \geq 0, a_2 \geq 0\}$ and by the natural binary order on $\mathbb{R}$. The lexicographic product $(G = \mathbb{C}\times\mathbb{R}, +, C)$ is a *lc*-group. Every two different elements $a = (\alpha, a_3)$ and $b = (\beta, b_3)$ of the group $(G, +)$ are cyclically comparable or sets of their cyclic limits exist. These sets are cycles

$$U_{a,b} = \{(\alpha \wedge \beta, x)/ \ \forall \ x \in \mathbb{R}\} \quad \text{and} \quad V_{a,b} = \{(\alpha \vee \beta, y)/ \ \forall \ y \in \mathbb{R}\},$$

CO-isomorphic onto the CO-set $(\mathbb{R}, C_2)$.

E x a m p l e 2.5. The lexicographic product $(G = \mathbb{R} \times \mathbb{C}, +, C')$ of the *lc*-groups $(\mathbb{R}, +, C_2)$ and $(\mathbb{C}, +, C_1)$ from Example 2.4 is a *lc*-group, for which the following holds: If $a_1 \neq b_1$, then the elements $a = (a_1, \alpha)$ and $b = (b_1, \beta)$ of G are cyclically comparable; If $a_1 = b_1$, then the cyclic limits $a \wedge_c b = (a_1, \alpha \wedge \beta)$ and $a \vee_c b = (a_1, \alpha \vee \beta)$ are uniquely determined for each $\alpha, \beta \in \mathbb{C}$.

E x a m p l e 2.6. Let $(G_0, +, C_0)$ be a group with card $G_0 = 2$ and with a trivial cyclic order C_0. Let $(\mathbb{R}, +C_2)$ be the CO-group from Example 2.4.

The lexicographic product $(G_1, +, C')$ of the CO-groups $(G_0, +, C_0)$ and $(\mathbb{R}, +, C_2)$ is a *lc*-group with exactly two noncrossing cycles.

The lexicographic product $(G_2, +, C'')$ of the CO-groups $(\mathbb{R}, +, C_2)$ and $(G_0, +, C_0)$ is a PCO-group, which is not a *lc*-group.

The lexicographic product $(G_3, +, C''')$ of the *lc*-groups $(\mathbb{R}, +, C_2)$ and $(G_1, +, C')$ is not a *lc*-group, either.

3. CO-automorphisms, orbits and stabilizers

Let (M, C) be a CO-set and let $\mu(M) = Aut(M, C)$ be the group of the CO-automorphisms of this set. We denote the unit of any group by e.

Definition 3.1. Let a be a fixed element of the CO-set (M, C). The set $Ob(a) = \{x \in M/x = af$ for each $f \in \mu(M)\}$ is said to be an orbit of the element a.

Proposition 3.1. *Every CO-set is a union of two by two noncrossing orbits.*

P r o o f. It follows from $a = ae$ that $a \in Ob(a)$ for each $a \in M$. If $Ob(a) \neq Ob(b)$ and $c = af = bg$ for some $f, g \in \mu(M)$, then CO-automorphisms h and t exist such that $x = bgf^{-1}h$, $y = afg^{-1}t$ for each $x, y \in Ob(a)$. This result implies the contradiction $Ob(a) = Ob(b)$. Thus we proved that just one of the possibilities $Ob(a) = Ob(b)$ or $Ob(a) \cap Ob(b) \neq \emptyset$ exists for each pair $(a, b) \in M^2$. ∎

Definition 3.2. Let C' be the induced cyclic order on the set $Ob(a)$. The set $St(a) = \{f \in Aut(Ob(a), C')/af = a\}$ will be called a stabilizer of the element a.

Definition 3.3. Let $\leq_a$ be the binary linear order on the set $Ob(a)$ with a least element a, induced by the cyclic order C', i.e.

$$\leq_a : \begin{cases} x <_a y, & \text{if } a \neq x \neq y \neq a \text{ and } (a, x, y) \in C'; \\ a <_a x & \text{for each } x \in Ob(a), x \neq a; \\ x = x & \text{for each } x \in Ob(a). \end{cases}$$

Let $Aut(Ob(a), \leq_a)$ be the group of all o-automorphisms on the set $(Ob(a), \leq_a)$.

