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Grid Approximation of Parabolic Equations with Small Parameters Multiplying the Space and Time Derivatives. Reaction-Diffusion Equations [1]

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In this paper we consider the Dirichlet problems on a strip and on a rectangle for singularly perturbed parabolic equations which contain no convective terms. The time derivative and/or the highest space derivatives of the equation are multiplied by parameters taking arbitrary values in the half-interval (0,1]. As the parameters tend to zero, the solutions of these problems have boundary and initial layers. The error in the numerical solution by classical finite difference schemes depends on the perturbation parameters and can be comparable with the exact solution. It is shown that for boundary value problems with initial layers, as well as for problems with parabolic boundary layers, there exist no difference schemes from the natural class of fitted operator methods that converge uniformly with respect to the parameters. Using a condensing mesh technique, we construct special difference schemes which approximate the boundary value problems uniformly in the parameters.

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1. Introduction

Boundary value problems for singularly perturbed parabolic equations, i.e. for equations with small parameters multiplying the highest-order space derivatives and/or the time derivatives, arise, for example, in the analysis of heat and mass transfer when the duration of the process and the coefficients of heat conductivity (diffusion) are small or large. For small values of the parameters in the equation (or some of them) boundary and/or initial layers appear. For such problems the error in the approximate solutions by classical finite difference schemes (on uniform grids) is comparable with the solution of the boundary

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value problem (see Theorem 1), just as for equations with a small parameter multiplying the highest space derivatives (see, e.g. [2, 4, 8, 11]).

To construct special difference schemes which converge uniformly with respect to the perturbation parameters, one can generally use a fitted operator method (see the description of this method, e.g. in [2, 4]) and a condensing mesh method (see its description, e.g., in [1, 11]). However, for parabolic equations with a small parameter ε multiplying the highest space derivatives, whose solutions have a parabolic boundary layer (i.e., a layer described by a parabolic equation), there does not exist a scheme of fitted operator type that converges ε-uniformly in a neighbourhood of the boundary layer [8, 11, 12, 13].

In the present paper we consider the Dirichlet problems for singularly perturbed linear parabolic equations on a strip and on a rectangle. The coefficients of the equation involve several parameters taking arbitrary values in the half-interval (0,1]. As these parameters (or one of them) tend to zero, boundary and initial layers appear. The issues are shown that arise when one attempts to solve such problems using classical difference approximations or fitted operator methods. Thus, an initial layer appears as the parameter multiplying the time derivative tends to zero. For this problem the solution of a classical difference scheme does not converge to the solution of the Dirichlet problem uniformly with respect to the perturbation parameter; in the natural class of fitted operator methods there exist no schemes convergent uniformly in the parameter (see Theorem 2).

To obtain uniform convergence in the discrete maximum norm, we use a piecewise uniform grid with nodes that are condensed by a special way in the neighbourhood of the boundary and initial layers. Quasilinear equations are discussed in Section 7.

Satisfactory grid approximations for parabolic equations with a small parameter only multiplying the highest-order space derivatives were considered, e.g. in [11, 12, 14, 15].

2. Problem formulation

2.1. On the strip $\overline{D}$, where

$$D = \{x : 0 < x_1 < d_1,\ x_2 \in \mathrm{R}\}, \tag{2.1a}$$

we consider the linear parabolic equation

$$L_{(2.2)}u(x,t) = F(x,t,\varepsilon), \quad (x,t) \in G, \tag{2.2a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2b}$$

Here

(2.1b) $$G = D \times (0,T], \quad S = S(G) = \overline{G} \setminus G,$$

$$L_{(2.2)} \equiv \varepsilon_1^2 \sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2} - \varepsilon_2^2\, p(x,t)\frac{\partial}{\partial t} - \varepsilon_3^2\, c(x,t).$$

The parameters ε_1, ε_2 and the parameter ε_3, i.e., the components of the vector-parameter ε (or, briefly, of the parameter ε), take any values from the half-interval $(0,1]$ and the segment $[0,1]$, respectively. The coefficients $a_s(x,t)$, $c(x,t)$, $p(x,t)$ and the right-hand side $F(x,t,\varepsilon)$ are sufficiently smooth on $\overline{G}$ for a fixed value of the parameter ε, moreover,

(2.3a) $$0 < a_0 \le a_s(x,t) \le a^0, \quad c(x,t) \ge 0, \quad 0 < p_0 \le p(x,t) \le p^0, \quad (x,t) \in \overline{G},$$

the boundary function $\varphi(x,t)$ is sufficiently smooth on the sets $S^L = \Gamma \times (0,T]$ and $S_0 = \overline{D} \times \{t = 0\}$, and it is continuous on S; $\Gamma = \overline{D} \setminus D$, $S = S^L \bigcup S_0$,

The solution of problem (2.2) is regarded as a function $u \in C(\overline{G}) \bigcap C^{2,1}(G)$ which satisfies the differential equation on G and the boundary condition on S.

The notation $L_{(j.k)}$ ($m_{(j.k)}$, $f_{(j.k)}(x,t)$) means that this operator (or constant, function) is first introduced in formula $(j.k)$. By M, M_i (m, m_i) we denote sufficiently large (small) positive constants which are independent of the parameters ε_s and the discretization parameters. We say that the discrete solution converges ε-uniformly if it converges with respect to each of the parameters ε_i, $i = 1, 2, 3$,.

