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# Complete Systems of Gegenbauer Polynomials in Spaces of Holomorphic Functions

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In the paper sufficient conditions are given for a sequence of Gegenbauer polynomials to be complete in suitable spaces of holomorphic functions.

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## 1. Introduction

**1.1** Let $B$ be a nonempty open set in the extended complex plane $\overline{C} = C\bigcup\{\infty\}$ and let $H(B)$ be the $C$-vector space of the complex functions holomorphic in $B$. As usually we consider $H(B)$ with the topolgy of uniform convergence on the compact subset of $B$.

A subset of $H(B)$ is said to be complete in $H(B)$ if its closed linear span is the whole space $H(B)$.

Let $B^* = \overline{C}\setminus B$ be nonempty and let $E$ be a closed subset of $B^*$ having a point of common with every (connected) component of $B^*$. Then the classical Runge's theorem sais that the set of all rational functions with poles in $E$ is complete in $H(B)$. In particular if $B \subset C$ does not separate $\overline{C}$ i.e. if $B^*$ is connected, then the set of all polynomials is complete in $H(B)$. Indeed in the latter case we can choose $E = \{\infty\}$.

**1.2** Let $\gamma \subset C$ be a rectifiable Jordan curve and let $C_\gamma$ be the closure of that component of $\overline{C}\setminus\gamma$ which contains the point $\infty$. Denote by $H_\gamma$ the $C$-vector space of the complex functions everyone of which is holomorphic in an open set containing $C_\gamma$ and vanishes at the point $\infty$.

There is a criterion for completeness of a sequence $\{h_n(z)\}_{n=0}^{\infty} \subset H(G)$ provided that $G \subset C$ is a simply connected region namely [1, p. 211, Theorem 17]:

(CC)*Let $G \subset C$ be a simply connected region. Then a sequence $\{h_n(z)\}_{n=0}^{\infty} \subset H(G)$ is complete in $H(G)$ iff whatever the rectifiable Jordan curve $\gamma \subset G$ be, the only function $F \in H_\gamma$ which is orthogonal to everyone of the functions $\{h_n(z)\}_{n=0}^{\infty}$ is identically zero i.e. the equalities*

$$\int_\gamma F(z)h_n(z)\,dz = 0, n = 0,1,2,\ldots$$

*imply* $F \equiv 0$.

## 2. Gegenbauer polynomials and associated functions

**2.1** Gegenbauer polynomials $\left\{P_n^{(\lambda)}(z)\right\}_{n=0}^{\infty}$ (called also ultraspherical) when $2\lambda \neq 0, -1, -2, \ldots$ are defined by the equalities

$$P_n^{(\lambda)}(z) = \frac{\Gamma(\lambda+1/2)\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+1/2)} P_n^{(\lambda-1/2,\lambda-1/2)}(z), n = 0,1,2,\ldots,$$

where $\left\{P_n^{(\alpha,\beta)}(z)\right\}_{n=0}^{\infty}$ are the Jacobi polynomials with parameters $\alpha, \beta$. By definition $P_0^{(0)}(z) \equiv 1$ and $P_n^{(0)}(z) = \lim_{\lambda\to 0} \lambda^{-1} P_n^{(\lambda)}(z)$ when $n = 0,1,2,\ldots$ i.e.

$$P_n^{(0)}(z) = \frac{2\Gamma(1/2)\Gamma(n)}{\Gamma(n+1/2)} P_n^{(-1/2,-1/2)}(z), n = 1,2,3,\ldots.$$

The above definition of Gegenbauer polynomials can be found in [2, 4.7] as well as in [3, 10.9]. Another way to define these polynomials is by means of a generating function. Namely, if $2\lambda \neq 0, -1, -2, \ldots$ then for every $z \in C$ and $w \in C$ we have

$$\sum_{n=0}^{\infty} \frac{\Gamma(2\lambda)P_n^{(\lambda)}(z)w^n}{2^{\lambda-1/2}\Gamma(\lambda+1/2)\Gamma(n+2\lambda)} \tag{2.1}$$

$$= ((1-z^2)^{1/2}w)^{1/2-\lambda} J_{\lambda-1/2}((1-z^2)^{1/2}) \exp(zw),$$

where $J_\alpha$ is the Bessel function of the first kind with index $\alpha$ [2, (4.10.6)].