Proposition 3.2. *$St(a) = Aut(Ob(a), \leq_a)$ for each $a \in M$.*

P r o o f. It is easy to prove that the mapping f is a CO-automorphism on $(Ob(a), C')$ iff f is a o-automorphism on $(Ob(a), \leq_a)$.

If g is an o-automorphism such that $ag \neq a$, then there are elements b and c on the set $Ob(a)$, for which $b = ag$ and $a = cg$. The inequalities $a = cg <_a ag = b$ imply the contradiction $c <_a a$. Hence, we proved that every o-automorphism on the set $(Ob(a), \leq_a)$ is an element of the group $(St(a), \circ)$. ∎

Proposition 3.3. *The set $\mu(Ob(a))/St(a)$ is a cycle, CO-isomorphic onto the set $Ob(a)$.*

P r o o f. Let β be the set $\mu(Ob(a))/St(a)$ and let $\overline{f}$ be the element of β with a representative the CO-automorphism f. The element $\overline{f}$ is the set of all CO-automorphisms, which map the element a onto the element af. The relation $<$, defined by: $\overline{f} < \overline{g}$ iff $af <_a ag$, is a binary linear order on the set β. Let C_β be the cyclic order on the set β, induced by this binary order. The mapping $\mathcal{Q}$, defined by $\overline{f}\mathcal{Q} = af$ for each $f \in \mu(Ob(a))$, is a CO-isomorphism of (β, C_β) onto $(Ob(a), C')$. ∎

Proposition 3.4. *$Ob(a) = Ob(af)$ for each $a \in M$ and each $f \in \mu(M)$.*

This fact follows directly from $a \in Ob(af)$ and Proposition 3.1. It indicates that every orbit is closed towards automorphisms on the CO-set (M, C).

Proposition 3.5. *If $a \in M$ and $f \in \mu(Ob(a))$, then $St(af) = f^{-1}St(a)f$.*

It is easy to show that $h = fgf^{-1} \in St(a)$ iff $g = f^{-1}hf \in St(af)$ for each $a \in M$ and for each $f \in \mu(Ob(a))$.

N o t e 3.1. It is well known that the group of o-automorphisms of a binary linear ordered set is a lattice orderable group. Therefore, $(St(a), \circ)$ is a l-group with a lattice order, defined by:

1) $f < g$ on $St(a)$ iff $xf \leq_a xg$ for each $x \in Ob(a)$ and there is an element $x_0 \in Ob(a)$ such that $x_0 f \neq x_0 g$; 2) $f = g$ on $St(a)$ iff $xf = xg$ for each $x \in Ob(a)$.

Proposition 3.6. *If $a \in M$ and $f \in \mu(Ob(a))$, then $g > e$ on $St(af)$ iff $fgf^{-1} > e$ on $St(a)$.*

P r o o f. Let $a \in M$, $f \in \mu(Ob(a))$ and $g \in St(af)$. Propositions 3.4 and 3.5 imply $Ob(a) = Ob(af)$ and $fgf^{-1} \in St(a)$.

Let $g > e$ on $St(af)$. The inequality $x \leq_{af} xg$ holds for each $x \in Ob(a)$ and there is an element $x_0 \in Ob(a)$ such that $x_0 g \neq x_0$. The element $y \in Ob(a)$ exists for each $x \in Ob(a)$ such that $x = yf$. The inequalities $af \leq_{af} yf \leq_{af} yfg$ hold for each $y \in Ob(a)$. If $af \neq yf \neq yfg \neq af$, then (af, yf, yfg), $(a, y, yfgf^{-1})$ and $a <_a y <_a yfgf^{-1}$. From $af = yf \neq yfg$ we conclude that $a = y \neq yfgf^{-1}$ and $a = y <_a yfgf^{-1}$. Therefore, the inequality $y \leq_a yfgf^{-1}$ holds for every $y \in Ob(a)$ and $fgf^{-1} > e$ on $St(a)$.

In the same way we prove that $h = fgf^{-1} > e$ on $St(a)$ implies $g > e$ on $St(af)$. ∎

N o t e 3.2. Let a be a fixed element of the CO-set (M, C), $S(a) = \{f \in \mu(M)/af = a\}$ and let $\leq_a$ be a linear order on the set M with a least element a, induced by the cyclic order C. Then $S(a) = Aut(M, \leq_a)$ and $S(a)$ is a l-group.