2.2. We shall also consider the boundary value problem

(2.4) $$\begin{aligned} L_{(2.2)}u(x,t) &= F(x,t,\varepsilon), \quad (x,t) \in G, \\ u(x,t) &= \varphi(x,t), \quad (x,t) \in S \end{aligned}$$

on the rectangle

(2.5a) $$D = \{x : \ 0 < x_s < d_s, \ s = 1,2\}.$$

Here

(2.5b) $$G = D \times (0,T], \quad S = \overline{G} \setminus G.$$

The coefficients of the equation and the function $F(x,t,\varepsilon)$ satisfy the same conditions as in (2.2), the function $\varphi(x,t)$ is continuous on S and sufficiently smooth on each of the sides and on the base of G.

2.3. Let us discuss a few conditions imposed on the functions $F(x,t,\varepsilon)$ and $c(x,t)$. For simplicity we suppose that

$$c(x,t) \geq c_0 > 0, \quad (x,t) \in \overline{G}, \tag{2.3b}$$

if $\varepsilon_3 \geq M(\varepsilon_1 + \varepsilon_2)$.

The a-priori estimates for the solutions of the boundary value problems (see Sections 4, 6) provide the ε-uniform boundedness of these solutions if the function $F(x,t,\varepsilon)$ satisfies the condition: $|F(x,t,\varepsilon)| \leq M(\varepsilon_1+\varepsilon_2+\varepsilon_3)^2$, $(x,t) \in \overline{G}$. We assume that the function $F(x,t,\varepsilon)$ takes the form

$$F(x,t,\varepsilon) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 f(x,t), \quad (x,t) \in \overline{G} \tag{2.3c}$$

where $f(x,t)$ is a sufficiently smooth bounded function. Then the differential equation can be written in the form

$$L_{(2.6)}u(x,t) \equiv \{\tilde{\varepsilon}_1^2 \sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2} - \tilde{\varepsilon}_2^2\, p(x,t)\frac{\partial}{\partial t} - \tag{2.6}$$

$$- \tilde{\varepsilon}_3^2\, c(x,t)\,\}\, u(x,t) = f(x,t), \quad (x,t) \in G$$

where $\tilde{\varepsilon}_i = \varepsilon_i(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^{-1}$, $i = 1,2,3$; $\tilde{\varepsilon}_i \in (0,1]$, and also $\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 = 1$.

2.4. As the parameter ε_1 (or ε_2) tends to zero under the condition $\varepsilon_1 = o(\varepsilon_2 + \varepsilon_3)$ (or $\varepsilon_2 = o(\varepsilon_1 + \varepsilon_3)$), boundary (either initial or inner) layers appear. Let the two parameters ε_1, ε_2 tend to zero. Then both boundary and initial layers arise provided that $\varepsilon_1 + \varepsilon_2 = o(\varepsilon_3)$. If $\varepsilon_3 \leq M(\varepsilon_1 + \varepsilon_2)$ then either only boundary layers (for $\varepsilon_1 = o(\varepsilon_2)$) or only initial ones (for $\varepsilon_2 = o(\varepsilon_1)$) appear; under the condition $m\varepsilon_1 \leq \varepsilon_2 \leq M\varepsilon_1$ there are no boundary and initial layers. The boundary and initial layers are described by equations of parabolic type.

In Section 3 we show difficulties arising if we solve problem (2.2), (2.1) by means of a classical difference scheme. The error in the discrete solution depends on the parameters ε_i and can be comparable with the exact solution for small ε_1, ε_2. In connection with this we come to the need for the construction of a special difference scheme whose solution converges ε-uniformly. As is well known, for parabolic equations with a small parameter multiplying the highest space derivatives, in the presence of parabolic boundary layers, the fitted operator method is inapplicable to construct these special schemes [11, 12, 13]. In Section 3 it is shown that the fitted operator method is also inapplicable to construct an ε-uniformly convergent scheme in the case if $\varepsilon_2 \in (0,1]$, $\varepsilon_1 = 1$, $\varepsilon_3 = 0$.

In Sections 5, 6, for problems (2.2), (2.1) and (2.4), (2.5) we construct finite difference schemes that converge ε-uniformly. For this we use the condensing mesh method. Necessary a-priori estimates for the solutions are considered in

Sections 4, 6.

3. Formulation of the objective for research

3.1. Now we discuss issues arising in the numerical solution of problem (2.2), (2.1). If the data in (2.2), (2.1) are independent of x_2, we come to the Dirichlet problem in one space dimension. If

$$\varepsilon_1 \in (0,1], \quad \varepsilon_2 = 1, \quad \varepsilon_3 = 0, \tag{3.1}$$

the solution of this problem on a segment has a boundary layer for small values of ε_1, moreover, the solution of a classical difference scheme does not converge to the solution of the boundary value problem ε_1-uniformly (see, e.g. [11, 14]).

