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Submanifolds of Some Almost Contact Manifolds with B-Metric with Codimension Two, II

G. Nakova [1], *K. Gribachev*[2]

Presented by P.Kenderov

In this paper, we study submanifolds of almost contact manifolds with B-metric of codimension 2, such that the vector field of the almost contact structure does not belong to the tangential space or to the normal space of the submanifold. Examples of such submanifolds are constructed.

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1. Introduction

Geometry of almost contact manifolds with B-metric can be considered as a natural extension of the geometry of almost complex manifolds with B-metric to the odd dimensional case. A classification of almost contact manifolds with B-metric $(M,\varphi,\varepsilon,\eta,g)$ with respect to the covariant derivative of the fundamental tensor φ of type (1.1) is given in [3].

Submanifolds of almost contact manifolds with B-metric of codimension 2 in the cases, when ε is a tangential vector field of the submanifold or ε belongs to the normal section are considered in [5]. In this paper we study submanifolds of codimension 2, when the vector ε is arbitrary, i.e. ε is not in the tangential space or in the normal space of the submanifold.

2. Preliminaries

Let $(M,\varphi,\varepsilon,\eta,g)$ be a $(2n+1)$-dimensional almost contact manifold with B-metric g, i.e. $(\varphi,\varepsilon,\eta)$ is an almost contact structure [1] and g is a metric on M [3] such that

$$\varphi^2 = -I + \eta\otimes\varepsilon, \quad \eta(\varepsilon) = 1, \tag{2.1}$$

where I denotes the identity transformation,

(2.2) $$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y),$$

for arbitrary vector fields X, Y on M. We denote by χM the Lie algebra of C^∞-vector fields on M.

The associated with g metric $\tilde{g}$ [3] on the manifold is given by

$$\tilde{g}(X,Y) = g(X, \varphi Y) + \eta(X)\eta(Y), \quad X, Y \in \chi M.$$

Both metrics g and $\tilde{g}$ are indefinite of signature $(n+1, n)$ [3].

Further X, Y, Z, W stand for arbitrary differentiable vector fields on M and x, y, z, w - for arbitrary vectors in the tangential space T_pM, $p \in M$.

Let ∇ be the Levi-Civita connection of the metric g. The tensor field F of type (0,3) on M is defined by

$$F(x,y,z) = g((\nabla_X \varphi)y, z) \quad [3]$$

The following 1-forms are associated with F :

$$\Theta(x) = g^{ij}F(e_i, e_j, x), \quad \Theta^*(x) = g^{ij}F(e_i, \varphi e_j, x),$$

(2.4) $$w(x) = F(\varepsilon, \varepsilon, x),$$

where $x \in T_pM$, $\{e_i, \varepsilon\}$, $(i = 1, \ldots, 2n)$ is a basis of T_pM, and (g^{ij}) is the inverse matrix of (g_{ij}) [3].

A classification of the almost contact manifold with B-metric with respect to the tensor F is given in [3], where are defined eleven basic classes $\mathcal{F}_i (i = 1, 2, \ldots, 11)$ of almost contact manifolds with B-metric. The class $\mathcal{F}_0$ is defined by the condition $F(x,y,z) = 0$. This special class belongs to everyone of the basic classes.

Let R be the curvature tensor field of type (1.3) of the Levi-Civita connection ∇ of g, i.e.

(2.5) $$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \chi M.$$

The corresponding to R tensor field of type (0,4) is given by

(2.6) $$R(X,Y,Z,W) = g(R(X,Y,Z)W);$$

If α is a section in the tangential space of $(M, \varphi, \varepsilon, \eta, g)$ with a basis $\{x, y\}$, in general we can define analogously as in [2] the following curvatures for section α :

(2.7) $$K(\alpha; p) = K(x,y) = \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)}$$

for every nondegenerate section α with respect to g. This is the usual Riemannian sectional curvature.

Let α be a 2-dimensional section in T_pM. A section α is said to be totally real if $\varphi\alpha\perp\alpha$. The totally real sections α are two types: α is orthogonal to ε, or α is not orthogonal to ε. A section α is said to be φ-holomorphic if $\varphi\alpha = \alpha$, i.e. $\alpha = \{\varphi x, \varphi^2 x\}, x \in T_pM$. A section α is said to be a ε-section if ε belongs to α, [5].

3. Submanifolds of an almost contact manifold with B-metric of codimension 2

In [5] submanifolds M of almost contact manifold $(\overline{M}, \overline{\varphi}, \overline{\varepsilon}, \overline{\eta}, g)$ ($\dim \overline{M} = 2n+3$) with B-metric g of codimension 2 of two basic types are studied. From the first type are the submanifolds such that the structure vector field $\overline{\varepsilon} \in T_pM$ and from the second type are the submanifolds such that $\overline{\varepsilon} \in (T_pM)^{\perp}$. In this section we consider submanifolds of almost contact manifold $(\overline{M}, \overline{\varphi}, \overline{\varepsilon}, \overline{\eta}, g)$ with B-metric g of codimension 2, when the structure vector field $\overline{\varepsilon}$ is not in T_pM or in $(T_pM)^{\perp}$ for every point p of the submanifold M.

Let $\overline{M}, \overline{\varphi}, \overline{\varepsilon}, \overline{\eta}, g)(\dim \overline{M} = 2n+3)$ be an almost contact manifold with B-metric g and let M be a submanifold of codimension 2 of $\overline{M}$. We assume that there exists a normal section $\alpha = \{N_1, N_2\}$ defined globally over the submanifold M such that:

$$g(N_1, N_1) = -g(N_2, N_2) = 1, \quad g(N_1, N_2) = 0. \tag{3.1}$$

We consider the following decomposition for $\overline{\varepsilon}, \overline{\varphi}X, \overline{\varphi}N_1, \overline{\varphi}N_2$ with respect to $\{N_1, N_2\}$ and T_pM.