Proposition 3.7 *If a and b are elements of the CO-set (M, C), $f, g \in \mu(M)$ and $af = ag$, $bf = bg$, then $fg^{-1} > e$ on $S(a)$ iff $fg^{-1} > e$ on $S(b)$.*

P r o o f. Assume that the conditions of this proposition are valid, i.e. $fg^{-1} \in S(a) \cap S(b)$. If $fg^{-1} > e$ on $S(a)$, then $x \leq xfg^{-1}$ for each $x \in M$ and there is an element $c \in M$ such that $c <_a cfg^{-1}$.

If (a, b, c) holds, then (a, b, cfg^{-1}) and $b <_b cfg^{-1} <_b a$. The inequalities $a <_a c <_a cfg^{-1}$ imply (a, c, cfg^{-1}). From (a, b, c) and (a, c, cfg^{-1}) we conclude that (a, b, cfg^{-1}), (b, c, cfg^{-1}) and $b <_b c <_b cfg^{-1} <_b a$.

If (a, c, b) is true, then (a, cfg^{-1}, b), $b <_b a \;\; <_b c$ and $b <_b a <_b cfg^{-1}$ are true, too. The relation (a, c, cfg^{-1}) implies (cfg^{-1}, a, c). The relation (cfg^{-1}, b, c) and $b <_b a <_b c <_b cfg^{-1}$ follow from (cfg^{-1}, b, a) and (cfg^{-1}, a, c).

In this way we have proved that $c <_b cfg^{-1}$ for each $c \in M$ such that $c \neq cfg^{-1}$, i.e. $fg^{-1} > e$ on $S(b)$.

Analogically, we prove that $fg^{-1} > e$ on $S(b)$ implies $fg^{-1} > e$ on $S(a)$. ∎

In the next propositions we use the following definitions.

Definition 3.4 We say that a is an isolated element of the CO-set (M, C) iff there is an element $b \in M$ such that (b, a, x) holds for each $x \in M$. In this case we say that b is a CO-predecessor of the element a and a is a CO-successor of the element b.

Definition 3.5. The element a is a boundary element of the CO-set (M, C) iff for each $b \in M$ there is an element $x \in M$ such that (b, x, a) holds.

Definition 3.6. We say that the set (M, C) is a CO-discrete set iff every element of this CO-set is an isolated element.

Definition 3.7. The set (M, C) is a CO-compact set iff every element of this CO-set is a boundary element.

Definition 3.8. A homogeneous CO-set is a CO-set, which is discrete or compact.

Proposition 3.8. *Every orbit is a homogeneous CO-set.*

Proof. If a is a fixed element of the CO-set (M, C) and the set $(Ob(a), \leq_a)$ has no largest element, then the elements af and af^{-1} are boundary elements for each $f \in \mu(M)$ (see Proposition 1.2, [3]). Every element $x \in Ob(a)$ is an image of the element a by some CO - automorphism $g \in \mu(Ob(a))$ and $x = ag$ is a boyndary element, too. In this case $Ob(a)$ is a CO-compact set. If the set $(Ob(a), \leq_a)$ has a largest element, then the elements af and af^{-1} are isolated elements for each $f \in \mu(M)$ (see Proposition 1.3, [3]). In this case $Ob(a)$ is a CO-discrete set. ∎

Proposition 3.9. *If $Ob(a)$ is a CO-discrete orbit, then the element a has a CO-successor.*

Proof. Let (M, C) is a cycle. If $Ob(a)$ is a CO-discrete orbit and card $Ob(a) \geq 3$, then a is an isolated element and the element $b \in Ob(a)$ exists such that b is the CO-predecessor of the element a. The fact $b \in Ob(a)$ implies $b = af$ for some $f \in \mu(M)$. Let $c = af^{-1}$. We assume that the element $d \in Ob(a)$ exists such that (a, d, c) holds. This fact implies (b, df, a) which is a contradicion. Therefore, we have proved that the element c is the CO-successor of the element a. ∎