**2.2** Let $\Re\lambda > -1$, then the functions $\left\{Q_n^{(\lambda)}(z)\right\}_{n=0}^{\infty}$ defined in the region $\overline{C} \setminus [-1,1]$ by means of the equalities

$$Q_n^{(\lambda)}(z) = -\int_{-1}^{1} \frac{(1-t^2)^\lambda P_n^{(\lambda)}(t)}{t-z}\,dt, n = 0,1,2,\ldots$$

are called Gegenbauer associated functions.

Let $\omega$ be that inverse of Žukowski function $z = (w + w^{-1})/2$ for which $\omega(\infty) = \infty$. For every $r \in (1, \infty)$, $e(r) := \{z \in C \setminus [-1, 1] : |\omega(z)| = r\}$ is an ellipse with focuses at the points $-1, 1$. We denote by $E(r)$ the interior of $e(r)$ and define $E^*(r) = \overline{C} \setminus \overline{E(r)}$. By definition $E(1)=\emptyset$and $E^*(1) = \overline{C} \setminus [-1, 1]$.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that

$$1 \leq R = \max\{1, \limsup_{n\to\infty} |a_n|^{1/n}\} < \infty. \tag{2.2}$$

Then the series

$$\sum_{n=0}^{\infty} a_n Q_n^{(\lambda)}(z) \tag{2.3}$$

is absolutely uniformly convregent on every closed subset of $E^*(R)$ and diverges in $E(R) \setminus [-1, 1]$. Further if $F$ is a complex function holomorphic in the region $E*(R)(1 \leq R < \infty)$ and $F(\infty) = 0$, then it has there an expansion in a series of the kind (2.3) i.e.

$$F(z) = \sum_{n=0}^{\infty} a_n^{(\lambda)}(F) Q_n^{(\lambda)}(z). \tag{2.4}$$

Moreover whatever $r \in (R, \infty)$ be, for the coefficients of the series in (2.4) we have the representations

$$a_n^{(\lambda)}(F) = (2\pi i K_n^{(\lambda)})^{-1} \int_{e(r)} F(z) P_n^{(\lambda)}(z)\, dz, n = 0, 1, 2, \ldots, \tag{2.5}$$

where

$$K_n^{(\lambda)} = \int_{-1}^{1} (1 - t^2)^{\lambda} \left\{P_n^{(\lambda)}(t)\right\}^2 dt, n = 0, 1, 2, \ldots.$$

## 3. The main result

**3.1** We define $a(z, t) = z + t(1 - z^2)^{1/2}$ provided that $z \in C \setminus [-1, 1]$ and $t \in [-1, 1]$. Further if $0 \leq \eta < 1$ and $\theta \in R$, then a region $G \subset C \setminus [-1, 1]$ is called $(\eta, \theta)$-admissible when for every $z \in G$ and every $t \in [-1, 1]$ the following inequality holds namely

$$|\arg(a(z, t) \exp i\theta)| < (1 - \eta)\frac{\pi}{2} \tag{3.1}$$

**Remark.** It is clear that if a region $G \subset C \setminus [-1, 1]$ is $(\eta, \theta)$-admissible then every its subregion has the same property.

Let us note that $(\eta, \theta)$-admissible regions really exist. Let $\rho > 1$ and $A(\eta, \theta, \rho)$ be the region defined by $|\arg(z \exp i\theta)| < (1-\eta)\pi/2$ and $|z| > \rho$. Since $\lim_{z\to\infty} z^{-1}a(z,t) = 1 + t$ uniformly when $t \in (-1, 1]$, for every $\epsilon \in (0, 1-\eta)$ there exists $\rho_0 = \rho_0(\epsilon) > 1$ such that $A(\eta + \epsilon, \theta, \rho)$ is $(\eta, \theta)$-admissible region provided that $\rho > \rho_0$.

**3.2** For an increasing sequence $k = \{k_n\}_{n=0}^{\infty}$ of nonnegative integers is said that it has density $\delta(k)$ if there exists $\delta(k) = \lim_{n\to\infty}(n/k_n)$.