(3.2) $\overline{\varepsilon} = \varepsilon_0 + aN_1 + bN_2;$

(3.3) $\overline{\varphi}X = \varphi X + \eta^1(X)N_1 + \eta^2(X)N_2, \quad X \in \chi M;$

(3.4) $\overline{\varphi}N_1 = \varepsilon_1 + cN_1 + dN_2;$

(3.5) $\overline{\varphi}N_2 = mN_1 + nN_2;$

where φ denotes a tensor field of type (1.1) on $M, \varepsilon_0, \varepsilon_1 \in \chi M, \eta^1$ and η^2 are 1-forms on M and a, b, c, d, m, n are functions on M. We denote the restriction of g on M by the same letter.

From (3.1), (3.2) and $g(\overline{\varepsilon}, \overline{\varepsilon}) = 1$ it follows

$$g(\varepsilon_0, \varepsilon_0) = 1 - a^2 + b^2; \tag{3.6}$$

Using (2.1), (2.2), (3.1) and the equalities (3.2)$\div$ (3.5) we find $\eta^1(X) = g(X, \varepsilon_1) = \frac{b}{d}\eta^0(X)$, where $\eta^0(X) = g(X, \varepsilon_0) \quad X \in \chi M$, i.e.

$$\varepsilon_1 = \frac{b}{d}\varepsilon_0, \quad \eta^2(X) = 0, \quad X \in \chi M; \tag{3.7}$$

(3.8)
$$\varphi^2 X = -X + \frac{d^2 - b^2}{d^2}\eta^0(X)\varepsilon_0;$$

$$\eta^0(\varphi X) = (\frac{ad}{b} - c)\eta^0(X), \quad X \in \chi M;$$

According to (2.2), (3.1), from (3.4) and (3.5) we get

(3.9)
$$m = -d$$

Then, taking into account (3.7), for (3.3), (3.4) and (3.5) we obtain

$$\overline{\varepsilon} = \varepsilon_0 + aN_1 + bN_2;$$

(3.10)
$$\overline{\varphi}X = \varphi X + \frac{b}{d}\eta^0(X)N_1, \quad X \in \chi M;$$

$$\overline{\varphi}N_1 = \frac{b}{d}\varepsilon_0 + cN_1 + dN_2;$$

$$\overline{\varphi}N_2 = -dN_1 + nN_2;$$

After direct computations of (3.10) by using $\overline{\varphi}\,\overline{\varepsilon} = 0$ we have

(3.11)
$$\phi\varepsilon_0 = -\frac{ab}{d}\,\varepsilon_0;$$

(3.12)
$$n = -\frac{ad}{b}.$$

Because of (2.2), (3.1) and (3.12), $g(\overline{\varphi}N_2, \overline{\varphi}N_2) = 1 + b^2 = \dfrac{d^2(b^2 - a^2)}{b^2}$ and hence,

(3.13)
$$|b| > |a|, \quad \text{i.e.} \quad g(\varepsilon_0, \varepsilon_0) > 0;$$

(3.14)
$$d = \frac{b\sqrt{b^2+1}}{\sqrt{b^2 - a^2}}.$$

Having in mind (2.2), (3.1), (3.9), (3.12) and (3.14) from the equalities (3.10) we compute

$$c = \frac{a(1-a^2+2b^2)}{\sqrt{b^{-}a^2}\sqrt{b^2+1}};$$

(3.15) $$m = -\frac{b\sqrt{b^2+1}}{\sqrt{b^2-a^2}};$$

$$n = -\frac{a\sqrt{b^2+1}}{\sqrt{b^2-a^2}}.$$

Substituting (3.14) and (3.15) in (3.8), (3.10), (3.11) we obtain

$$\overline{\varepsilon} = \varepsilon_0 + aN_1 + bN_2;$$

$$\overline{\varphi}X = \varphi X + \frac{\sqrt{b^2-a^2}}{\sqrt{b^2+1}}\,\eta^0(X)N_1, \quad X \in \chi M;$$

(3.16) $$\overline{\varphi}N_1 = \frac{\sqrt{b^2-a^2}}{\sqrt{b^2+1}}\,\varepsilon_0 + \frac{a(1-a^2+2b^2)}{\sqrt{b^2-a^2}\sqrt{b^2+1}}\,N_1 + \frac{b\sqrt{b^2+1}}{\sqrt{b^2-a^2}}\,N_2;$$

$$\overline{\varphi}N_2 = -\frac{b\sqrt{b^2+1}}{\sqrt{b^2-a^2}}\,N_1 - \frac{a\sqrt{b^2+1}}{\sqrt{b^2-a^2}}\,N_2;$$

$$\varphi^2 = -I + \frac{1+a^2}{1+b^2}\,\eta^0 \otimes \varepsilon_0;$$

$$\varphi\varepsilon_0 = -dfraca\sqrt{b^2-a^2}\sqrt{b^2+1}\,\varepsilon_0;$$

(3.17) $$\eta^0(\varphi X) = -\frac{a\sqrt{b^2-a^2}}{\sqrt{b^2+1}}\,\eta^0(X), \quad X \in \chi M;$$

$$g(\varphi X, \varphi Y) = -g(X,Y) + \frac{1+a^2}{1+b^2}\,\eta^0(X)\eta^0(Y), \quad X, Y \in \chi M.$$

Now we define a vector field ε, a 1-form η and a tensor field ϕ of type (1.1) on M by

$$\varepsilon = \frac{1}{\sqrt{1-a^2+b^2}}\,\varepsilon_0;$$

(3.18) $$\eta(X) = \frac{1}{\sqrt{1-a^2+b^2}}\,\eta^0(X), \quad X \in \chi M;$$

$$\phi(X) = \varphi X + \frac{a\sqrt{b^2-a^2}}{\sqrt{b^2+1}}\,\eta(X)\varepsilon, \quad X \in \chi M.$$