Definition 3.9. Let $(\mathbb{Z}, C_z)$ be the CO-set of integers, where C_z is the cyclic order, induced by the natural bynary order $<$ on this set. Let

$$A = \bigcup_{i=1}^{n} Z_i, \ (n \in \mathbb{N}) \text{ or } A = \bigcup_{i \in \mathbb{Z}} Z_i,$$

where $(Z_i, <_i)$ is an o-isomorphic set onto the set $(\mathbb{Z}, <)$ and (Z_i, C_i) is a CO-isomorphic set onto the set $(\mathbb{Z}, C_z)$. We denote by C_A the following cyclic order on the set A:

$$(a_i, b_j, c_k) \in C_A \text{ iff } \begin{cases} (a_i, b_j, c_k) \in C_i & \text{, if } i = j = k; \\ a_i <_i b_i & \text{, if } i = j \neq k; \\ b_j <_j c_j & \text{, if } i \neq j = k; \\ c_k <_k a_k & \text{, if } k = i \neq j; \\ (i, j, k) \in C_z & \text{, if } i \neq j \neq k \neq i, \end{cases}$$

where a_i, b_j, c_k are respectively elements of the sets Z_i, Z_j, Z_k.

The following fact follows from Propositions 3.8 and 3.9.

Proposition 3.10 *For each CO-discrete orbit (M, C) just one of the following conditions is valid:*

1) (M, C) is a finite CO-set;

2) (M, C) is a CO-isomorphic set onto $(\mathbb{Z}, C_z)$;

3) (M, C) is a CO-isomorphic set onto (A, C_A).

4. Transitive groups of automorphisms of a cyclically ordered set

In this section we prove that any transitive group of automorphisms of a cyclically ordered set is a *lc*-orderable group.

The group $\mu(M)$ of automorphisms of a CO-set (M, C) is said to be a transitive group iff there is an element $a \in M$ such that $M = Ob(a)$.

Theorem 4.1 *Any transitive group of automorphisms of a CO-set (M, C) with card $M \geq 3$ is a lc-orderable group.*

Proof. Let C_μ be the following ternary relation:

4.1. $(f, g, h) \in C_\mu$ iff $f, g, h \in \mu(M)$ and just one of the conditions is valid:

1) $(xf, xg, xh) \in C$, if $xf \neq xg \neq xh \neq xf$;

2) $gf^{-1} > e$ on $St(x)$, if $xf = xg \neq xh$;

3) $hg^{-1} > e$ on $St(x)$, if $xf \neq xg = xh$;

4) $fh^{-1} > e$ on $St(x)$, if $xh = xf \neq xg$;

5) $gf^{-1} > e$ & $hg^{-1} > e$, or $hg^{-1} > e$ & $fh^{-1} > e$, or $fh^{-1} > e$ & $gf^{-1} > e$ on $St(x)$, if $xf = xg = xh$.

It is easy to prove that $(\mu(M), \circ, C_\mu)$ is a PCO-group.

Let (f, g) be any pair of different CO-automorphisms on the CO-set (M, C). We prove that C_μ is a *lc*-order by analyzing the following cases:

I. Let $xf \neq xg$ for any $x \in M$.

1. If (M, C) is a CO-compact set, then f and g are cyclically comparable elements of the PCO-group $\mu(M)$.

In fact, if $a <_a xf <_a xg$ (or $a <_a xg <_a xf$) for some $x \in M$, then an element $y \in M$ and a CO-automorphism h exists such that $a <_a y <_a xf$ (or $xg <_a y <_a xf$) and $y = xh$. Hence, $(f, g, h) \in C_\alpha$.

2. Let (M, C) be a CO-discrete set.

a) If card $M = n \in \mathbb{N}$, then the group $(\mu(M), \circ, C_\mu)$ is CO-isomorphic onto the finite cyclic CO-group $\mathbb{C}(n)$ (see Theorem 4, [2]).

b) If the set (M, C) is CO-isomorphic onto the set $(\mathbb{Z}, C_z)$, then the group $(\mu(M), \circ, C_\mu)$ is CO-isomorphic onto the infinite cyclic CO-group $(\mathbb{Z}, +, C_z)$, (see Proposition 2.3, [3]).

c) If (M, C) is a CO-isomorhpic set onto the set (A, C_A) from Definition 3.9, then

$$M = \bigcup_{i \in I} Z_i,$$

where card $I = m \in \mathbb{N}$ or $I = \mathbb{Z}$, and any set $(Z_i = \{a_n^{(i)}/n \in \mathbb{Z}\}, C_i)$ is CO-isomorphic onto the set $(\mathbb{Z}, C_z)$ for $i \in I$.