Lemma 1. *Suppose that we use classical finite difference schemes for the solution of boundary value problem (2.2), (2.1). Then, under condition (3.1), the approximate solution does not converge ε_1-uniformly to the exact one.*

3.2. We will consider the boundary value problem on a segment provided that

$$\varepsilon_2 \in (0,1], \quad \varepsilon_1 = 1, \quad \varepsilon_3 = 0. \tag{3.2}$$

Then the solution has an initial layer for small values of ε_2. To solve such a model problem

$$L_{(3.3)} u(x,t) \equiv \left\{ \frac{\partial^2}{\partial x^2} - \varepsilon_2^2 \frac{\partial}{\partial t} \right\} u(x,t) = 0, \quad (x,t) \in G, \tag{3.3}$$

$$u(x,t) = \sin(\pi x), \quad (x,t) \in S,$$

where

$$G = D \times (0,T], \quad \overline{D} = [0,d], \tag{3.4}$$

we use the classical finite difference scheme [9]

$$\Lambda_{(3.5)} z(x,t) \equiv \{\delta_{x\overline{x}} - \varepsilon_2^2 \delta_{\overline{t}}\} z(x,t) = 0, \quad (x,t) \in G_h, \tag{3.5}$$

$$z(x,t) = \sin(\pi x), \quad (x,t) \in S_h.$$

Here

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad G_h = G \cap \overline{G}_h, \quad S_h = S \cap \overline{G}_h, \tag{3.6}$$

$\overline{D}_h$ and $\overline{\omega}_0$ are uniform grids on the segments $\overline{D}$ and $[0,T]$ with step-sizes $h = dN^{-1}$ and $h_0 = TN_0^{-1}$, respectively, $N+1$ and N_0+1 are the number of nodes in the grids $\overline{D}_h$ and $\overline{\omega}_0$; $\delta_{x\overline{x}}z(x,t)$ and $\delta_{\overline{t}}z(x,t)$ are the second and the first (backward) difference derivatives.

The comparison of the explicit solutions for problems (3.3), (3.4) and (3.5), (3.6) implies: $\max_{\overline{G}_h} |u_{(3.3)}(x,t) - z_{(3.5)}(x,t)| \geq m$ for $h,\ h_0 \to 0$, $\varepsilon_2 = \varepsilon_2(h_0) = h_0^{1/2}$, that is, the solution of the classical difference scheme under condition (3.2) does not converge ε_2-uniformly. The following assertions hold.

Lemma 2. *Let classical finite difference schemes be used for the solution of boundary value problem (2.2), (2.1). Then, under condition (3.2), the approximate solution does not converge ε_2-uniformly to the exact one.*

Theorem 1. *Let classical finite difference schemes be used for the solution of boundary value problem (2.2), (2.1). Then the approximate solution does not converge to the exact one ε-uniformly; the schemes do not converge ε_1-uniformly under condition (3.1) and ε_2-uniformly under condition (3.2).*

Thus, in the case of problem (2.2), (2.1) it is our concern to develop special difference schemes that are convergent ε-uniformly. A similar problem arises in the case of problem (2.4), (2.5) as well.

3.3. In the area of ε-uniform methods fitted operator methods are more attractive and have been sufficiently widely developed. These methods give an opportunity to obtain a numerical solution on very simple meshes (e.g. uniform meshes). Bit it turned out that for problems with parabolic boundary layers there is no fitted operator method on a uniform mesh that gives satisfactory numerical solutions.

In [11, 12, 13] it was shown that, in the case of the Dirichlet problem on a segment under condition (3.1), no fitted scheme can be constructed on four-point stencils of implicit finite difference schemes, for which the solutions do converge ε_1-uniformly.

3.4. Let us show issues arising in the construction of fitted operator schemes on a uniform mesh under condition (3.2).

We are interested in grid approximations of the Dirichlet problem for the singularly perturbed heat equation

$$L_{(3.7)}u(x,t) \equiv \left\{ \frac{\partial^2}{\partial x^2} - \varepsilon_0^2 \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \quad (x,t) \in G, \tag{3.7}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S.$$

Here $f(x,t)$, $(x,t) \in \overline{G}$, $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth functions,

$\overline{G} = \overline{G}_{(3.4)}$, $\varepsilon_0 \in (0,1]$. As $\varepsilon_0 \to 0$, an initial layer appears in the neighbourhood of the set S_0.

Let $\overline{G}_h = \overline{\omega} \times \overline{\omega}_0$ be a rectangular grid on $\overline{G}$, were $\overline{\omega}$ and $\overline{\omega}_0$ are, generally speaking, nonuniform grids on $\overline{D}$ and $[0,T]$. By $N+1$ and N_0+1 we denote respectively the number of nodes in the grids $\overline{\omega}$ and $\overline{\omega}_0$. By h, h_0 we denote the maximal step-size of $\overline{\omega}$ and $\overline{\omega}_0$, $h \le MN^{-1}$, $h_0 \le MN_0^{-1}$. Let $z(x,t)$, $(x,t) \in \overline{G}_h$ be the solution of some finite difference scheme on the grid set $\overline{G}_h$. We say that the solution of this scheme converges ε_0-uniformly if the function $z(x,t)$ satisfies: $\max_{\overline{G}_h} |u(x,t) - z(x,t)| \le \lambda(N^{-1}, N_0^{-1})$, where $\lambda(N^{-1}, N_0^{-1}) \to 0$ uniformly in ε_0 as N, $N_0 \to \infty$.

We describe a class of finite difference scheme (the class **A**) defined by sufficiently natural conditions, on which we try to construct a fitted scheme for such a particular problem

$$L_{(3.7)}u(x,t) = 0, \quad (x,t) \in G, \tag{3.8a}$$

$$u(x,t) = \varphi_0(x), \quad (x,t) \in S, \tag{3.8b}$$

where $\varphi_0(x) = \varphi_{(3.7)}(x,0)$, and also $\varphi_0(0) = \varphi_0(1) = 0$. The solution of this problem is an initial layer function $W(x,t)$.