Taking into account the equalities (3.6), (3.17) and (3.18) we have

$$\phi^2 = -I + \eta \otimes \varepsilon,\ g(\varepsilon,\varepsilon) = 1,\ \eta(\varepsilon) = 1,\ \eta(\phi X) = 0 \tag{3.19}$$
$$g(\phi X, \phi Y) = -(X,Y) + \eta(X)\eta(Y), \quad X, Y \in \chi M.$$

Hence, $(\phi, \varepsilon, \eta)$ is an almost contact structure on M and the restriction of g on M is B-metric. Thus, the submanifold $(M, \phi, \varepsilon, \eta, g)$ ($\dim M = 2n+1$) of $(\overline{M}, \overline{\phi}, \overline{\varepsilon}, \overline{\eta}, g)$ ($\dim \overline{M} = 2n+3$) is an almost contact manifold with B-metric.

Denoting by $\overline{\nabla}$ and ∇ the Levi-Civita connections of the metric g in $\overline{M}$ and M, respectively, the formulas of Gauss and Weingarten are

$$\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y + \sigma(X,Y), && X, Y \in \chi M;\\
\overline{\nabla}_X N_1 &= -A_{N_1}X + D_X N_1, && X \in \chi M;\\
\overline{\nabla}_X N_2 &= -A_{N_2}X + D_X N_2, && X \in \chi M,
\end{aligned}$$

where σ is the second fundamental form on M, A_{N_1} is the second fundamental tensor with respect to N_1, A_{N_2}- with respect to N_2 and D is the normal connection on M. Having in mind the properties of $\overline{\nabla}$ and (3.1), from the formulas of Gauss and Weingarten we compute

$$\begin{aligned}
\sigma(X,Y) &= g(A_{N_1}X, Y)N_1 - g((A_{N_2}X, Y)N_2\\
&= g(X, A_{N_1}Y)N_1 - g(X, A_{N_2}Y)N_2, \quad X, Y \in \chi M;
\end{aligned}$$

$$D_X N_1 = \alpha(X)N_2, \quad X \in \chi M;$$

$$D_X N_2 = \alpha(X)N_1, \quad X \in \chi M;$$

where α is a 1-form on M.

From now on, in this section $(\overline{M}, \overline{\varphi}, \overline{\varepsilon}, \overline{\eta}, g)$ ($\dim \overline{M} = 2n+3$) will be an $\mathcal{F}_0$-manifold and $(M, \phi, \varepsilon, \eta, g)$ ($\dim M = 2n+1$) will be a submanifold of $\overline{M}$, such that $\overline{\varepsilon} \notin (T_pM)$ and $\overline{\varepsilon} \notin (T_pM)^{\perp}$.

If $\overline{M} \in \mathcal{F}_0$, i.e. $\overline{\nabla}\,\overline{\varphi} = \overline{\nabla}\,\overline{\eta} = \overline{\nabla}\,\overline{\varepsilon} = \overline{\nabla} g = 0$, then the formulas of Gauss and Weingarten become

$$\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y + g(A_{N_1}X, Y)N_1 - g(A_{N_2}X, Y)N_2, && X, Y \in \chi M;\\
\overline{\nabla}_X N_1 &= -A_{N_1}X + \alpha(X)N_2, && X \in \chi M;\\
\overline{\nabla}_X N_2 &= -A_{N_2}X + \alpha(X)N_1, && X \in \chi M.
\end{aligned} \tag{3.20}$$

From $(\overline{\nabla}_X\overline{\varphi})N_1 = (\overline{\nabla}_X\overline{\varphi})N_2 = 0$, we find

$$\begin{aligned}
\alpha(X) &= \frac{1}{2(b^2-a^2)}\left(-b(X \circ a) + \frac{a(1+a^2)}{1+b^2}(X \circ b)\right)\\
&= \frac{1}{2(b^2-a^2)}\left(-b\,da(X) + \frac{a(1+a^2)}{1+b^2}db(X)\right)
\end{aligned}$$

$$AX = A_{N_1} X = -\frac{a}{b} A_{N_2} X - \frac{\sqrt{b^2 - a^2}}{b\sqrt{b^2+1}} \phi(A_{N_2} X)$$
$$+ \frac{1}{2\sqrt{1 - a^2 + b^2}} (-da(X) + \frac{a(1 - a^2 + 2b^2)}{b(1+b^2)} db(X))\varepsilon, \quad \text{for } X \in \chi M.$$

Let $\overline{R}$ and R be the curvature tensors of $\overline{M}$ and M respectively. Then for the equation of Gauss we have

(3.21)
$$\overline{R}(x,y,z,w) = R(x,y,z,w) - \pi_1(A_{N_1}x, A_{N_1}y, z, w) + \pi_1(A_{N_2}x, A_{N_2}y, z, w)$$

for arbitrary $x, y, z, w \in T_pM$.

