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Resolvability of g-Othogonality in Normed Spaces

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The orthogonality of two vectors in a normed space can be defined in numerous different ways. In the present paper we consider the g-orthogonality, defined as a particular case of the so-called g-angle between two vectors [8]. This orthogonality has many desirable properties. Thus, in smooth space this orthogonality is uniquely resolvable. In l^4-space it is equivalent to the Singer orthogonality. In such spaces the estimate (25) holds for the g-projection of vector y onto vector x. Amog these are some geometric properties of the parallelogram with verices at $0, x, y$ and $x + y$.

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Let X be a real normed space and $g : X^2 \to R$ the natural generatizotion of the inner product (.,.) [6][1] The functional g has the following properties:

(1) $$g(x, x) = \|x\|^2,$$

(2) $$g(\alpha x, \beta y) = \alpha\beta g(x, y) \quad (\alpha\beta \in R),$$

(3) $$g(x, x + y) = \|x\|^2 + g(x + y),$$

(4) $$|g(x, y)| = \|x\| \, \|y\|,$$

(5) $$(X, \|\cdot\|)$$

is an inner product space iff $g(x, y)$ is an innner product of vectors x and y, for all $x, y \in X$.

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[1]The functionsl g always exist on X^2 .

If the space X is the smooth (S), then the expression $[y,x] := g(x,y)$ is a semi-inner product in the sense of Lumer (see [4]). Then the functional g is linear in the second argument. The normed space in wich the functional g is linear in the second argument, is denoted by (G). (For instance in l^1 the functional g is linear in the second argument, although l^1 is not smooth).

In [8] the g-angle between the vectors x and y $(x \neq 0, y \neq 0)$ has been defined by

$$\angle(x,y)_g = \arccos \frac{g(x,y)+g(y,x)}{2\|x\|\|y\|} \tag{6}$$

To what extent this definition is natural, the following result shows [8]. $(X, \|\cdot\|)$ is an inner product space iff

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos \angle(x,y)_g \quad (x,y \in X \setminus \{0\}), \tag{7}$$

i.e. iff the cosine-theorem holds in X.

By use of functional g the orthogonality may be defined in several ways:

$$x \perp_g y \iff g(x,y) = 0,$$

$$x \perp_g ?y \iff g(x,y) = g(y,x) = 0,$$
$$x \perp^g y \iff g(x,y)g(y,x) = 0,$$
$$x^g \perp y \iff \|x\|^2 g(x,y) + \|y\|^2 g(y,x) = 0 \quad ([9]).$$

Using the g-angle, in the present paper we define the following orthogonality relation:

$$x \overset{g}{\perp} y \iff \angle(x,y)_g = \pi/2, \tag{8}$$

or equivalently

$$x \overset{g}{\perp} y \iff g(x,y) + g(y,x) = 0$$

All these orthogonalities, defined by g, will be called g-orthogonalities.

We remark that $\overset{g}{\perp} \cap \perp^g = \overset{\perp}{g}$.

We mention yet certain well-known orthogonalities and their denotation:

$$x \perp_B y \iff (\forall t \in R)\ \|x\| \leq \|x + tx\| \quad \text{(Birkhoff orthogonality)},$$

$$x \perp_P y \iff \|x+y\|^2 = \|x\|^2 + \|y\|^2 \quad \text{(Pythagorean orthogonality)},$$
$$x \perp_J y \iff \|x-y\| = \|x+y\| \quad \text{(James issceles orthogonality)},$$

$$x \perp_S y \iff \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| = \|\frac{x}{\|x\|} + \frac{y}{\|y\|}\| \quad \text{(Singer orthogonality)}.$$

For an orthogonality relation $\perp$ some desirable properties are given below (they holds for the orthogonality in Euclidean space):

