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On a Generalization of Jackson's Theorem in R^m

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In this paper an estimate of the best Hausdorff approximation of bounded functions of many variables by trigonometric polynomials is obtained.

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1. Notations, definitions, some properties of the Hausdorff distance in R^m

In this paper we use the following notations:

$$\Delta^m = \Delta^m_{[0,l]} = \{x = (x_1, x_2, ..., x_m) : 0 \le x_i \le l, i = 1, 2, \dots, m;\ m \ge 1\},$$

R - the set of all natural numbers, A_{Δ^m} - the set of all bounded functions f defined on Δ^m,

$$\max\{|g(x)| : x \in \Delta^m\} \le M,$$

$\rho(A(x_1, ..., x_m), B(y_1, ..., y_m)) = \max\{|x_1 - y_1|, ..., |x_m - y_m|\}$ - the distance on Δ^m,

$$w(\Delta^m, f; \delta) = \sup\{|f(x) - f(y)|; \rho(x, y) \le \delta, x, y \in \Delta^m\},$$

$\delta > 0$ - the modulus of continuity, $f \in A^M_{\Delta^m}$,

$$R(\Delta^m; f, g) = \sup\{|f(x) - g(x)|, x \in \Delta^m\}$$

the uniform distance between $f, g \in A^M_{\Delta^m}$, and finally $r[\Delta^m, \alpha; f, g]$ – the Hausdorff distance between $f, g \in A_{\Delta^m}$ with a parameter $\alpha > 0$ (see [1, 4]).

The following lemma is an analog of a proposition proved for $m = 1$ (see [2]). Using a similar method, it is not difficult to prove the following.

Lemma 1. *Let $f, g \in A^M_{\Delta^m}$ and $f(g)$ have the property:*

$$\lim_{\delta\to 0} w(\Delta^m, f; \delta) = 0, \qquad \left(\lim_{\delta\to 0} w(\Delta^m, g; \delta) = 0\right).$$

Then it holds:

$$\lim_{\alpha\to 0} r(\Delta^m, \alpha; f, g) = R(\Delta^m; f, g).$$

2. Some basic statements

First we have to define the following function

$$w(x,\delta,f;y) = \begin{cases} 0; \\ \left\{y \in \Delta^m_{[0,l]};\ |(x_i - y_i)| \le \delta, i = 1,\dots,m\right\}; \\ h(f; 0, \dots, |x_{i_1} - y_{i_1}| - \delta, \dots, 0); \\ i_1 = 1, 2, \dots, m; \\ \begin{cases} y \in \Delta^m_{[0,l]}; \\ |x_{i_1} - y_{i_1}| > \delta \\ |x_j - y_j| \le \delta, j \ne i_1, i = 1,\dots,m; \end{cases} \\ \dots \\ h(f; 0, \dots, |x_{i_1} - y_{i_1}| - \delta, \dots, |x_{i_s} - y_{i_s}| - \delta, \dots, 0); \\ i_1, \dots, i_s = 1, 2, \dots, m; \\ (i_1 \ne i_2 \ne \dots \ne i_s); \\ \begin{cases} y \in \Delta^m_{[0,l]}; \\ |x_k - y_k| > \delta, k = i_1, \dots, i_s; \\ |x_j - y_j| \le \delta, j \ne i_1, \dots, i_s; \\ j = 1, 2, \dots, m. \end{cases} \\ \cdots s = 1, 2, \dots, m. \\ \dots \\ h(f; 0, \dots, |x_1 - y_1| - \delta, \dots, |x_m - y_m| - \delta); \\ \begin{cases} y \in \Delta^m_{[0,l]}; \\ |x_j - y_j| > \delta \\ j = 1, 2, \dots, m. \end{cases} \end{cases}$$

where $h(f; \delta_1, \delta_2, \dots, \delta_m) = \sup\{|f(x) - f(y)|\ |x_k - y_k| \le \delta_k; k = 1, 2, \dots, m; x, y \in \Delta^m$

Theorem 1. *Let $L(f)$ be linear and positive operator, defined on $A^M_{\Delta^m}$. Then for every $\delta > 0$, $\alpha > 0$ the inequality*

(1) $$r(\Delta^m, \alpha; L(f), f) \le (\Delta^m, \alpha, f; 2\delta)$$

$$+\sup\{L(w(x,\delta,f);x), x\in\Delta^m\} + M\sup\{\mathrm{mod}(1-L(1;x)), x\in\Delta^m\},$$

holds.