Using $\overline{\nabla\varphi} = 0$ and (3.18) we calculate

$$(3.22) \qquad (\nabla_X \phi)Y = \frac{\sqrt{b^2 - a^2}\sqrt{b^2+1}}{(1+a^2)\sqrt{1 - a^2 + b^2}} (\eta(y)Ax + g(Ax, y)\varepsilon)$$
$$+ \frac{a(b^2 - a^2)}{(1+a^2)\sqrt{1 - a^2 + b^2}} (\eta(y)\phi(Ax) + g(Ax, \phi y)\varepsilon)$$
$$+ \frac{\sqrt{b^2+1}}{\sqrt{b^2 - a^2}(1 - a^2 + b^2} (\frac{b^2 - 2a^2}{1+a^2}(x \circ a) + \frac{ab}{1+b^2}(x \circ b))\, \eta(y)\varepsilon.$$

From (3.22) we obtain the following assertion:

Theorem 3.1 *Let M be a submanifold of the $\mathcal{F}_0$-manifold $\overline{M}$. Then*

$$(3.23) \qquad F(x,y,z) = \frac{\sqrt{b^2 - a^2}\sqrt{b^2+1}}{(1+a^2)\sqrt{1 - a^2 + b^2}} (\eta(y)g(Ax, z) + \eta(z)g(Ax, y))$$
$$+ \frac{a(b^2 - a^2)}{(1+a^2)\sqrt{1 - a^2 + b^2}} (\eta(y)g(Ax, \phi z) + \eta(z)g(Ax, \phi y))$$
$$+ \frac{\sqrt{b^2+1}}{\sqrt{b^2 - a^2}(1 - a^2 + b^2)} (\frac{b^2 - 2a^2}{1+a^2}(x \circ a) + \frac{ab}{1+b^2}(x \circ b))\, \eta(y)\eta(z)$$

for arbitrary vectors x, y, z in T_pM.

According to (3.23) we compute

(3.24)
$$F(x,y,\varepsilon) = \frac{\sqrt{b^2 - a^2}\sqrt{b^2+1}}{(1+a^2)\sqrt{1 - a^2 + b^2}} g(Ax, y) + \frac{a(b^2 - a^2)}{(1+a^2)\sqrt{1 - a^2 + b^2}} g(Ax, \phi y)$$
$$+ \frac{\sqrt{b^2+1}}{\sqrt{b^2 - a^2}(1 - a^2 + b^2)} (\frac{b^2 - 2a^2}{1+a^2}(x \circ a) + \frac{ab}{1+b^2}(x \circ b))\, \eta(y).$$

From (3.23) and (3.24) we get $F(x,y,z)=\eta(y)F(x,z,\varepsilon)+\eta(z)F(x,y,\varepsilon)$. The last equality is the characterization condition for the direct sum $\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_7\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{11}$. [5]. Thus it follows the next

Proposition 3.2. *Let M be a submanifold of the $\mathcal{F}_0$-manifold $\overline{M}$. Then $M\in\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_7\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{11}$.*

4. Examples of submanifolds of an almost contact manifold with B-metric of codimension 2

Example 4.1. Let $\overline{M}=(\mathbb{R}^{2n+3},\overline{\varphi},\overline{\varepsilon},\overline{\eta},g)$ [3]. Identifying the point $p=(u^1,\dots,u^{n+1},v^1,\dots,v^{n+1},t)$ in $\overline{M}$ with its position vector Z, we define a submanifold M by the equalities:

$$(4.1)\qquad \begin{aligned} g(Z,Z)&=1;\\ \tilde{g}(Z,Z)&=-1. \end{aligned}$$

M is a $(2n+1)$-dimensional submanifold of $\overline{M}$ and $Z,\overline{\varphi}Z+\eta(Z)\overline{\varepsilon}$ are normal to T_pM^{2n+1}. We choose the unit normal vector fields $N_1=Z$ and $N_2=\dfrac{\text{ch}t}{\sqrt{2}}(Z+\overline{\varphi}Z+\eta(Z)\overline{\varepsilon})$. Then $g(N_1,N_1)=-g(N_2,N_2)=-1$ and $g(N_1,N_2)=0$. It is clear, that $\overline{\varepsilon}\notin T_pM$ and $\overline{\varepsilon}\notin(T_pM)^{\perp}$.

From (3.16) and (3.18) by $a=\text{th}t, b=-\sqrt{2}\text{sh}t,\quad t\neq\text{const}$, we obtain respectively

$$(4.2)\qquad \begin{aligned} &\overline{\varepsilon}=\varepsilon_0+\text{th}tN_1-\sqrt{2}\text{sh}tN_2;\\ &\overline{\varphi}=\varphi X-\text{th}t\eta^0(X)N_1\quad X\in\chi M;\\ &\overline{\varphi}N_1=-\text{th}t\varepsilon_0-(1+\text{th}^2t)N_1+\sqrt{2}\text{ch}tN_2;\\ &\overline{\varphi}=-\sqrt{2}\text{ch}tN_1+N_2. \end{aligned}\qquad \text{and}$$

$$(4.3)\qquad \begin{aligned} \varepsilon&=\frac{\text{ch}t}{\sqrt{1+2\text{sh}^2t\text{ch}^2t}}\varepsilon_0;\\ \eta(X)&=\frac{\text{ch}t}{\sqrt{1+2\text{sh}^2t\text{ch}^2t}}\eta^0(X),\quad X\in\chi M;\\ \phi X&=\varphi X-\text{th}^2t\eta(X)\varepsilon,\quad X\in\chi M. \end{aligned}$$

Taking into account (4.3) and (3.19), we can conclude that $(M,\phi,\varepsilon,\eta,g)$ is an almost contact manifold with B-metric.