(9) $$x \perp y \Rightarrow x = 0 \quad (\text{ nondegeneracy }),$$

(10) $$x \perp \Rightarrow y \perp x \quad (\text{ symmetry }),$$

(11) $$x \perp \rightarrow \alpha x \perp \beta y \, (\alpha\beta \neq 0) \quad (\text{ homogenity }),$$

(12) $$x \perp y \wedge x \perp z \Rightarrow x \perp y + z \quad (\text{ right additivity }),$$

(12′) $$y \perp x \wedge x \perp z \Rightarrow y + z \perp x \quad (\text{ left additivity }),$$

(13) $$(\forall x, y \in X)(\exists a \in R) x \perp ax + y \quad (\text{ right resolvability })$$

(13′) $$(\forall y, x \in X)(\exists a \in R) \; ax + y \perp x \quad (\text{ left resolvability })$$

(14)
$$(\exists\, C > 0)(\forall x, y \in X) x \perp y \Rightarrow \|x\| \not< C\|x + y\| \quad (\text{ uniform boundedness }),$$

(15) If $x_n \to x, y_n \to y (n \to \infty)$ and $x_n \perp y_n$ for all $n \in N$,

then $x \perp y$ (continuity).

It is easy to see that the orthogonality $\perp_g$ has the properties (9), (11), (13) and (14) in an arbitrary normed space. The next theorem describes some other desirable properties of the orthogonality $\perp_g$.

Theorem 1. *For the orthogonality $\perp_g$ the following statements hold:*
1^0 *X is an inner product space iff X is smooth and the relation $\perp_g$ is symmetric.*
2^0 *X has the property (G) iff if the relation $\perp_g$ right additive.*
3^0 *Let X has the property (G). Then the expression $g(x, y)$ is an inner product iff the relation $\perp_g$ is left additive.*
4^0 *If X is smooth then the relation $\perp_g$ is continuous.*

Proof. 1^0 If X is smooth then the orthogonality $\perp_g$ is equivalent to the orthogonality $\perp_b$ (Theorem 2 in [7]). The rest of the proof follows from the

Assertion 18.7 in [2].
2^0 This is Assertion e) of Theorem 1 in [11].
3^0 This is Assertion b) of Theorem 2 in [11].
4^0 The proof follows from the Lemma 3 in [10].

It follows immediately from the definition of the functional g that the orthogonality $\overset{g}{\perp}$ has the properties (9), (10) and (11). The following theorem has been proved by Lemma 2 and Lemma 3 in [10]. ■

Theorem 2. 1^0 *The orthogonality* $\overset{g}{\perp}$ *is uniformly bounded.*
2^0 *It X is smooth, then* $\overset{g}{\perp}$ *is continuous.*

In this paper we make a further study of the orthogonality $\overset{g}{\perp}$ and $\overset{\perp}{g}$ in conection with the property (13).

For the proof of our main result we shall need the following auxiliary results.

Lemma 1. *If X is smooth, then the orthogonality* $\overset{g}{\perp}$ *is resolvable, i.e.*

$$\|x\|\|y\| \neq 0 \Rightarrow (\exists a \in R)\, x \overset{g}{\perp} ax + y.$$

Proof. For fixed vectors $x, y \in X \setminus \{0\}$ we consider the function f defined as

$$f(x) = g(x, tx + y) + g(tx + y, x) \quad (t \in R).$$

Let $t \neq 0$. From (1),(2) and (3) we have

$$f(t) = t\|x\|^2 + g(x, y) + t\|x + y/t\|^2 - g(x + y/t, y)\ (t \neq 0). \tag{16}$$

Since X is smooth, the expression $[y, x] = g(x, y)$ is a semi-inner product. Thus, the function $t \mapsto g(tx + y, x)$ is continuous and

$$\lim_{t \to \pm\infty} g(x + y/t, y) = g(x, y) \quad (see[4]).$$

Hence, from (16) the function f is continuous on $(-\infty, +\infty)$ and we have

$$\lim_{t \to \pm\infty} f(x) = \pm\infty$$

Therefore there exsts $a \in R$ such that $f(a) = 0$, i.e. $x \overset{g}{\perp} ax + y$. ■

Lemma 2. *Let* $u, v \in S(X)$[1] *and* $u \overset{g}{\perp} v$. *If there exists* $t > 0$ *such that* $\|tu + v\| \leq 1$, *then* $g(u, v) \geq 0$.