Proof. First we shall prove that for every $x,y\in\Delta^m_{[0,l]}$ it holds

$$I(\delta,f;x)-w(x,\delta,f;y)\le f(y), \tag{2}$$

where

$$f(y)\le S(\delta,f;x)+w(x,\delta,f;y).$$

Let $x=(x_1,\dots,x_m)\in\Delta^m_{[0,l]}$ be an arbitrary point.

1) We consider $y=(y_1,\dots,y_m)\in\Delta^m_{[0,l]}$ such that

$$\mathrm{mod}[x_j-y_j]\le\delta,\quad i=1,2,\dots,m.$$

By the definition of $I(\delta,f;\cdot)$, and $S(\delta,f;\cdot)$ we get

$$I(\delta,f;x)\le f(y)\le S(\delta,f;x) \tag{3}$$

Hence, (2) is true.

2) We consider $y=(y_1,\dots,y_m)\in\Delta^m_{[0,l]}$ such that

$$\begin{aligned}\mathrm{mod}[x_k-y_k] &> \delta, \quad k=i_1,i_2,\dots,i_s,\\ \mathrm{mod}[x_j-y_j] &\le \delta, \quad j=1,2,\dots,m;\ j=i_1,i_2,\dots,i_s,\ s=1,2,\dots,m.\end{aligned}$$

Without any restriction we can assume that $y_k>x_k+\delta;\ j\ne i_1,i_2,\dots,i_s$. We can consider the other possible cases just in the same way. From the definition of the modulus of continuity and the definition of $w(x,\delta,f;\cdot)$ we have:

$$\begin{aligned}f(y) &\le f(y_1,\dots,x_{i_1}+\delta,\dots,x_{i_s}+\delta,\dots,y_m)\\ &\quad +h(f;0,\dots,|x_{i_1}-y_{i_1}|-\delta,\dots,|x_{i_s}-y_{i_s}|-\delta,\dots,0)\\ &\le S(\delta,f;x)+w(x,\delta,f;y)\end{aligned} \tag{4}$$

and

$$\begin{aligned}&h(f;0,\dots,|x_{i_1}-y_{i_1}|-\delta,\dots,|x_{i_s}-y_{i_s}|-\delta,\dots,0)\\ &-h(f;0,\dots,|x_{i_1}-y_{i_1}|-\delta,\dots,|x_{i_s}-y_{i_s}|-\delta,\dots,0)\\ &\ge I(\delta,f;x)-w(x,\delta,f;y).\end{aligned} \tag{5}$$

The inequalities (4) and (5) prove (2).

Further we use that the operator $L(f)$ is a linear and positive. Then, in view of (2) it follows:

$$I(\delta,f;x)L(1;x)-L(w(x,\delta,f;y);x)\le L(f;x),$$

$$L(f;x) \leq S(\delta, f; x) L(1;x) + L(w(x,\delta, f; y); x).$$

or

$$\begin{aligned} L(f;x) &\leq S(\delta, f;x) + \sup\{L(w(x,\delta,f);x), x \in \Delta^m\} \\ &+M \sup\{\mathrm{mod}(1 - L(1;x)), x \in \Delta^m\}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} L(f;x) &\leq I(\delta, f;x) + \sup\{L(w(x,\delta,f);x), x \in \Delta^m\} \\ &-M \sup\{\mathrm{mod}(1 - L(1;x)), x \in \Delta^m\}. \end{aligned} \tag{7}$$

Finally, in view of Lemma 1, (3), (6) and (7) yield (1). Thus, the theorem is proved. ■

For $m = 1$ this statement is proved in [2].

Let $K = (k_1, k_2, ..., k_m) \in R^m$; q - natural number;

$$\sigma_{K,q} = \left\{x = (x_1, ..., x_m) \in \Delta^m; x_i \in \left[\frac{l}{6q}.k_i, \frac{l}{6q}.(k_i + 6)\right], \ i = 1, 2, ..., m\right\},$$

$k_1; k_2; ..., k_m = 0, 1, ..., 6(q-1)$ – cubes having side at most $\frac{l}{q}$; $\sum_{m,q}$ is a covering of Δ^m by the cubes of the type $\sigma_{K,q}$.