**Theorem.** *Let $0 \le \eta < 1$, $\theta \in R$, $\Re\lambda > 0$ and let $k = \{k_n\}_{n=0}^{\infty}$ be an increasing sequence of nonnegative integers with density $\delta(k) > (1-\eta)/2$. Then the system $\left\{P_{k_n}^{(\lambda)}(z)\right\}_{n=0}^{\infty}$ is complete in every space $H(G)$ provided that $G \subset C \setminus [-1, 1]$ is a simply connected $(\eta, \theta)$-admissible region.*

Proof. We define the system of polynomials $\left\{\tilde{P}_n^{(\lambda)}(z)\right\}_{n=0}^{\infty}$ by the equalities

$$\tilde{P}_n^{(\lambda)}(z) = \frac{\Gamma(\lambda)\Gamma(2\lambda)\pi^{1/2}(-1)^n}{\Gamma(\lambda+1/2)\Gamma(n+2\lambda)} P_n^{(\lambda)}(z), n = 0, 1, 2, \ldots \tag{3.2}$$

provided that $2\lambda \ne 0, -1, -2, \ldots$.

Under the assumption $\Re\lambda > 0$ we define the function $P^{(\lambda)}(z, w)$ for $z \in C \setminus [-1, 1]$ and $w \in C$ by

$$P^{(\lambda)}(z, w) = \int_{-1}^{1} (1-t^2)^{\lambda-1} \exp(-a(z,t)w)\, dt. \tag{3.3}$$

Then as a corollary of (2.3), (3.2) and the Poisson integral representation of the function $J_\alpha$ [3, 7.12, (7)] we find that

$$\sum_{n=0}^{\infty} \tilde{P}_n^{(\lambda)}(z) w^n = P^{(\lambda)}(z, w). \tag{3.4}$$

Suppose now the statement we wish to prove is not true. By the completeness criterion (CC) that means there exist a $(\eta, \theta)$-admissible simply connected region $G \subset C \setminus [-1, 1]$, a Jordan rectifiable positively oriented curve $\gamma \subset G$ and a function $F \in H_\gamma$ which is not identically zero but

$$\int_\gamma P_{k_n}^{(\lambda)}(z) F(z)\, dz = 0, n = 0, 1, 2, \ldots$$

In view of the equalities (3.2) that means that

$$\int_\gamma \tilde{P}_{k_n}^{(\lambda)}(z) F(z)\, dz = 0, n = 0, 1, 2, \ldots. \tag{3.5}$$

We define the function $f$ by

$$f(w) = \int_\gamma P^{(\lambda)}(z,w)F(z)\,dz. \tag{3.6}$$

It is clear that $f$ is holomorphic in the whole complex plane i.e. it is an entire function. We will see that its Taylor expansion centered at the origin is

$$f(w) = \sum_{n=0}^{\infty} A_n^{(\lambda)}(F)w^n, \tag{3.7}$$

where

$$A_n^{(\lambda)}(F) = \int_\gamma \tilde{P}_n^{(\lambda)}(z)F(z), n = 0,1,2,\ldots. \tag{3.8}$$

Indeed, (3.3), (3.4) and (3.6) give that for every $n = 0,1,2,\ldots$,

$$A_n^{(\lambda)}(F) = (n!)^{-1}f^{(n)}(0)$$

$$= (n!)^{-1}(-1)^n \int_\gamma \left\{\int_{-1}^{1}(1-t^2)^{\lambda-1}(z+t(z^2-1)^{1/2})^n\,dt\right\}F(z)\,dz.$$

Since $\Gamma(2\lambda) = 2^{2\lambda-1}\pi^{-1/2}\Gamma(\lambda)\Gamma(\lambda+1/2)$ [3, 1.2, (15)], the valididty of the equalities (3.8) is a corollary of the integral representation [3, 10.9, (31)] of Gegenbauer polynomils as well as of the equalities (3.2).