Denoting by $\overline{\nabla}$ and ∇ the Levi-Civita connections of the metric g in $\overline{M}$ and M, respectively, the formulas of Gauss and Weingarten are

$$
(4.4)\quad
\begin{aligned}
&\overline{\nabla}_X Y = \nabla_X Y + g(A_{N_1}X, Y)N_1 - g(A_{N_2}X, Y)N_2, \quad X, Y \in \chi M;\\
&\overline{\nabla}_X N_1 = -A_{N_1}X, \quad X \in \chi M;\\
&\overline{\nabla}_X N_2 = -A_{N_2}X, \quad X \in \chi M.
\end{aligned}
$$

Since $\overline{\nabla}$ is flat, then $\overline{\nabla}_X Z = X$, Z being the position vector field and X being an arbitrary vector field on M. Using (4.4) and the definitions of N_1 and N_2, we get

$$
\begin{aligned}
&A_{N_1}X = -X \quad X \in \chi M;\\
&A_{N_2}X = -\frac{\mathrm{ch}t}{\sqrt{2}}(X + \phi X + \mathrm{ch}2t\eta(X)\varepsilon), \quad X \in \chi M;\\
&\eta(X) = \frac{1}{\mathrm{ch}t\sqrt{1+2\mathrm{sh}^2t\mathrm{ch}^2t}}(x \circ t), \quad X \in \chi M.
\end{aligned}
$$

Then the formulas (4.4) become

$$
(4.5)\quad
\begin{aligned}
&\overline{\nabla}_X Y = \nabla_X Y - g(X, Y)N_1 + \frac{\mathrm{ch}t}{\sqrt{2}}(g(X, Y) + g(\phi X, Y) + \mathrm{ch}2t\eta(X)\eta(Y))N_2;\\
&\overline{\nabla}_X N_1 = X;\\
&\overline{\nabla}_X N_2 = \frac{\mathrm{ch}t}{\sqrt{2}}(X + \phi X + \mathrm{ch}2t\eta(X)\varepsilon).
\end{aligned}
$$

Having in mind (4.2), (4.3) and (4.5) we compute

$$
\begin{aligned}
(\nabla_X\phi)y = & -\frac{\mathrm{sh}^3t}{\sqrt{1+2\mathrm{sh}^2t\mathrm{ch}^2t}}(\eta(y)\phi x + g(x, \phi y)\varepsilon)\\
& -\frac{\mathrm{sh}t\mathrm{ch}^2t}{\sqrt{1+2\mathrm{sh}^2t\mathrm{ch}^2t}}(\eta(y)\phi^2 x + g(\phi x, \phi y)\varepsilon), \quad x, y \in T_pM.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
F(x, y, z) = & -\frac{\mathrm{sh}^3t}{\sqrt{1+2\mathrm{sh}^2t\mathrm{ch}^2t}}(\eta(y)g(\phi x, z) + \eta(z)g(\phi x, y))\\
& -\frac{\mathrm{sh}t\mathrm{ch}^2t}{\sqrt{1+2\mathrm{sh}^2t\mathrm{ch}^2t}}(\eta(y)g(\phi x, \phi z) + \eta(z)g(\phi x, \phi y))
\end{aligned}
$$

for $x, y, z \in T_pM$.

From the last equality and from (2.4) we have $= \dfrac{\Theta(\xi)}{2n} = -\dfrac{\mathrm{sh}t\mathrm{ch}^2t}{\sqrt{1+2\mathrm{sh}^2t\mathrm{ch}^2t}}$,

$-\dfrac{\Theta^*(\varepsilon)}{2n} = -\dfrac{\mathrm{sh}^3 t}{\sqrt{1+2\mathrm{sh}^2 t\,\mathrm{ch}^2 t}}$. Then

$$F(x,y,z) = -\frac{\Theta(\varepsilon)}{2n}(\eta(y)g(\phi x,\phi z)+\eta(z)g(\phi x,\phi y)) - \frac{\Theta^*(\varepsilon)}{2n}(\eta(y)g(\phi x,z)+\eta(z)g(\phi x,y)).$$

Thus, the submanifold $(M,\phi,\varepsilon,\eta,g)$ is an almost contact manifold with B-metric in the class $\mathcal{F}_4\oplus\mathcal{F}_5$.

Now, from (2.5), (4.5) and $\overline{R}=0$ for the curvatuve tensor R of the submanifold M we find

$$R(x,y,z,w) = (1-\frac{\mathrm{ch}^2 t}{2})\pi_1(x,y,z,w) - \frac{\mathrm{ch}^2 t}{2}(\pi_2(x,y,z,w) \tag{4.6}$$

$$-\pi_3(x,y,z,w)+\mathrm{ch}2t\pi_4(x,y,z,w)+\mathrm{ch}2t\pi_5(x,y,z,w)),\quad x,y,z,w\in T_pM.$$

The tensors π_1,π_2,π_3,π_5 are given in [5]. The tensor π_4 is defined by:

$$\begin{aligned}\pi_4(x,y,z,w) &= g(y,z)\eta(x)\eta(w)-g(x,z)\eta(y)\eta(w)\\ &+g(x,w)\eta(y)\eta(z)-g(y,w)\eta(x)\eta(z).\end{aligned}$$

Proposition 4.1. *Let $(M,\phi,\varepsilon,\eta,g)(\dim M=2n+1)$ be a submanifold of the flat $\mathcal{F}_0$-manifold $\overline{M}=(\mathbb{R}^{2n+3},\overline{\varphi},\overline{\varepsilon},\overline{\eta},g)$, defined by the equalities (4.1). Then M has pointwise constant curvatures $K(\alpha;p)=1-\dfrac{\mathrm{ch}^2 t}{2}$ of the totally real orthogonal to ε sections α and pointwise constant holomorphic sectional curvatures $K(\alpha;p)=-\mathrm{sh}^2 t$.*

Proof. Let $\alpha=\{x,y\}$, $x,y\in T_pM$ be a totally real orthogonal to ε section. From (4.6) it follows that

$$R(x,y,y,x) = (1-\frac{\mathrm{ch}^2 t}{2})\pi_1(x,y,y,x).$$

Then the formula (2.7) implies immediately $K(\alpha;p)=1-\dfrac{\mathrm{ch}^2 t}{2}$. In the case, when α is a holomorphic section, i.e. $\alpha=\{\phi x,\phi^2 x\}$, $x\in T_pM$ we have $R(\phi x,\phi^2 x,\phi^2 x,\phi x)=-\mathrm{sh}^2 t\pi_1(\phi x,\phi^2 x,\phi^2 x,\phi x)$ and $K(\alpha;p)=-\mathrm{sh}^2 t$. ■