On the uniform grid $\overline{G}_{h(3.6)}$ we construct the difference scheme on a four-point stencil of implicit finite difference schemes

$$\Lambda_{(3.9)}z(x,t) \equiv \{\delta_{x\overline{x}} - P\delta_{\overline{t}}\}\, z(x,t) = 0, \quad (x,t) \in G_h, \tag{3.9}$$

where the coefficient P is a functional of the coefficients of Eq. (3.8a) and depends on x, t, h, h_t, ε_0; $h_t = h_0$.

From the variables x, t we pass to the stretched variables x, τ; $\tau = \tau(t, \varepsilon_0) = \varepsilon_0^{-2}t$. The singularly perturbed equation (3.8a) is transformed into the regular equation

$$L^0_{(3.10)}u^0(x,\tau) \equiv \left\{\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau}\right\} u^0(x,\tau) = 0, \quad (x,\tau) \in G_\tau; \tag{3.10a}$$

the sets G^0_τ in the variables x, τ correspond to the sets $G^0 \subseteq \overline{G}$. We denote $v(x, t(\tau)) = v^0(x,\tau)$. On the boundary S_τ the function $u^0(x,\tau)$ takes the given values

$$u^0(x,\tau) = \varphi_0(x), \quad (x,\tau) \in S_\tau. \tag{3.10b}$$

The relation $u(x,t) = u^0(x, \varepsilon_0^{-2}t)$ is true where $u(x,t)$ is the solution of problem (3.8).

On the grid $G_{h\tau}$, $G_h = G_{h(3.6)}$, Eq. (3.9) takes the form

$$\Lambda_{(3.11)} z^0(x,\tau) \equiv \{\delta_{x\bar{x}} - \gamma^0 \delta_{\bar{\tau}}\} z^0(x,\tau) = 0, \quad (x,\tau) \in G_{h\tau} \tag{3.11}$$

where $z^0(x,\tau) = z(x, \varepsilon_0^2 \tau)$, $\gamma^0(x,\tau,h,h_\tau,\varepsilon_0) = \varepsilon_0^{-2} P(x, \varepsilon_0^2 \tau, h, \varepsilon_0^2 h_\tau, \varepsilon_0)$, $h_\tau = \varepsilon_0^{-2} h_t$.

Note that the parameter ε_0 does not enter into the formulation of problem (3.10). The grid sets $G_{h\tau}$ and $S_{h\tau}$ are also independent of ε_0; these sets are defined only by the steps h, h_τ. Because problem (3.10), and hence its solution, and also the grid $\overline{G}_{h\tau}$ do depend on the parameter ε_0, therefore it is natural to look for the coefficients of the difference equation (3.11), corresponding to differential equation (3.10a), in the form independent of ε_0 as well:

$$\Lambda^0_{(3.12)} z^0(x,\tau) \equiv \{\delta_{x\bar{x}} - \gamma^0(x,\tau,h,h_\tau) \delta_{\bar{\tau}}\} z^0(x,\tau) = 0, \quad (x,\tau) \in G_{h\tau}. \tag{3.12}$$

Equation (3.9) on the grid $G_{h(3.6)}$, being written in the variables x, τ, and Eq. (3.12) on the grid $G_{h\tau}$ are equivalent.

The condition of pointwise approximation, on a smooth function, of the operator $L^0_{(3.10)}$ by the operator $\Lambda^0_{(3.12)}$ [9] results in the relation

$$|\gamma^0(x,\tau,h,h_\tau) - 1| \le \mu(h, h_\tau, x, \tau) \tag{3.13a}$$

where $\mu(h, h_\tau, x, \tau) \to 0$ at a point $(x,\tau) \in G_{h\tau}$ as $h, h_\tau \to 0$, $\mu(h, h_\tau, x, \tau)$ is independent of the parameter ε_0.

We say that the operator $\Lambda^0_{(3.12)}$ approximates the operator $L^0_{(3.10)}$ uniformly on the set $G^*_\tau \subset \overline{G}_\tau$, if $\mu(h, h_\tau, x, \tau)$ is independent of x, τ for $(x,\tau) \in G^*_\tau$, that is,

$$\mu(h, h_\tau, x, \tau) = \lambda(h, h_\tau) \quad \text{for} \quad (x,\tau) \in G^*_\tau \cap G_{h\tau}. \tag{3.13b}$$

We assume that (3.13) is fulfilled, where G^*_τ belongs to an m-neighbourhood of the set $S_{0\tau}$. In this class **A** we will construct fitted schemes. The following theorem is valid.

Theorem 2. *In the class of finite difference schemes* **A** *there exist no difference scheme the solution of which converges to the solution of boundary value problem (3.8)* ε_0*-uniformly for* h, $h_t \to 0$.

Thus, in the case of problem (2.2), (2.1) there are no fitted schemes in the natural classes of finite difference schemes (neither under condition (3.1) nor under condition (3.2)) that converge ε-uniformly. So we come to such a theoretical problem:

Construct a condensing rectangular mesh and a standard finite difference operator that give an ε-uniformly convergent numerical method.

The same problem appears in the case of problem (2.4), (2.5).

3.5. The p r o o f of Theorem 2 is performed by contradiction according to the plan of proving the non-existence of ε_1-uniform fitted schemes in the case of condition (3.1) (cf. [11, 12, 13]). Assume that on the grid $\overline{G}_{h(3.6)}$ there exists a finite difference scheme which is convergent ε_0-uniformly. Let us investigate this scheme.