Let us note that $f$ is not identically zero. Otherwise we would have $A_n^{(\lambda)}(F) = 0$ for every $n = 0,1,2,\ldots$ and since the system $\{P_n^{(\lambda)}(z)\}_{n=0}^{\infty}$ is complete in any space $H(G)$ with $G$ simply connected, the completeness criterion (CC) would imply $F \equiv 0$.

We define $R_0 = \inf\{R : F \in E^*(R)\}$. It is clear that $R_0 < \infty$. The assumption $R_0 = 1$ leads to the conclusion that the function $F$ is holomorphic in $\overline{C}\setminus[-1,1]$. Since it is holomorphic in a neighbourhood of the segment $[-1,1]$, it follows that $F$ is holomorphic on $\overline{C}$ i.e. $F$ is a constant function. But $F(\infty) = 0$ i.e. $F \equiv 0$ which contradicts the assumption we made. Thus we have proved that $1 < R_0 < \infty$.

For the function $F$ we have the representation (2.4) with coefficients given by (2.5). If we assume that the ellipse $e(r)$ is positively oriented, then we can assert that for every $n = 0,1,2,\ldots$,

$$\int_\gamma P_n^{(\lambda)}(z)F(z)\,dz = \int_{e(r)} P_n^{(\lambda)}(z)F(z)\,dz.$$

Therefore in viev of (3.2) and (3.8) we will have that

$$A_n^{(\lambda)}(F) = \frac{2\pi^2 i}{(n+\lambda)\Gamma(n+1)} a_n^{(\lambda)}(F), n = 0, 1, 2, \ldots \tag{3.9}$$

The set of $n's$ for which $|a_n^{(\lambda)}(F)| > 1$ is not finite. Otherwise we would have $\limsup_{n\to\infty} |a_n^{(\lambda)}(F)| \leq 1$ and in view of (3.5) the series in (3.7) would be convergent in the region $E^*(1)$ i.e. the function $F$ would be holomorphic in $E^*(1)$ which as we have already seen is not possible.

Now we are going to prove that the entire function (3.6) is of order one and type $R_0$. Let $p = \{p_n\}_{n=0}^{\infty}$ be the nonnegative integers for which $|a_{p_n}^{(\lambda)}(F)| > 1$ and let $q$ be the complementary sequence of $p$ with respect to the set of all nonnegative integers.

Since $R_0 < \infty$, the sequence $\{|a_n^{(\lambda)}(F)|^{1/n}\}_{n=0}^{\infty}$ is bounded and in particular we have $1 < |a_{p_n}^{(\lambda)}(F)|^{1/p_n} \leq B < \infty (n = 0, 1, 2, \ldots)$. Therefore

$$\lim_{n\to\infty} (p_n \log p_n)^{-1} \log |a_{p_n}^{(\lambda)}(F)| = 0$$

and then from (3.9) it follows that

$$\lim_{n\to\infty} p_n \log p_n (\log(|A_{p_n}^{\lambda}(F)|^{-1}))^{-1} = 1.$$

If $q$ is finite, then [1, Chapter I: (1.05), (1.06)] give that $f$ is of order one and type $R_0$.

Suppose $q$ is not finite and let $q = \{q_n\}_{n=0}^{\infty}$. Then we have

$$\frac{q_n \log q_n}{\log(|A_{q_n}^{(\lambda)}(F)|^{-1})} \leq \frac{q_n \log q_n}{\log\Gamma(q_n+1) + \log|q_n + \lambda| + \log(2\pi)}.$$

Therefore

$$\limsup_{n\to\infty} q_n \log q_n (\log(|A_{q_n}^{(\lambda)}(F)|^{-1}))^{-1} \leq 1$$

and again [1, Chapter I: (1.05), (1.06)] yield that $f$ is of order one and type $R_0$.

Let us note that $a(z,t)$ is a continuous and nonvanishing function of the variables $z \in C \setminus [-1,1]$ and $t \in [-1,1]$. Since $\gamma \subset G$ is a compact set we can assert that the real number $\mu$ defined by

$$\mu = \min\{|a(z,t)| : z \in \gamma, t \in [-1,1]\} \tag{3.10}$$

is positive.