E x a m p l e 4.2. Let M be the submanifold of $\overline{M} = (\mathbb{R}^{2n+3}, \overline{\varphi}, \overline{\varepsilon}, \overline{\eta}, g)$, determined by

$$(4.7) \qquad \begin{aligned} &\tilde{g}(Z,Z) = -1; \\ &\tilde{g}(Z,\overline{\varphi}Z) = 0; \end{aligned}$$

The vector fields $\overline{\varphi}^2 Z$ and $\overline{\varphi}Z + \overline{\eta}(Z)\overline{\varepsilon}$ are normal to M. Hence, the structure vector field $\overline{\varepsilon}$ on $\overline{M}$ is not in T_pM and $(T_pM)^{\perp}$. At every point $p \in M$ we can put $\overline{\eta}(Z) = \mathrm{sh}t$, $t \neq$ const. Then for the vector fields $N_1 = -\dfrac{\mathrm{th}t}{\mathrm{ch}t}Z - \dfrac{1}{\mathrm{sh}t}\varphi Z - \dfrac{1}{\mathrm{ch}^2 t}\overline{\varepsilon}$ and $N_2 = \dfrac{1}{\mathrm{sh}t}\overline{\varphi}Z + \overline{\varepsilon}$ we have $g(N_1,N_1) = -g(N_2,N_2) = -1, \quad g(N_1,N_2) = 0.$

The following decomposition for $\overline{\varepsilon}, \overline{\varphi}N_1, \overline{\varphi}N_2$ and $\overline{\varphi}X,\ X \in \chi M$ with respect to $\{N_1, N_2\}$ and T_pM is valid:

$$(4.8) \qquad \begin{aligned} &\overline{\varepsilon} = \varepsilon_0 + N_1 + N_2; \\ &\overline{\varphi}X = \varphi X - \mathrm{th}^2 t\eta^0(X)N_1, \quad X \in \chi M; \\ &\overline{\varphi}N_1 = \mathrm{th}^2 t\varepsilon_0 - \frac{4\mathrm{ch}2t}{\mathrm{sh}^2 2t}N_1 - \mathrm{cth}^2 tN_2; \\ &\overline{\varphi}N_2 = \mathrm{cth}^2 tN_1 + \mathrm{cth}^2 tN_2, \end{aligned}$$

where φ denotes a tensor field of type (1.1) on $M, \varepsilon_0 \in \chi M$, η^0 is 1-form on M and $\eta^0(X) = g(\varepsilon_0, X)$, $X \in \chi M$.

From the equalities (4.8) we obtain

$$(4.9) \qquad \begin{aligned} &g(\varepsilon_0,\varepsilon_0) = 1; \\ &\varphi^2 = -I + (1 + \mathrm{th}^4 t)\eta^0 \otimes \varepsilon_0; \\ &\varphi\varepsilon_0 = -\mathrm{th}^2 t\varepsilon_0; \\ &\eta^0(\varphi X) = -\mathrm{th}^2 t\eta^0(X), \quad X \in \chi M. \end{aligned}$$

Now, we define a vector field ε, a 1-form η and a tensor field ϕ of type (1.1) on M by

$$(4.10) \qquad \begin{aligned} &\varepsilon = \varepsilon_0; \\ &\eta(X) = \eta^0(X), \quad X \in \chi M; \\ &\phi X = \varphi X + \mathrm{th}^2 t\eta(X)\varepsilon, \quad X \in \chi M. \end{aligned}$$

Using (4.9), (4.10) we check immediately that $(\phi, \varepsilon, \eta, g)$ is an almost contact structure with B-metric g on the submanifold M of $\overline{M}$.

Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections of the metric g in $\overline{M}$ and M, respectively. Then the formulas of Gauss and Weingarten are

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y - g(A_{N_1}X, Y)N_1 + g(A_{N_2}X, Y)N_2, \quad X, Y \in \chi M; \\ (4.11)\quad &\overline{\nabla}_X N_1 = -A_{N_1}X + \beta(X)N_2, \quad X \in \chi M; \\ &\overline{\nabla}_X N_2 = -A_{N_2}X + \beta(X)N_1, \quad X \in \chi M; \end{aligned}$$

From (4.11), the definitions of N_1, N_2 and $\overline{\nabla}_X Z = X$, Z being the position vector field, we find

$$\begin{aligned} A_{N_1}X &= \frac{\text{th}t}{\text{ch}t}X + \frac{1}{\text{sh}t}\phi X + \frac{1-\text{sh}^2 t}{\text{sh}t\text{ch}^2 t}\eta(X)\varepsilon, \quad X \in \chi M; \\ A_{N_2}X &= -\frac{1}{\text{sh}t}\phi X + \frac{1}{\text{sh}t\text{ch}^2 t}\eta(X)\varepsilon, \quad X \in \chi M; \\ \beta(X) &= \frac{\eta(X)}{\text{sh}t\text{ch}^2 t}, \quad X \in \chi M; \\ \eta(X) &= \text{ch}t(X \circ t), \quad X \in \chi M. \end{aligned}$$

Then the formulas (4.11) become

$$\begin{aligned} \overline{\nabla}_X Y = \nabla_X Y &- (\frac{\text{th}t}{\text{ch}t}g(X,Y) + \frac{1}{\text{sh}t}g(\phi X, Y) + \frac{1-\text{sh}^2 t}{\text{sh}t\text{ch}^2 t}\eta(X)\eta(Y))N_1 \\ &-(\frac{1}{\text{sh}t}g(\phi X, Y) + \frac{1}{\text{sh}t\text{ch}^2 t}\eta(X)\eta(Y))N_2; \end{aligned}$$