Condition (3.13) implies such a property of the function $\gamma^0(x,\tau,h,h_\tau)$ (we shall name it by the property (*)). Let $(x_0,\tau_0) \in G_\tau$ be some point from the neighbourhood of G^*_τ, and also the set $\overline{G}_{0\tau} = [x_0-\delta, x_0+\delta] \times [0,\tau_0]$, $\delta > 0$ belong to $\overline{G}^*_\tau$. For any sufficiently small $\delta_0 > 0$ we can find an $\delta^0 = \delta^0(\delta_0)$ – neighbourhood of a point $(x_0, 0)$, belonging to $\overline{G}_{0\tau}$ (namely, the set $G^0_\tau = (x^0_1, x^0_2) \times (0,\tau^0)$, where $x^0_1 = x_0 - \delta^0$, $x^0_2 = x_0 + \delta^0$, $\tau^0 = \delta^0$, $\overline{G}^0_\tau \subseteq \overline{G}_{0\tau}$), such that for any $(x_1,\tau), (x_2,\tau) \in G^0_\tau$ and any h, $h_\tau \le m_1$, $m_1 = m_1(\delta_0)$, we have

$$|\gamma^0(x_1,\tau,h,h_\tau) - 1| \le m_2, \tag{3.14}$$
$$|\gamma^0(x_1,\tau,h,h_\tau) - \gamma^0(x_2,\tau,h,h_\tau)| \le \delta_0, \quad (x_1,\tau),\ (x_2,\tau) \in G^0_\tau.$$

The property (*) ensures the validity of the maximum principle for the Dirichlet problem on the set $\overline{G}^0_{h\tau} = \overline{G}^0_\tau \cap \overline{G}_{h\tau}$ in the case of Eqs. (3.9) (or (3.12)).

To prove Theorem 2 we estimate the functions $\omega^i(x,t) = u^i(x,t) - z^i(x,t)$, $i = 1,3$, where $u^i(x,t) = \Phi^i_0(x,t)$ is the solution of problem (3.8), $z^i(x,t)$ is the corresponding solution of the difference scheme; $\Phi^i_0(x,t) = \Phi^i(x,\varepsilon_0^{-2}t)$ are auxiliary functions. The functions $\Phi^i(x,\tau)$ are defined by: $\Phi^i(x,\tau) = \sin(ik\pi x)\exp(-i^2k^2\pi^2\tau)$, $i = 1,3$, where k is an integer and $\sin(k\pi x_0) = 1$. Let $\delta^0 \le 12^{-1}k^{-1}$.

The functions $\Phi^i(x,\tau)$ satisfy the differential equation

$$\left\{\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau}\right\}\Phi^i(x,\tau) = 0, \quad (x,\tau) \in G_\tau$$

and such conditions on the boundary S_τ: $\Phi^i(0,\tau) = \Phi^i(1,\tau) = 0$, $\Phi^i(x,0) = \sin(ik\pi x)$. We define auxiliary "fitting coefficients" $\gamma^1(x,\tau,h,h_\tau)$, $\gamma^3(x,\tau,h,h_\tau)$ and the function $\gamma^*(x,\tau,h,h_\tau)$, i.e., the mean value of these coefficients, by

$$(\delta_{x\overline{x}} - \gamma^i\delta_{\overline{\tau}})\,\Phi^i(x,\tau) = 0, \quad i = 1,3,$$
$$\gamma^*(x,\tau,h,h_\tau) = 2^{-1}(\gamma^1(x,\tau,h,h_\tau) + \gamma^3(x,\tau,h,h_\tau)), \quad (x,\tau) \in G_{h\tau}.$$

The functions $\Phi^i(x,\tau)$ and the coefficients $\gamma^j(x,\tau,h,h_\tau)$ obey the relations

$$(3.15)\ \delta_{x\bar{x}}\Phi^j(x,\tau) = \frac{\partial^2}{\partial x^2}\Phi^j(x,\tau)+\frac{1}{4!}\frac{\partial^4}{\partial x^4}\Phi^j(x_1,\tau)h^2+\frac{1}{4!}\frac{\partial^4}{\partial x^4}\Phi^j(x_2,\tau)h^2,$$
$$x-h\le x_1,\ x_2\le x+h,$$
$$\delta_{\bar{\tau}}\Phi^j(x,\tau) = \frac{\partial}{\partial\tau}\Phi^j(x,\tau)-\frac{1}{2}\frac{\partial^2}{\partial\tau^2}\Phi^j(x,\tau)h_\tau+\frac{1}{6}\frac{\partial^3}{\partial\tau^3}\Phi^j(x,\tau_1)h_\tau^2,$$
$$\tau-h_\tau\le\tau_1\le\tau,$$

$$|\,\delta_{\bar{\tau}}\Phi^j(x,\tau)+j^2k^2\pi^2\sin(jk\pi x)\exp(-j^2k^2\pi^2\tau)[1+2^{-1}j^2k^2h_\tau]\,|\le Mh_\tau^2,$$
$$|\,\delta_{x\bar{x}}\Phi^j(x,\tau)+j^2k^2\pi^2\sin(jk\pi x)\exp(-j^2k^2\pi^2\tau)\,|\le Mh^2,$$
$$|\,\gamma^i(x,\tau,h,h_\tau)-[1-2^{-1}i^2k^2\pi^2h_\tau]\,|\le M_1[h^2+h_\tau^2],$$
$$|\,\gamma^1(x,\tau,h,h_\tau)-\gamma^3(x,\tau,h,h_\tau)-4\pi^2k^2h_\tau\,|\le M_2[h^2+h_\tau^2],$$
$$|\,x-x_0\,|\le\delta^0,\quad \tau\le\delta^0,\quad i,j=1,3.$$