Further since $G$ is $(\eta, \theta)$-admissible there exists $\sigma \in (0, 1-\eta)$ such that the inequality

$$|\arg(a(z,t)\exp i\theta)| \leq (1-\eta-\sigma)\frac{\pi}{2} \tag{3.11}$$

holds whenever $z \in \gamma$ and $t \in [-1.1]$.

Let $W(\eta, \theta)$ be the angular domain defined by $|\arg(w\exp(-i\theta)| < \eta\pi/2$. Then from (3.10) and (3.11) it follows that $\Re(a(z,t)w) \geq -\mu\sin(\sigma\pi/2)|w|$ provided that $z \in \gamma$, $t \in [-1,1]$ and $w \in \overline{W(\eta,\theta)}$ and as a corollary of (3.3) and (3.6) we obtain

$$|f(w)| \leq L(\lambda)\exp(-\mu\sin(\sigma\pi/2)|w|) \tag{3.12}$$

with

$$L(\lambda) = M(F)l(\gamma)\int_{-1}^{1}(1-t^2)^{\Re\lambda-1}\,dt,$$

where $M(F) = \max\{|F(z)| : z \in \gamma\}$ and $l(\gamma)$ is the lenght of $\gamma$.

That means $f$ is of order one and nonpositive type on the closed angular domain $\overline{W(\eta,\theta)}$.

Let $k^*$ be the complementary sequence of $k = \{k_n\}_{n=0}^{\infty}$ with respect to the nonnegative integers. Since it is supposed that $A_{k_n}(F) = 0$ for every $n = 0, 1, 2, \ldots$, $k^*$ is not finite. Otherwise $f$ would be a nonzero polynomial which because of (3.12) is not possible.

Further we denote by $\tilde{k}$ the sequence of indices of the nonzero coefficients in the Taylor expansion (3.7). The assumption that $\tilde{k}$ is finite leads again to a contradiction.

Let $\Delta(\tilde{k})$ be the maximal density of the sequence $\tilde{k}$ [4]. Then by a theorem of G. Pólya [4, p. 626, Satz VII], $\Delta(\tilde{k}) \geq (1+\eta)/2$.

It is not difficult to prove that the sequence $k^*$ also has density and moreover $\delta(k^*) = 1 - \delta(k)$. Since $\tilde{k}$ is a subsequence of $k^*$, we have $\Delta(\tilde{k}) \leq \delta(\tilde{k})$ [5, Note I, 2.] and therefore $\Delta(\tilde{k}) \leq 1 - \delta(k) < (1+\eta)/2$. We come to a contradiction and thus our statement is proved.

## 4. Comments

Let $\Gamma_1 \subset C$ be a continuous curve joining the points 0 and $\infty$ and such that every circle centered at the origin intersects $\Gamma_1$ in a single point. Suppose that $0 < \tau \leq 1$ and define $\Gamma_2 = \exp(2\pi i)\Gamma_1$. It is clear that $\Gamma_1$ and $\Gamma_2$ have no point in common except 0 and $\infty$. Let further $D$ be the region "between" $\Gamma_1$ and $\Gamma_2$ and let call it curvilinear angular domain of angle $2\pi\tau$.

In [6] it is proved that if the sequence $\{k_n\}_{n=0}^{\infty}$ has density $\tau$, then the system of Jacobi polynomials $\left\{P_{k_n}^{(\alpha,\beta)}(z)\right\}_{n=0}^{\infty}$ is complete in every space $H(G)$ with $G \subset C \setminus [-1,1]$ provided $\omega(G)$ is the part of a curvilinear domain of angle $2\pi\tau$ lying outside of the unit disk $U(0;1)$. In view of (2.1) the same is true for the system $\left\{P_{k_n}^{(\lambda)}(z)\right\}_{n=0}^{\infty}$.

As it was mentioned above the angular domain $A(\eta+\epsilon,\theta,\rho)$ is $(\eta,\theta)$-admissible when $0<\epsilon<1-\eta$ and $\rho>1$ is large enough. Since any point of the unit circle is not a boundary point of the region $A(\eta+\epsilon,\theta,\rho)$, it can't be of the kind $D \setminus D \bigcap \overline{U(0;1)}$.

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