(4.12)

$$\begin{aligned} \overline{\nabla}_X N_1 &= -\frac{\text{th}t}{\text{ch}t}X - \frac{1}{\text{sh}t}\phi X + \frac{\text{sh}^2 t - 1}{\text{sh}t\text{ch}^2 t}\eta(X)\varepsilon + \frac{1}{\text{sh}t\text{ch}^2 t}\eta(X)N_2; \\ \overline{\nabla}_X N_2 &= \frac{1}{\text{sh}t}\phi X + \frac{1}{\text{sh}t\text{ch}^2 t}\eta(X)\varepsilon + \frac{1}{\text{sh}t\text{ch}^2 t}\eta(X)N_1; \end{aligned}$$

Using (4.8), (4.10) and (4.12) we compute

$$(\nabla_x \phi)y = -\frac{\text{th}t}{\text{ch}t}(g(x,\phi y)\varepsilon + \eta(y)\phi x), \quad x, y \in T_p M.$$

Hence, $F(x,y,z) = -\frac{\text{th}t}{\text{ch}t}(\eta(z)g(x,\phi y) + \eta(y)g(x,\phi z))$, $x, y, z \in T_p M$. From the last equality and from (2.4) we get $\Theta(\varepsilon) = 0$, $-\frac{\Theta^*(\varepsilon)}{2n} = -\frac{\text{th}t}{\text{ch}t}$. Therefore, $F(x,y,z) = -\frac{\Theta^*(\varepsilon)}{2n}(\eta(z)g(x,\phi y) + \eta(y)g(x,\phi z))$. According to [4], $M \in \mathcal{F}_5$

iff $F(x,y,z) = -\frac{\Theta^*(\varepsilon)}{2n}(\eta(z)g(x,\phi y) + \eta(y)g(x,\phi z))$ and consequently the submanifold $(M,\phi,\varepsilon,\eta,g)$ is an almost contact manifold with B-metric in the class $\mathcal{F}_5$.

Taking into account (2.5), (4.12) and $\overline{R} = 0$ for the curvature tensor R of the submanifold M we obtain

$$(4.13)\qquad \mathrm{R(x,y,z,w)} = \frac{1}{\mathrm{ch}^2 t}(-\mathrm{th}^2 t\pi_1(x,y,z,w) + \pi_3(x,y,z,w)$$
$$+\frac{\mathrm{sh}^2 t - 1}{\mathrm{ch}^2 t}\pi_4(x,y,z,w) + \pi_5(x,y,z,w));\ x,y,z,w \in T_pM.$$

Proposition 4.2 *Let* $(M,\phi,\varepsilon,\eta,g)$ $(\dim M = 2n+1)$ *be a submanifold of the flat* $\mathcal{F}_0$*-manifold* $\overline{M} = (\overline{\mathbb{R}}^{2n+3},\overline{\varphi},\overline{\varepsilon},\overline{\eta},g)$, *defined by the equalities (4.8). Then* M *has pointwise constant curvatures of the totally real orthogonal to* ε *sections and pointwise constant holomorphic and* ε*- sectional curvatures.*

P r o o f. Using (2.7) and (4.13) we have:

$K(\alpha;p) = -\frac{\mathrm{th}^2 t}{\mathrm{ch}^2 t}$, where $\alpha = \{x,y\}$, $x,y \in T_pM$ is a totally real orthogonal to ε section;

$K(\alpha;p) = -\frac{\mathrm{th}^2 t}{\mathrm{ch}^2 t}$, where $\alpha = \{\phi x,\phi^2 x\}$, $x \in T_pM$ is a ϕ-holomorphic section;

$K(\alpha;p) = -\frac{1}{\mathrm{ch}^4 t}$, where $\alpha = \{x,y\}$, $x \in T_pM$ is a ε-section. ■

E x a m p l e 4.3. Let M be the submanifold of $\overline{M} = (\overline{\mathbb{E}}^{2n+3},\overline{\varphi},\overline{\varepsilon},\overline{\eta},g)$, determined by

$$(4.14)\qquad \begin{gathered} g(Z,Z) = -1; \\ g(Z,\overline{\varphi}Z) = 0. \end{gathered}$$

The vector fields Z and $\overline{\varphi}Z$ are normal to M. Therefore $\overline{\varepsilon} \notin T_pM$ and $\overline{\varepsilon} \notin (T_pM)^\perp$, $p \in M$. If at every point $p \in M$ we set $\overline{\eta}(Z) = \mathrm{sh}t$, $t \neq$ const, then the vector fields $N_1 = Z$; $N_2 = \frac{1}{\mathrm{ch}t}\overline{\varphi}Z$ fulfil the conditions $g(N_1,N_1) = -g(N_2,N_2) = -1$ and $g(N_1,N_2) = 0$.

Further, we have the following decomposition for $\overline{\varepsilon},\overline{\varphi}N_1,\overline{\varphi}N_2$, and $\overline{\varphi}X, X \in \chi M$ with respect to $\{N_1,N_2\}$ and T_pM :

$$(4.15)\qquad \begin{aligned} &\overline{\varepsilon} = \varepsilon_0 - \mathrm{sh}t N_1; \\ &\overline{\varphi}X = \varphi X + \mathrm{th}t\eta^0(X)N_2, \quad X \in \chi M; \\ &\overline{\varphi}N_1 = \mathrm{ch}t N_2; \\ &\overline{\varphi}N_2 = \mathrm{th}t\varepsilon_0 - \mathrm{ch}t N_1, \end{aligned}$$

where φ is a tensor field of type (1.1) on M, $\varepsilon_0 \in \chi M$, η^0 is 1-form on M and $\eta^0(x) = g(\varepsilon_0, X)$, $X \in \chi M$.