The values of γ^1 and γ^3 differ from each other by a positive quantity of the order of the value $\beta_{(3.15)}=4\pi^2k^2h_\tau$ for arbitrarily small values of h and h_τ. The constants h_1 and $h_{\tau1}$, $h_1=h_1(h_{\tau1})$, $h_{\tau1}=h_{\tau1}(x_0)$, are chosen sufficiently small so that

$$(3.16)\qquad M_3\,[\,h_1^2+h_{\tau1}^2\,]\le 8^{-1}m_{(3.16)},$$

where $m_{(3.16)}=4\pi^2k^2h_{\tau1}$, $M_3=2(1+T^2)(M_{1(3.15)}+M_{2(3.15)})$.

The value δ^0 is chosen sufficiently small such that $\delta_0\le 8^{-1}m_{(3.16)}$, and, besides this, (3.14) and also the following inequality are true

$$|\,\gamma^i(x_1,\tau,h,h_t)-\gamma^i(x_2,\tau,h,h_t)\,|\le\delta_0,\quad (x_1,\tau),\ (x_2,\tau)\in\overline{G}_\tau^0,\quad i=1,3.$$

The quantity $h_\tau=h_{\tau2}$ is chosen to satisfy the condition $h_{\tau2}\le h_{\tau1}=4^{-1}\pi^{-2}k^{-2}m_{(3}$ $h_{\tau2}\le\delta^0$, and further we keep it fixed.

We construct the set G_τ^0

$$(3.17)\qquad G_\tau^0=\{(x,\tau):\ |\,x-x_0\,|<\delta^0,\ 0<\tau\le h_{\tau2}\},$$

on which we shall analyze the fitted scheme. Note that $h_{\tau2}$ and δ^0 are independent of the parameter ε_0.

For $x=x_0$, $\tau=h_{\tau2}$ at least one of the following inequalities is valid:

$$(3.18a)\qquad \gamma^0(x_0,h_{\tau2},h,h_{\tau2})\ge\gamma^*(x_0,h_{\tau2},h,h_{\tau2}),$$

$$(3.18b)\qquad \gamma^0(x_0,h_{\tau2},h,h_{\tau2})\le\gamma^*(x_0,h_{\tau2},h,h_{\tau2}).$$

Assume that (3.18a) is realized. Then

(3.19) $\gamma^0(x, h_{\tau 2}, h, h_{\tau 2}) \geq \gamma^3(x, h_{\tau 2}, h, h_{\tau 2}) + 8^{-1} m_{(3.19)}, \quad (x,\tau) \in G^0_{h\tau},$

where $G^0_\tau = G^0_{\tau(3.17)}$, $m_{(3.19)} = 4\pi^2 k^2 h_{\tau 2}$. In this case the function $\omega^{03}(x,\tau) = \omega^3(x, t(\tau))$ satisfies the relation

(3.20a)

$$\Lambda^0_{(3.12)}\omega^{03}(x,\tau) = (\gamma^3(x,\tau,h,h_{\tau 2}) - \gamma^0(x,\tau,h,h_{\tau 2}))\delta_{\overline{\tau}}\Phi^3(x,\tau) \geq m^1_{(3.20)},$$

$$(x,\tau) \in G^0_{h\tau}.$$

In the case of (3.18b) for the function $\omega^{01}(x,\tau) = \omega^1(x, t(\tau))$ we have

(3.20b) $\Lambda^0_{(3.12)}\omega^{01}(x,\tau) \leq -m^2_{(3.20)}, \quad (x,\tau) \in G^0_{h\tau}.$

The finite difference scheme is assumed to converge ε_0-uniformly:

(3.21) $|\omega^i(x,t)| \leq \lambda(h, h_t), \quad (x,t) \in \overline{G}_h, \quad \varepsilon_0 \in (0,1], \quad i = 1,3.$

Then the following inequality is satisfied on the set $S^0_{h\tau}$, $S^0 = \overline{G}^0 \setminus G^0$:

(3.22) $|\omega^{0i}(x,\tau)| \leq \lambda(h, \varepsilon_0^2 h_{\tau 2}), \quad (x,\tau) \in S^0_{h\tau}, \ i = 1,3$

where $\lambda(h, \varepsilon_0^2 h_{\tau 2}) \to 0$ for ε_0, $h \to 0$, $h_{\tau 2} = \text{const}$. Taking into account (3.20), (3.22) and the maximum principle, we show that, at least for one $i = j$, the following bound is valid:

(3.23) $|u^{0j}(x_0, h_{\tau 2}) - z^{0j}(x_0, h_{\tau 2})| \geq m$ *for* $h \leq h_1$, $\varepsilon_0 \in (0, \varepsilon_0^1]$,

where h_1, ε_0^1 are sufficiently small numbers. It follows from (3.23) that

(3.24) $\max_{\overline{G}_h} |u^j(x,t) - z^j(x,t)| \geq m$

for any $h \leq m_{1(3.14)}$, $h_t \leq \varepsilon_0^2 m_{1(3.14)}$, $h_t \leq (\varepsilon_0^1)^2 h_{\tau 2}$, $\varepsilon_0^1 = \varepsilon^1_{0(3.23)}$, $h \leq h_{1(3.23)}$; $h_t = \varepsilon_0^2 h_{\tau 2}$. Inequality (3.24) contradicts (3.21) for $\varepsilon_0 \in (0, \varepsilon_0^1]$. This concludes the proof of Theorem 2.