It is not difficult to see that the following statement is valid (see [1]).

Lemma 2. *Let $f, g \in G^M_{\Delta^m}$ be a function such that:*

$$\max_{\sigma_{K,q} \in \sum_{m,q}} \left| \sup_{x \in \sigma_{K,q}} f(x) - \sup_{x \in \sigma_{K,q}} g(x) \right| \leq \varepsilon;$$

$$\max_{\sigma_{K,q} \in \sum_{m,q}} \left| \inf_{x \in \sigma_{K,q}} f(x) - \inf_{x \in \sigma_{K,q}} g(x) \right| \leq \varepsilon.$$

Then for $\alpha > 0$ it holds:

$$r(\Delta^m, \alpha; f, g) \leq \max\left(\varepsilon, \frac{l}{\alpha q}\right).$$

Now we prove the following theorem.

Theorem 2. *Let $f \in G^M_{\Delta^m}, \delta > 0$. Then there exists a function $\phi(\cdot)$ such that:*

$$r(\Delta^m, \alpha; f, \phi) \leq \frac{l}{\alpha q}, (\alpha > 0, q - \text{natural number}), \tag{8}$$

$$r(\Delta^m, \alpha, \phi; \delta) \leq \frac{l}{\alpha}\delta, \tag{9}$$

$$w(\Delta^m, \phi; \delta) \leq 25 w(\Delta^m, f; \delta), \tag{10}$$

where $\frac{l}{24q} \le \delta \le \frac{l}{6q}$.

Proof. For every $\sigma_{K,q} \in \Delta^m_{[0,l]}$, $(k_1; k_2; ..., k_m = 0, 1, ..., 6(q-1))$ we denote

$$m_k = \min\{f(x) : x \in \sigma_{K,q}\}, \quad M_k = \min\{f(x) : x \in \sigma_{K,q}\},$$

and define $\phi(\cdot)$ as follows

$$\phi(x) = \begin{cases} m_K, x \in \widehat{\sigma}_{K,q}; \\ \widehat{\sigma}_{K,q} = \left\{ x \in \Delta^m : \left| x_i - \frac{l(k_i+3)}{6q} \right| \le \frac{l}{6q}, \quad i = 1, 2, ..., m \right. \\ M_K, x \in \sigma_{K,q} \backslash \widehat{\sigma}_{K,q}. \end{cases}$$

It is not difficult to see that from Lemma 2 follows that $\phi(\cdot)$ satisfies (8) and (9).

Further we have to prove (10). Let

$$d \ge \max\{\max|M_K - m_K|, ..., \max|M_K - M_{k_1}, ..., k_i+1, ..., k_m|,$$
$$..., \max|M_K - M_{k_1}, ..., k_i+1, , , ..., k_s+1, ..., k_m|, ...,$$
$$..., \max|M_K - M_{k_1+1}, ..., k_i+1, ..., k_s+1, ..., k_m+1|, \},$$

where $k_1; k_2; ..., k_m = 0, 1, ..., 6(q-1)$.

By the definition of modulus of continuity we get $d \le w\left(\Delta^m, f; \frac{l}{q}\right)$. Then, if $0 \le \delta \le l/6q$, for $\phi(\cdot)$ will be true $w(\Delta^m, \phi; \delta) \le d$. Hence the inequality is valid:

$$w(\Delta^m, \phi; \delta) \le w\left(\Delta^m, f; \frac{l}{q}\right). \tag{11}$$

But in [5], it is proved that

$$w(\Delta^m, f; \lambda\delta) \le (1 + [\lambda])\, w(\Delta^m, f; \delta), \tag{12}$$

where $[\lambda]$ is the most integer $\le \lambda$. Then, if $\frac{l}{24q} \le \delta \le \frac{l}{6q}$, from (11) and (12) we have:

$$w(\Delta^m, \phi; \delta) \le 25w(\Delta^m, f; \delta).$$

The theorem is proved. ■

For $m = 1$ this statement is proved in ([2], page 133).