After direct computations from (4.15) we get

$$\begin{aligned} & g(\varepsilon_0, \varepsilon_0) = \mathrm{ch}^2 t; \\ & \varphi^2 = -I + \frac{1}{\mathrm{ch}^2 t}\eta^0 \otimes \varepsilon_0; \\ (4.16) \qquad & \varphi\varepsilon_0 = 0; \\ & \eta^0(\varphi X) = 0, \quad X \in \chi M. \end{aligned}$$

We define a vector field ε, a 1-form η and a tensor field ϕ of type (1.1) on M by

$$\begin{aligned} & \varepsilon = \frac{1}{\mathrm{ch} t}\varepsilon_0; \\ (4.17) \qquad & \eta(X) = \frac{1}{\mathrm{ch} t}\eta^0(X), \quad X \in \chi M; \\ & \phi X = \varphi X, \quad X \in \chi M. \end{aligned}$$

It is easy to check that $(\phi, \varepsilon, \eta, g)$ is an almost contact structure with B-metric g on the submanifold M of $\overline{M}$.

Denoting by $\overline{\nabla}$ and ∇ the Levi-Civita connections of the metric g in $\overline{M}$ and M, respectively, the formulas of Gauss and Weingarten are

$$\begin{aligned} & \overline{\nabla}_X Y = \nabla_X Y - g(A_{N_1}X, Y)N_1 + g(A_{N_2}X, Y)N_2, \quad X, Y \in \chi M; \\ (4.18) \qquad & \overline{\nabla}_X N_1 = -A_{N_1}, \quad X \in \chi M; \\ & \overline{\nabla}_X N_2 = -A_{N_2}X, \quad X \in \chi M. \end{aligned}$$

The definitions of N_1 and N_2, (4.18) and $\overline{\nabla}_X Z = X$, Z being the position vector field and X being an arbitrary vector field on M, imply

$$\begin{aligned} & A_{N_1}X = -X, \quad X \in \chi M; \\ & A_{N_2}X = -\frac{1}{\mathrm{ch} t}\phi X, \quad X \in \chi M; \\ & \eta(X) = (X \circ t), \quad X \in \chi M. \end{aligned}$$

Hence, the formulas (4.18) become

$$\begin{aligned} & \overline{\nabla}_X Y = \nabla_X Y + g(X, Y)N_1 - \frac{1}{\mathrm{ch} t}g(X, \phi Y)N_2; \\ (4.19) \qquad & \overline{\nabla}_X N_1 = X; \\ & \overline{\nabla}_X N_2 = \frac{1}{\mathrm{ch} t}\phi X. \end{aligned}$$

Now, using (4.15), (4.17) and (4.19) we calculate

$$(\nabla_X \phi)y = -\mathrm{th} t(\eta(y)\phi x + g(x, \phi y)\varepsilon), \quad x, y \in T_pM.$$

Hence, $F(x,y,z) = -\mathrm{th}t(\eta(y)g(x,\phi z) + \eta(z)g(x,\phi y))$, $x,y,z \in T_pM$. From this equality and (2.4) we compute $\Theta(\varepsilon) = 0$, $-\frac{\Theta^*(\varepsilon)}{2n} = -\mathrm{th}t$. Then

$$F(x,y,z) = -\frac{\Theta^*(\varepsilon)}{2n}(\eta(y)g(x,\phi z) + \eta(z)g(x,\phi y)).$$

Thus, $(M,\phi,\varepsilon,\eta,g)$ is an almost contact manifold with B-metric in class $\mathcal{F}_5$.

Finally, for the curvature tensor R of the submanifold M we have

$$(4.20) \quad R(x,y,z,w) = -\pi_1(x,y,z,w) + \frac{1}{\mathrm{ch}^2 t}\pi_2(x,y,z,w),\ x,y,z,w \in T_pM.$$

Proposition 4.3. *Let* $(M,\phi,\varepsilon,\eta,g)$ (dim $M = 2n+1$) *be a submanifold of the flat* $\mathcal{F}_0$*-manifold* $\overline{M} = (\overline{\mathbb{R}}^{2n+3}, \overline{\varphi}, \overline{\varepsilon}, \overline{\eta}, g)$, , *defined by the equalities (4.8). Then M is of constant totally real and* ε*-sectional curvatures and pointwise constant holomorphic sectional curvatures.*

P r o o f. Using (2.7) and (4.20) we get:

$K(\alpha) = -1$, where $\alpha = \{x,y\}$, $x,y \in T_pM$ is an arbitrary totally real section;

$K(\alpha) = -1$, where $\alpha = \{x,\varepsilon\}$, $x \in T_pM$ is a ε-section;

$K(\alpha;p) = -\mathrm{th}^2 t$, where $\alpha = \{\phi x, \phi^2 x\}$, $x \in T_pM$ is a ϕ-holomorphic section. ■

R e m a r k. Note, that as in the Example 4.2., the manifold $M \in \mathcal{F}_5$ in the Example 4.3. has negative pointwise constant characteristic sectional curvatures. But in the Example 4.3. the curvatures of totally real and ε-sections are global constans equal to -1. Moreover in the Example 4.2. the totally real sections are orthogonal to ε and in the Example 4.3. the totally real sections are arbitrary.

An example of a manifold $M \in \mathcal{F}_5$, which has vanishing totally real and ε-sectional curvatures is given in [5].

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[1] *Department of Mathematics*
"Vasil Levski" Higher Military School
Veliko Tarnovo, BULGARIA

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[2] *Faculty of Mathematics, University of Plovdiv*
"Zar Asen" 24
Plovdiv, BULGARIA