[1] $S(X) := \{x \in X \| \|x\| = 1\}$

Proof. Assume $t > 0$ and $\|tu + v\| \leq 1$. Then we have

$$\|v\|\|tu + v\| \leq 1 \Rightarrow g(v, tu + v) \leq 1 = tg(v, u) \leq 0.$$

Combining $g(u, v) + g(v, u) = 0$ and $tg(v, u) \leq 0$ we obtain $g(u, v) \geq 0$. ■

Lemma 3. *Let X be smooth and $u, v \in S(X)$. Then for every $t > 0$*

$$g(v, u) \leq \frac{\|tu + v\| - 1}{t} \leq \frac{g(tu + v, u)}{\|tu + v\|}. \tag{17}$$

Proof. In [3] p.21, the next assertion has been proved: If a mapping $x \mapsto f_x$ of $X \setminus \{0\}$ to $X^* \setminus \{0\}$[2] satisfyng the conditions:

$$f_u(u) = \|f_u\| = 1 \text{ for } u \in S(X), \tag{i}$$

$$f_{tu} = tf_u \text{ for } t \geq 0 \text{ and } u \in S(X); \tag{ii}$$

then

$$\frac{f_v(u)}{\|v\|} \leq \frac{\|v + tu\| - \|v\|}{t} \leq \frac{f_{v+tu}(u)}{\|nv + tu\|} \quad (t > 0, v \in S(X)). \tag{18}$$

In a smooth space, for every $x \in X \setminus \{0\}, g(x, .) \in X^*$ and the functional $f_u = g(, .)$, for $u \in S(X)$ satisfies the conditions (i) and (ii). Thus from (18) we get (17). ■

Lemma 4. *Let X be smooth and $u \overset{g}{\perp} v$ $(u, v \in S(X)$. Then there is no t, $t \neq 0$, such that $u \overset{g}{\perp} tu + v$.*

Proof. Putting $f(t) = g(u, tu + v) + g(tu + v, u)$ we have $f(0) = 0$ and

$$f(t) = t + g(u, v) + (tu + v, u) \quad (t \neq 0). \tag{19}$$

First, we shall prove that $f(t) > 0$ for $t > 0$. Let $t > 0$. If $\|tu + v\| \leq 1$, then combining (18) and (19) we obtain

$$f(t) \geq t + g(u, v) + \|tu + v\|g(v, u)$$

Therefore, from $g(u, v) + g(v, u) = 0$ it follows that

$$f(t) \geq t + (1 - \|tu + v\|)g(u, v).$$

[2] X^* is the topological dual of X.

Thus, by Lemma 2, we get $f(t) > 0$. Now suppose that $\|tu + v\| \geq 1$. Then, by (17) we get

$$f(t) \geq t + g(u,v) + \frac{\|tu + v\| - 1}{t}\|tu + v\|. \tag{20}$$

If $g(u,v) \geq 0$, we obtain $f(t) > 0$ from (20). If $g(u,v) < 0$ then $g(v,u) > 0$ and from (17) we get Therefore from (20) we have

$$f(t) \geq t + (\|tu + v\| - 1)g(v,u).$$

Thus $f(t) > 0$ for $t > 0$.