Theorem 3. *Let* $f \in G^M_{\Delta^m}$ *be* 2π*-periodic and integrable function. Then for* $\delta > 0, \alpha > 0$ *it is true:*

$$r(\Delta^m, \alpha; T^m_n(f), f) \le \tau(\Delta^m, \alpha, f; 2\delta)$$

$$+2w\left(\Delta^m,,f;\delta\right)\cdot\sum_{i=1}^{m}\int_{\delta}^{\pi}\delta^{-1}x_iK_{s,p}\left(x_i\right)dx_i,$$

where s,p - are natural numbers, $sp\geq 2$;

$$T_n^m\left(f;y\right)=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}f\left(y+x\right)\prod_{i=1}^{m}K_{s,p}\left(x_i\right)dx_i;$$

$$K_{s,p}\left(\theta\right)=C_{s,p}\left(\frac{\sin\frac{s\theta}{2}}{s\sin\frac{\theta}{2}}\right)^{2p};\qquad\left(\theta\in R\right),$$

$C_{s,p}$ is a such a constant that $\int_{-\pi}^{\pi}K_{s,p}\left(\theta\right)d\theta=1$ is a trigonometric polynomial of a degree at most n, $n=msp$.

Proof. Using Theorem 1 for $T_n^m\left(\cdot\right)$ we have:

$$\begin{aligned}r\left(\Delta^m,\alpha;T_n^m\left(f\right),f\right)&\leq\tau\left(\Delta^m,\alpha,f;2\delta\right)\\&+\sup\left\{T_n^m\left(w\left(x,\delta,f\right)\right),\ x\in\Delta^m\right\}.\end{aligned}$$

Our purpose is to estimate the second term of the last inequality's sum. In view of the definition of $w\left(y,\delta,f;\cdot\right)$ and (12), we obtain:

$$\begin{aligned}T_n^m\left(w\left(y,\delta,f\right);y\right)&=2^m\int_0^{\pi}\cdots\int_0^{\pi}\left(w\left(y,\delta,f\right);x+y\right)\prod_{i=1}^{m}K_{s,p}\left(x_i\right)dx_i\\&=2^mh\left(\Delta^m,f;\delta,\delta,...,\delta\right)\left\{\sum_{k=1}^{m}\left[\int_0^{\delta}\cdots\int_{\delta}^{\pi}\cdots\int_0^{\delta}\delta^{-1}x_kdx_k\right]\right.\\&+\sum_{j_1,\dots,j_s=1}^{m}\left[\int_0^{\delta}\cdots\int_{\delta}^{\pi}\cdots\int_{\delta}^{\pi}\cdots\int_0^{\delta}\left(\delta^{-1}x_{j_1}+\cdots+\delta^{-1}x_{j_s}\right)\right]\\&\left.+\int_{\delta}^{\pi}\cdots\int_{\delta}^{\pi}\left[\sum_{i=1}^{m}\delta^{-1}x_i\right]\right\}\prod_{i=1}^{m}K_{s,p}\left(x_i\right)dx_i.\end{aligned}$$

Further we rearrange the addends and after some calculations we have

$$\begin{aligned}T_n^m\left(w\left(y,\delta,f\right);y\right)&=2^mw\left(\Delta^m,f;\delta\right)\times\sum_{k=1}^{m}\left[\int_{\delta}^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}\delta^{-1}x_k\prod_{i=1}^{m}K_{s,p}\left(x_i\right)dx_i\right]\\&\leq 2w\left(\Delta^m,f;\delta\right)\sum_{k=1}^{m}\left[\int_{\delta}^{\pi}\delta^{-1}x_kK_{s,p}\left(x_i\right)dx_i\right].\end{aligned}\tag{13}$$

Thus, the theorem is proved. ■

3. Main result

It is well known the classic Jackson's theorem of the best uniform approximation of continuous, 2π-periodic functions by trigonometric polynomials, which gives an estimate with the modulus of continuity of the function.