4.2. In this case $a_n^{(i)} f = a_{n+k_i}^{(i+s)}$ and $a_n^{(i)} g = a_{n+l_i}^{(i+t)}$ are valid for any $i \in I$ and any $n \in \mathbb{Z}$, where k_i, l_i, s, t are fixed integers.

If $s \neq t$, then a CO-automorphism h_1 exists such that $a_n^{(i)} h_1 = a_{n+r_i}^{(i+t)}$ for any $i \in I$ and any $n \in \mathbb{Z}$, where r_i is a fixed integer and $r_i > l_i$. Hence, $(a_{n+k_i}^{(i+s)}, a_{n+l_i}^{(i+t)}, a_{n+r_i}^{(i+t)}) \in C_A$ for any $i \in I$ and any $n \in \mathbb{Z}$. This fact implies $(f, g, h_1) \in C_\mu$.

If $s = t$ and $k_i < l_i$ (or $l_i < k_i$) for each $i \in I$, then a CO-automorphism h_2 exists such that $a_n^{(i)} h_2 = a_{n+m_i+1}^{(i+s)}$ for any $i \in I$ and any $n \in \mathbb{Z}$, where $m_i = \max(k_i, l_i)$. The relation $(a_{n+k_i}^{(i+s)}, a_{n+l_i}^{(i+s)}, a_{n+l_i+1}^{(i+s)})$ (or $(a_{n+l_i}^{(I+s)}, a_{n+k_i}^{(i+s)}, a_{n+k_i+1}^{(i+s)})$ implies $(f, g, h_2) \in C_\mu$) or $(f, h_2, g) \in C_\mu$).

If $s = t, k_i < l_i$ for some $i \in I_1$ and $l_j < k_j$ for other $j \in I_2$, where $I_1 \cup I_2 = I$, then a CO-automorphism h_3 exists such that $a_n^{(i)} h_3 = a_n^{(i)} g = a_{n+l_i}^{(i+s)}$ and $a_n^{(j)} h_3 = a_{n-1}^{(j)} g = a_{n+l_j-1}^{(j+s)}$ for any $i \in I_1$ and any $j \in I_2$. The conditions $a_n^{(i)} f \neq a_n^{(i)} h_3 = a_n^{(i)} g, gh_3^{-1} > e$ on $St(a_n^{(i)})$, $i \in I_1$ and $(a_n^{(j)} f, a_n^{(j)} h_3, a_n^{(j)} g) \in C_A$, $j \in I_2$ imply $(f, h_3, g) \in C_\mu$.

We have proved in case **I** that every two different CO-automorphisms f and g are comparable elements of the PCO-group $\mu(M)$.

II. Let x_0 be an element on the CO-set (M, C) such that $x_0 f = x_0 g$, i.e. $gf^{-1} \in St(x_0)$.

1. If gf^{-1} and e are comparable elements of the l-group $St(x_0)$, then the CO-automorphisms f and g are comparable elements of the PCO-group $\mu(M)$.

In fact, $gf^{-1} > e$ on $St(x_0)$ implies just one of the possibilities $xfg^{-1} <_{x_0} x <_{x_0} xgf^{-1}$ or $xfg^{-1} = x = xgf^{-1}$ for each $x \in M$. Every one of them implies respectively $(xfg^{-1}, x, xgf^{-1}) \in C$ or $gf^{-1} > e$ on $St(x)$ by Proposition 3.7. Hence, $(fg^{-1}, e, gf^{-1}) \in C_\mu$ is true by Definition 4.1 and $(f, g, gf^{-1}g) \in C_\mu$.

2. If gf^{-1} and e are incomparable elements of the l-group $St(x_0)$, then the elements $u = e \wedge gf^{-1}$, $v = \vee gf^{-1}$ exist and (uf, f, vf), (uf, g, vf) hold on the PCO-group $\mu(M)$.

We denote the cycle, containing the elements uf, f, vf by C_f and the cycle, containing the elements uf, g, vf by C_g. It is easy to show that $f \overline{\in} C_g$ and $g \overline{\in} C_f$. We will prove that (t, uf, f) and (t, uf, g) hold for each $t \in C_f \cap C_g$ such that $t \neq uf$ and $t \neq vf$. We assume that (uf, t, f) is valid.