We now consider the case $t < 0$. By (11) we have $(-u)\overset{g}{\perp}v$. By the above proof, we conclude that

$$0 < g(-tu + v, -u) + g(-u, -tu + v) =$$

$$= -[g(u, -tu + v) + g(-tu + v, u)] = -f(-t),\ t > 0$$

So, $f(-t) < 0$ for $t > 0$, that is $f(t) < 0$ for all $t < 0$. ■

Theorem 3. *Let X be a smooth space and $\|x\|\|y\| \neq 0$. Then there exists a unique $a \in R$ such that $x \overset{g}{\perp} ax + y$.*

Proof. The assertion of this theorem is equivalent with the assertion: If $\|x\|\|y\| \neq 0$ and $x\overset{g}{\perp}y$, then x is not orthogonal, in the sense of orthogonality $\overset{g}{\perp}$, on $tx + y$ for all $t \neq 0$, i.e. $g(x, tx + y) + g(tx + y, x) \neq 0$ for $t \neq 0$, shorterly $x\overset{g}{\perp}ax + y$ for $t \neq 0$.

The existence of a number a is shown in Lemma 1. We prove that the number a is unique. Let $x\overset{g}{\perp}y$ and $x \in S(X)$. Then $x\overset{g}{\perp}y/\|y\|$. By Lemma 4 we have $x\overset{g}{\perp}(t/\|y\|)x + y/\|y\|$ for each $t \neq 0$, or equivalently

$$(\forall t \neq 0)g(x, \frac{tx}{\|y\|} + \frac{y}{\|y\|}) + g(\frac{tx}{\|y\|} + \frac{y}{\|y\|}, x) \neq 0 \iff$$

$$(\forall t \neq 0)g(x, tx + y) + g(tx + y, x) \neq 0 \iff (\forall t \neq 0)x\overset{g}{\perp}tx + y.$$

For any $x \neq 0$ we have $(x/\|x\|)\overset{g}{\perp}(y/\|x\|)$. By the above argument we get $(x/\|x\|)\overset{g}{\perp}(tx/\|x\|) + (y/\|x\|)$ for each $t \neq 0$, i.e.

$$(\forall t \neq 0)g(x, \frac{tx}{\|x\|} + \frac{y}{\|x\|}) + g(\frac{tx}{\|x\|} + \frac{y}{\|x\|}, x \neq 0) \iff$$

$$(\forall t \neq 0) g(x, tx + y) + g(tx + y, x) \neq 0 \iff (\forall t \neq 0) x \overset{g}{\perp} tx + y.$$

The relation $\overset{\perp}{g}$ is a subrelation of the relation $\overset{g}{\perp}$. However, the orthogonality $\overset{\perp}{g}$ is not resolvable in all smooth spaces. ■

Theorem 4. *X is an inner product space iff X is smooth and the orthogonality $\overset{\perp}{g}$ is resolvable in X.*

Proof. If $(.,.)$ is an inner product on X^2, then X is smooth and, for $x \neq 0$, we have $(x, ax + y) = (ax + y, x) = 0 \iff a = (x, y)/\|x\|^2$. This proves the resolvability of the orthogonality in $(X(.,.))$. Conversely, if X is smooth and the orthogonality $\overset{\perp}{g}$ is resolvable, then for all $u, v \in S(X)$ there exists $a \in R$ such that

$$g(u, au + v) + g(au + v, u) = 0.$$

From the equality $g(u, au + v) = 0$ we obtain $a = -g(u, v)$. Then by equality $g(au + v, u) = 0$, we obtain

$$g(-g(u, v)u + v, u) = 0.$$

By Theorem 2 [7], the last equality can be written as $-g(u, v)u + v \perp_B u \iff (\forall t \in R)\|v - g(u, v)u\| \leq \|v - g(u, v)u + tu\|$. Therefore, for $t = g(u, v)$, we get

$$\|v - g(u, v)u\| \leq 1 \quad (u, v \in S(X)). \tag{21}$$

Since X is smooth, the condition (21) characteizes the existence of an inner product on X^2 (see the assertion 18.17 in [2]).