4. A-priori estimates for the solution of problem (2.2), (2.1)

In this section we estimate the solution and its derivatives for problem (2.2), (2.1). For this we use the technique like that in [3, 5, 7] and [8, 11]. Using the comparison theorems we find

(4.1) $|u(x,t)| \leq M[(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-2} \max_{\overline{G}} |F(x,t,\varepsilon)| + \max_S |\varphi(x,t)|], \quad (x,t) \in \overline{G}.$

Estimate (4.1) is sharp with respect to the entering parameters ε_i, $i = 1, 2, 3$.

The data of the boundary value problem are assumed to satisfy the condition

$$(4.2a)\qquad \begin{gathered} a_s,\ c,\ p,\ f\ \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}),\\ \varphi \in C^{l+2+\alpha}(S_0) \cap C^{l+2+\alpha,(l+2+\alpha)/2}(\overline{S}^L) \cap C(S),\\ s = 1, 2,\ l \geq K,\ \alpha > 0,\ K \geq 2, \end{gathered}$$

moreover, on the set $\gamma_0 = \overline{S}^L \cap S_0$ the compatibility conditions [7] are fulfilled that ensure the inclusion

$$(4.2b)\qquad u \in C^{l+2+\alpha,(l+2+\alpha)/2}(\overline{G})$$

for each fixed set of the parameters ε_i. A series of additional assumptions is given below.

4.1. Using a-priori estimates up to the boundary [3, 7] we find the estimates for the solution of problem (2.2), (2.1). This problem in the new variables ξ, τ, $\xi_s = \tilde{\varepsilon}_1^{-2} x_s$, $s = 1, 2$, $\tau = \tilde{\varepsilon}_2^{-1} t$, where $\tilde{\varepsilon}_i = \varepsilon_i(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^{-1}$, $i = 1, 2$, transforms into the problem

$$(4.3a)\qquad \tilde{L}\,\tilde{u}(\xi,\tau) = \tilde{f}(\xi,\tau,\varepsilon), \quad (\xi,\tau) \in \tilde{G},$$

$$(4.3b)\qquad \tilde{u}(\xi,\tau) = \tilde{\varphi}(\xi,\tau), \quad (\xi,\tau) \in \tilde{S}.$$

Here $\tilde{v}(\xi,\tau) = v(x(\xi), t(\tau))$, $v(x,t)$ is one of the functions $u(x,t), \ldots, \varphi(x,t)$, $\tilde{G}^0 = \{(\xi,\tau) : \ \xi = \xi(x),\ \tau = \tau(t),\ (x,t) \in G^0\}$, G^0 is one of the sets G, S. Thus, Eqs. (4.3a) and (4.3b) are the regular (with respect to the parameters $\tilde{\varepsilon}_i$ involved) differential equation in $\tilde{G}$ and the boundary condition on $\tilde{S}$. Using the a-priori estimates up to the boundary, we find

$$\left| \frac{\partial^{k+k_0}}{\partial \xi_1^{k_1} \partial \xi_2^{k_2} \partial \tau^{k_0}} \tilde{u}(\xi,\tau) \right| \leq M, \quad (\xi,\tau) \in \overline{\tilde{G}}.$$

In the variables x, t we have

$$(4.4)\qquad \left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} u(x,t) \right| \leq M \tilde{\varepsilon}_1^{-k} \tilde{\varepsilon}_2^{-2k_0}, \quad (x,t) \in \overline{G},\ k + 2k_0 \leq K + 2.$$

Let us make estimate (4.4) more precise. For simplicity, the coefficients $a_s(x,t)$, $p(x,t)$ are assumed to be constant:

$$(4.5)\qquad a_s(x,t),\ p(x,t) = \mathring{c}onst, \quad (x,t) \in \overline{G},\ s = 1, 2.$$

We differentiate the equation and the boundary condition with respect to x_2. The function $\overline{u}(x,t) \equiv (\partial/\partial x_2)u(x,t)$ satisfies the boundary value problem similar to problem (2.2). For $\overline{u}(x,t)$ we have the estimate like (4.4) and, therefore, the estimate valid for the function $u(x,t)$ is as follows:

$$\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} u(x,t) \right| \le M \widetilde{\varepsilon}_1^{-k_1} \widetilde{\varepsilon}_2^{-2k_0} [1 + \widetilde{\varepsilon}_1^{-k_2+1}], \quad (x,t) \in \overline{G}, \ k + 2k_0 \le K + 1.$$

In a similar way we obtain

(4.6)
$$\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} u(x,t) \right| \le M \widetilde{\varepsilon}_1^{-k_1} \widetilde{\varepsilon}_2^{-2k_0} [1 + \widetilde{\varepsilon}_1^{-k_2+2}], \quad (x,t) \in \overline{G}, \ k + 2k_0 \le K.$$

Theorem 3. *For the solution of boundary value problem (2.2), (2.1) the estimate (4.1) holds. In the case of conditions (4.2), (4.5), estimates (4.4), (4.6) are satisfied.*

4.2. Now we find estimates for the smooth and singular components of the solution provided that

(4.7)
$$\varepsilon_3 \le M(\varepsilon_1 + \varepsilon_2).$$

Let

(4.8)
$$\varepsilon_2 \le M \varepsilon_1.$$

The solution of the boundary value problem can be represented as a sum of functions