If (uf, g, t) holds, then (uf, t, f) implies (uf, g, f) and (u, gf^{-1}, e). The inequalities $x_0 u = x_0 gf^{-1} = x_0$, $u < gf^{-1}$ on $St(x_0)$ and (u, gf^{-1}, e) imply the contradiction $gf^{-1} < e$.

If (uf, t, g) holds, then (u, h, e) and (u, h, gf^{-1}) are true, where $h = tf^{-1}$. If $h \in St(x_0)$, then $u < h < e$ and $u < h < gf^{-1}$ on the l-group $St(x_0)$. From (u, h, e), (u, h, gf^{-1}) and $h \overline{\in} St(x_0)$ we conclude that $x_0 = x_0 gf^{-1} = x_0 u \neq x_0 h$ and $e < u$, $gf^{-1} < u$. We come to a contradiction with $u = e \wedge gf^{-1}$ in both cases.

In the same way it is proved that (w, f, vf) and (w, g, vf) are true for each CO-automorphism $w \in C_f \cap C_g$ such that $w \neq uf$ and $w \neq vf$.

Thus we have proved that the CO-automorphisms uf and vf are respectively the maximal left cyclic limit and the mimimal right cyclic limit of the CO-automorphisms f and g.

Therefore, the group $(\mu(M), \circ, C_\mu)$ is a lc-group, in which the cyclic limits are uniquely determined. ∎

N o t e: If the CO-set (M, C) is CO-isomorphic onto the set (A, C_A) in Definition 3.9, then CO-automorphisms f and g are incomparable elements of the lc-group $(\mu(M), \circ, C_\mu)$ iff in the formulae 4.2 $s = t$ and there is a triple $(i_1, i_2, i_3) \in I^3$ such that $k_{i_1} < l_{i_1}$, $k_{i_2} > l_{i_2}$, $k_{i_3} = l_{i_3}$. In this case the CO-automorphisms uf and vf are defined by:

$$a_n^{(i)} uf = a_{n+u_i}^{(i+s)} \quad \text{and} \quad a_n^{(i)} vf = a_{n+v_i}^{(i+s)},$$

where $u_i = \min(k_i, l_i)$ and $v_i = \max(k_i, l_i)$ for each $i \in I$.

5. Main result

In this section we consider the CO-set (M, C) as a union of noncrossing orbits, i.e.

$$M = \bigcup_{i \in I} Ob(a_i),$$

where $Ob(a_i) \cap Ob(a_j) = \phi$ for each $i, j \in I$ such that $i \neq j$ and $(I, \prec)$ is a well ordered set.

5.1. We denote the group $\mu(Ob(a_i))$ by μ_i and the restriction of the CO-automorphism $f \in \mu(M)$ on μ_i by f_i for each $i \in I$.

Lemma 5.1. *If $i \in I$, card $\mu_i = 2$ and $f_i \neq g_i$, then $xf_j \neq xg_j$ for each $j \in I$ and each $x \in Ob(a_j)$.*

Proof. We assume that $Ob(a_i) = \{a_i, a'_i\}$, $f_i \neq g_i = e_i$ and $x \in Ob(a_j)$, where $i, j \in I$ and $i \neq j$. Then (x, a_i, a'_i) or (x, a'_i, a_i) holds on CO-set (M, C). This fact implies respectively $(xf_j, , a'_i, a_i)$ & (xg_j, a_i, a'_i) or (xf_j, a_i, a'_i) & (xg_j, a'_i, a_i). If an element $y \in Ob(a_j)$ exists such that $yf_j = yg_j = z$, then (z, a_i, a'_i) and (z, a'_i, a_i) are valid at the same time, wich is a contradition. ■

Main theorem. *The automorphism's group of a CO-set is a lattice cyclically orderable group.*

Proof. Let (M, C) be a CO-set and let $(\mu_i, \circ, C_i)$ be the *lc*-group with cyclic order C_i, defined by Definition 4.1. for each $i \in I$. Let C_μ be the following ternary relation:

5.2. $(f, g, h) \in C_\mu$ iff $f, g, h \in \mu(M)$ and there is $\alpha \in I$ such that $(f_\alpha, g_\alpha, h_\alpha) \in C_\alpha$, where $f_\alpha \neq g_\alpha \neq h_\alpha \neq f_\alpha$ and $f_\beta = g_\beta = h_\beta$ for each $\beta \in I$, $\beta \prec \alpha$.