We shall pay now attention to some geometric problems in normed spaces. Namely, many geometrical statements concerning orthogonality which hold in an product space, do not hold in an arbitrary normed space for a cetrain type of orthogonality. So, the implication

$$u \perp_b v \Rightarrow (u + v) \perp_b (u - v) \quad (u, v \in S(X)),$$

(i.e. The diagonals of a quadrangle are perpendicular) holds only in an inner product space (the assertion 10.1 in [2]). Simillary, the implication

$$x \perp y \Rightarrow \|x + y\| = \|x - y\| \quad (x, y \in X \setminus \{0\}) \tag{22}$$

(i.e. The lengths of the diagonals of a rectangle are equal) do not hold in an arbitrary normed space for arbitrary orthogonality. For the James orthogonality (22) holds, but for the Singer orthogonality (22) holds only if X is an inner product space. Therefore we consider the question: Is there a normed space in which (22) holds for orthogonality $\overset{g}{\perp}$? ■

Lemma 5. *In l^4-space the following equalities hold:*

$$(23)\qquad \|x+y\|^4 - \|x-y\|^4 = 8(\|x\|^2 g(x,y) + \|y\|^2 g(y,x)) \quad (x, y \in l^4),$$

$$(24)\ \left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^4 - \left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^4 = 8\|x\|\|y\|(g(x,y)+g(y,x))(x, y \in l^4\setminus\{0\}).$$

Proof. Using the definition of the functional g for l^p-space ($p \geq 1$) we get

$$g(x,y) = \|x\|^{2-p}\sum_k |x_k|^{p-1} sgn x_k y_k \quad (x = (x_1, x_2, \ldots) \in l^p \setminus \{0\}).$$

Therefore, for $p = 4$ we have

$$g(x,y) = \|x\|^{-2}\sum_k x_k^3 y_k.$$

Immediate consequences of the last equality are the identities (23) and (24). ■

Corollary 1. *In l^4-space we have:*

$$x^g \perp y \iff x \perp_I y,$$

$$x \overset{g}{\perp} y \iff x \perp_\varrho y.$$

Corollary 2. *There is a normed space, which is not necesserily an inner product space, in which the following assertions hold:*
1^0 *The lengths of the diagonals of a parallelogram are equal iff it is a rectangle in the diagonals of the orthogonality $\overset{g}{\perp}$.*
2^0 *The lengths of the diagonals of a rhomb are equal iff it is a rectangle in a rectangle in the sense of the orthogonality $\overset{g}{\perp}$.*

Combining cetain results from [1] and [5] for the Singer orthogonality we can derive the following.

Theorem 5. *Let X be a normed space with the propery*

$$(\forall x, y \in X \setminus \{0\}) x \overset{g}{\perp} y \iff x \perp_\varrho y.$$

Then the following properties hold:

1^0 *The orthogonality* $\overset{g}{\perp}$ *is uniquely resolvable, i.e. for all* $x, y \in X \setminus \{0\}$ *there exists a unique* $a \in R$ *such that* $x \overset{g}{\perp} ax + y$.

2^0 *Let* $y_x = ax$ *with* a *from* 1^0. *Then the following estimate holds*

$$\|y_x\| \leq \frac{1+\sqrt{2}}{2}\|y\| \tag{25}$$

If $\|y_x\| \leq \|y\|$ *for all* $x, y \in X \setminus \{0\}$ *then* X *is an inner product space.*

3^0 *The diagonals of a quadrangle, in the sense of the orthogonality* $\overset{g}{\perp}$, *are perpendicular in the sense of the orthogonality* $\overset{g}{\perp}$.

Proof. 1^0 See [5]. 2^0 See [1]. 3^0 Let $u \overset{g}{\perp} v$. Putting $u + v = x$ and $u - v = y$ we get $\|x + y\| = 2$. Hence

$$\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \iff x \perp_S y \iff x \overset{g}{\perp} y.$$

■

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