Sendov and Popov ([3]; [2], page 146) generalize this classical result. They prove

Theorem 4. ([3], [2]) *There exists an absolute constant $C > 0$ such that, for every $f \in A_{\Delta_{[0,2\pi]}}$ and $\alpha > 0$ it holds*

$$\inf\left\{r\left(\Delta_{[0,2\pi]}; t_n(f), f\right),\ t_n \in T_n\right\}$$

$$\leq Cw\left(\Delta_{[0,2\pi]}, f; n^{-1}\right)\frac{\ln\left(e + \alpha n w\left(\Delta_{[0,2\pi]}, f; n^{-1}\right)\right)}{1 + \alpha n w\left(\Delta_{[0,2\pi]}, f; n^{-1}\right)},$$

where T_n is the set of all trigonometric polynomials $t_n(\cdot)$.

Now we show that the following statement is true.

Lemma 3. *If $K = (k_1, k_2, ..., k_m)$, k_i–natural numbers, $n_0 = [n/m]$, $\beta = e + \alpha n w\left(\Delta^m_, f; n^{-1}\right)$, $q = \left[n_0/6e^{1/2}\pi \ln\beta\right] + 1$, then*

$$r\left(\Delta^m_, \alpha; T^m_n(\phi), \phi\right) \leq 8me^{1/2}\pi^2\frac{\ln\beta}{\alpha n} + 160me^2\pi^3\frac{1}{\alpha n},$$

holds, where

$$\phi(\xi) = \begin{cases} m_K, \xi \in \hat{\sigma}_{K,q}; \\ \hat{\sigma}_{K,q} = \left\{\xi \in \Delta^m_{[-\pi,\pi]} : \left|x_i - \dfrac{2\pi(k_i+3)}{6q}\right| \leq \dfrac{2\pi}{6q}, \qquad i = 1, 2, ..., m \right. \\ M_K, \xi \in \sigma_{K,q}\backslash\hat{\sigma}_{K,q} \end{cases}$$

P r o o f. Indeed in view of definition of $\phi(\cdot)$ and Theorem 3, we have

$$r(\Delta^m, \alpha; T^m_n(\phi), \phi) \leq \tau(\Delta^m, \alpha, \phi; 2\delta)$$

$$(14) \qquad + 2w(\Delta^m, \phi; \delta)\sum_{i=1}^m \int_\delta^\pi \frac{x_i}{\delta}K_{s,p}(x_i)\,dx_i.$$

Now we set:

$$(15) \qquad s = [n_0/\ln\beta]; p = [\ln\beta]; \delta = e^{1/2}\pi^2/(2s);$$

and talking into account the inequality ([2], page 72),

$$(16) \qquad \int_\delta^\pi \frac{x_i}{\delta}K_{s,p}(x_i)\,dx_i \leq \frac{\pi\left(\pi^2/(2s\delta)\right)^{2p-1}}{8(2p-1)},$$

we obtain:

$$\int_{\delta}^{\pi} \frac{x_i}{\delta} K_{s,p}(x_i)\,dx_i \le \frac{\pi e^{1/2-[\ln\beta]}}{8\ln\beta} \le \frac{\pi e^{3/2}}{8\beta\ln\beta} \tag{17}$$

Hence, for second term of the right part of (14), in view of (10), (15) and (17) we have:

$$\begin{aligned} 2w(\Delta^m,\phi;\delta) \cdot \sum_{i=1}^{m}\left[\int_{\delta}^{\pi}\frac{x_i}{\delta}K_{s,p}(x_i)\,dx_i\right] &\le 50mw(\Delta^m,\phi;\delta)\frac{\pi e^{3/2}}{8\beta\ln\beta} \\ &\le 50m\pi e^{3/2}(1+n\delta)\frac{w\left(\Delta^m_{[-\pi,\pi]},f;n^{-1}\right)}{8\beta\ln\beta} \\ &\le 50m\pi e^{3/2}\left(1+n\frac{e^{1/2}\pi^2\ln\beta}{n_0}\right)\frac{w(\Delta^m,f;n^{-1})}{8\ln\beta(e+\alpha nw(\Delta^m,f;n^{-1}))} \\ &\le \frac{50\pi^3e^2}{4\alpha n_0} \le \frac{50m\pi^3e^2}{\alpha n}. \end{aligned} \tag{18}$$

Further (9), (14) and (18) yield:

$$\begin{aligned} r\left(\Delta^m_{,}\alpha;T_n^m(\phi),\phi\right) &\le \frac{4\pi}{6\alpha q} + \frac{50m\pi^3e^2}{\alpha n} \\ &\le \frac{4\pi^2e^{1/2}\ln\beta}{\alpha n_0} + \frac{50m\pi^3e^2}{\alpha n} \\ &\le \frac{8\pi^2e^{1/2}\ln\beta}{\alpha n} + \frac{50m\pi^3e^2}{\alpha n}. \end{aligned} \tag{19}$$

The lemma is proved. ∎

Now we are ready to prove the main result.