It is easy to verify that $\mu(M), \circ, C_\mu)$ is a PCO-group.

Let $f, g \in \mu(M)$ and $f \neq g$. Let N be the set of all elements $x \in M$, for which $xf \neq xg$ and let J be the set of all elements $j \in I$ with the quality $x \in N \cap Ob(a_j)$. If we denote the least element on the set $(J, \prec)$ by α, then $f_\alpha \neq g_\alpha$ and $f_\beta = g_\beta$ for each $\beta \in I$ and $\beta \prec \alpha$.

There are the following possibilities:

1. If card $\mu_\alpha = 2$, then according to Lemma 5.1 $\alpha = 1$ and $\mu_1 = \{e_1, f_1\}$. The CO-automorphisms f and g are incomparable automorphisms. If $h \in C_f \cap C_g$, where C_f and C_g are cycles, containing f and g respectively, then CO-automorphisms w and t exist such that (h, f, w) or (h, w, f); (h, g, t) or (h, t, g) hold on $(\mu(M), \circ, C_\mu)$. Definition 5.2 implies $h_1 = f_1 = w_1$ and $h_1 = e_1 = t_1$, i.e. $f_1 = e_1$. Therefore, every two cycles C_f and C_g are noncrossing cycles.

2. If card $\mu_\alpha \geq 3$, then the CO-automorphisms f_α and g_α are elements of the *lc*-group μ_α.

a) If f_α and g_α are cyclically comparable elements of the group μ_α, then a CO-automorphism $h_\alpha \in \mu_\alpha$ exists and just one of $(f_\alpha, g_\alpha, h_\alpha)$ or $(f_\alpha, h_\alpha, g_\alpha)$ is true.

Let h be the mapping, defined by:

$$xh = \begin{cases} xf_\beta & \text{, if } x \in Ob(a_\beta) \text{ and } \beta \prec \alpha; \\ xh_\alpha & \text{, if } x \in Ob(a_\alpha); \\ xt_\gamma & \text{, if } x \in Ob(a_\gamma), \alpha \prec \gamma \text{ and } t \in \mu(M). \end{cases}$$

(The CO-automorphism t is freely appointed.)

The mapping h is a CO-automorhpism by Proposition 3.4. According to Definition 5.2 the automorphisms f, g and h are comparable elements of the PCO-group $\mu(M)$.

b) If f_α and g_α are cyclically incomparable elements of the lc-group μ_α, then f and g have cyclic limits $u = f \wedge_c g$ and $v = f \vee_c g$, which may be nonuniquely determined by:

$$xu = \begin{cases} xf_\beta & \text{, if } x \in Ob(a_\beta) & \text{, where } \beta \prec \alpha; \\ xu_\alpha & \text{, if } x \in Ob(a_\alpha) & \text{; where } u_\alpha = f_\alpha \wedge_c g_\alpha; \\ xw_\delta & \text{, if } x \in Ob(a_\delta) & \text{, where } \alpha \prec \delta, \end{cases}$$

$$xv = \begin{cases} xf_\beta & \text{, if } x \in Ob(a_\beta) & \text{, where } \beta \prec \alpha; \\ xv_\alpha & \text{, if } x \in Ob(a_\alpha) & \text{; where } v_\alpha = f_\alpha \vee_c g_\alpha; \\ xt_\delta & \text{, if } x \in Ob(a_\delta) & \text{, where } \alpha \prec \delta \end{cases}$$

and automorphisms w and t are freely appointed elements of $\mu(M)$.

Thus we have proved that $(\mu(M), \circ, C_\mu)$ is a lc-group. ■

N o t e: We have proved that at most one orbit $Ob(a_i)$ may exist such that card $\mu_i = 2$ and then $i = 1$ holds. The CO-automorphisms f and g are elements of noncrossing cycles iff $f_1 \neq g_1$ and card $\mu_1 = 2$. In all other cases f and g are CO-comparable elements or they have cyclic limits.

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Received 27.11.1995