Theorem 5. *There exists an absolute constant $C_0 > 0$ such that for every $f \in A_{\Delta^m}, \alpha > 0$ and sufficiently large n it holds*

$$\begin{aligned} &\inf\left\{r\left(\Delta_{[-\pi,\pi]},\alpha;T_n^m(f),f\right),T_n^m \in T_{m,n}\right\} \\ &\le C_0mw\left(\Delta^m,f;n^{-1}\right)\frac{\ln\left(e+\alpha nw\left(\Delta^m_{,}f;n^{-1}\right)\right)}{1+\alpha nw\left(\Delta^m_{,}f;n^{-1}\right)}, \end{aligned} \tag{20}$$

where T_n is the set of all trigonometric polynomials $t_n(\cdot)$ of m variables of a degree at most $n, n \ge 2m$.

Proof. Let $T_n^m(f;\cdot) : A_{\Delta^m} \to R$ be a linear positive operator, such that

$$T_n^m(f;y) = \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(x+y)\prod_{i=1}^{m}K_{s,p}(x_i)\,dx_i;$$

$$K_{s,p}(\theta) = C_{s,p}\left(\frac{\sin\frac{s\theta}{2}}{s\sin\frac{\theta}{2}}\right)^{2p}; \qquad (ps \geq 2),$$

$$\int_{-\pi}^{\pi} K_{s,p}(\theta)\,dx\theta = 1.$$

Let us note as well, that in view of Theorem 2 for every $f \in G^M_{[-\pi,\pi]}$ there exists a function $\Phi(\cdot)$, such that (8), (9) and (10) are true.

Further, using the properties of the Hausdorff distance, (8) and (19), we obtain:

$$\begin{aligned} r\left(\Delta_{[-\pi,\pi]}, \alpha; T_n^m(\phi), f\right) &\leq r(\Delta^m, \alpha; T_n^m(\phi), \phi) + r(\Delta^m, \alpha; \phi, f) \\ &\leq \frac{16\pi}{6\alpha q} + \frac{50m\pi^3 e^2}{\alpha n} \\ &\leq \frac{8m\pi^2 e^{1/2}\ln\beta}{\alpha n} + \frac{50m\pi^3 e^2}{\alpha n}. \end{aligned} \tag{21}$$

Hence for every $f \in A_{\Delta^m}$, there exists a trigonometric polynomial of m variables of a degree at most n such that

$$r(\Delta^m, \alpha; T_n^m(\phi), f) \leq C_1 m \frac{\ln(e + \alpha n w(\Delta^m, f; n^{-1}))}{2\alpha n} + C_2 m \frac{1}{\alpha n}$$

holds, where $C_1 = 16\pi^2 e^{1/2}$, $C_2 > 0$ is an absolute constant. It is evident, that if

$$w\left(\Delta^m, f; n^{-1}\right) \sim \frac{1}{\alpha n},$$

then there exists $C_0 > 0$ such that for sufficiently large n the statement (20) is true. If this is not true, then for sufficiently large n

$$2\alpha n w\left(\Delta^m, f; n^{-1}\right) > 1 + \alpha n w\left(\Delta^m, f; n^{-1}\right),$$

will be right, which proved (20). The theorem is proved. ■

This statement is proved for $m = 1$ in ([3]; [2], pp.148). Finally, using Lemma 1 we obtain the following statement.

Corollary 1. *There exists an absolute constant $C_0^* > 0$ such that for every $f \in A_{\Delta^m}$ and sufficiently large n it holds*

$$\inf\{R(\Delta^m, \alpha; T_n^m(f), f), T_n^m \in T_{m,n}\} \leq C_0^* m w\left(\Delta^m, f; n^{-1}\right),$$

where T_n is the set of all trigonometric polynomials $t_n(\cdot)$ of m variables of a degree at most $n, n \geq 2m$.

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