(4.9)
$$u(x,t) = U(x,t) + W(x,t), \quad (x,t) \in \overline{G},$$

where $U(x,t)$, $W(x,t)$ are the regular and singular parts of the solution. The function $U(x,t)$ is the restriction on $\overline{G}$ of the function $U^*(x,t)$, $(x,t) \in \overline{G}^*$, which is the solution of the extended problem

$$L^*_{(2.2)} U^*(x,t) = F^*(x,t,\varepsilon), \ (x,t) \in G^*, \quad U^*(x,t) = \varphi^*(x,t), \ (x,t) \in S^*.$$

Here $S^* = S(G^*)$. The domain G^* is an extension of G beyond the set S_0 (G^* contains G together with an m-neighbourhood). The coefficients of the operator $L^*_{(2.2)}$ and the function $F^*(x,t,\varepsilon)$ are smooth continuations (on G^*) of the corresponding data in (2.2); $\varphi^*(x,t)$ is a smooth function, moreover, $\varphi^*(x,t) = \varphi(x,t)$, $(x,t) \in S^L$. The function $W(x,t)$ is the solution of the homogeneous problem

$$L_{(2.2)} W(x,t) = 0, \ (x,t) \in G, \quad W(x,t) = \varphi(x,t) - U(x,t), \ (x,t) \in S.$$

A further decomposition of $U^*(x,t)$ is defined by $U^*(x,t) = U^{*0}(x,t) + v^*(x,t)$, $(x,t) \in \overline{G}^*$, where the two components are the solutions of the problems

$$
\begin{aligned}
(4.10)\quad L^*_{(4.10)}U^{*0}(x,t) &\equiv \left\{\varepsilon_1^2 \sum_{s=1,2} a_s^*(x,t)\frac{\partial^2}{\partial x_s^2} - \varepsilon_3^2 c^*(x,t)\right\} U^{*0}(x,t) = \\
&= F^*(x,t,\varepsilon), \quad (x,t) \in \overline{G}^* \setminus \overline{S}^{*L}, \\
U^{*0}(x,t) &= \varphi^*(x,t), \quad (x,t) \in \overline{S}^{*L}; \\
L^*_{(2.2)}v^*(x,t) &= \varepsilon_2^2 p^*(x,t)\frac{\partial}{\partial t}U^{*0}(x,t), \quad (x,t) \in G^*, \\
v^*(x,t) &= \varphi^*(x,t) - U^{*0}(x,t), \quad (x,t) \in S^*
\end{aligned}
$$

with $\varphi^*(x,t) = U^{*0}(x,t)$, $(x,t) \in S_0^*$. For the components $U^{*0}(x,t)$, $v^*(x,t)$ we have the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}U^{*0}(x,t)\right| \le M,$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}v^*(x,t)\right| \le M\tilde{\varepsilon}_2^{-2k_0+2}, \quad (x,t) \in \overline{G}, \quad k + 2k_0 \le K.$$

These estimates implies the inequality

$$(4.11a)\quad \left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}U(x,t)\right| \le M\left[1 + \tilde{\varepsilon}_2^{-2k_0+2}\right], \quad (x,t) \in \overline{G}, \ k + 2k_0 \le K.$$

For the function $W(x,t)$ we derive the estimate

(4.11b)

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}W(x,t)\right| \le M\tilde{\varepsilon}_2^{-2k_0}\exp(-m\tilde{\varepsilon}_2^{-2}t), \quad (x,t) \in \overline{G}, \ k + 2k_0 \le K,$$

where m is an arbitrary number satisfying $m < m^0_{(4.11)}$, $m^0_{(4.11)} = (1+M_{(4.7)})^{-2} \times (1 + M_{(4.8)})^{-2}\pi^2 d_1^{-2} a_0 (p^0)^{-1}$.

Let

$$(4.12)\qquad \varepsilon_1 \le M\varepsilon_2.$$

We decompose the solution u into a regular component U and a singular component V as follows:

$$(4.13)\qquad u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}.$$

The function $U(x,t)$ is the restriction on $\overline{G}$ of the function $U^{**}(x,t)$, $(x,t) \in \overline{G}^{**}$, which is the solution of the extended problem

(4.14)

$$L^{**}_{(2.2)}U^{**}(x,t) = F^{**}(x,t,\varepsilon),\ (x,t) \in G^{**},\quad U^{**}(x,t) = \varphi^{**}(x,t),\ (x,t) \in S^{**}.$$

Here $S^{**} = S(G^{**})$. The domain G^{**} is an extension of G beyond the set S^L (G^{**} contains G together with its m-neighbourhood). The coefficients of the operator $L^{**}_{(2.2)}$ and the right-hand side $F^{**}(x,t,\varepsilon)$ are the same as for Eq. (2.2a) smoothly continued onto G^{**}; $\varphi^{**}(x,t)$ is a smooth function, moreover, $\varphi^{**}(x,t) = \varphi(x,t)$, $(x,t) \in S_0$. The function $V(x,t)$ is the solution of the problem

$$L_{(2.2)}V(x,t) = 0,\ (x,t) \in G,\quad V(x,t) = \varphi(x,t) - U(x,t),\ (x,t) \in S.$$

For the functions $U(x,t)$ and $V(x,t)$ we obtain the estimates

$$\text{(4.15a)}\quad \left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}U(x,t)\right| \le M\left[1+\tilde{\varepsilon}_1^{\,-k+2}\right],\quad (x,t)\in\overline{G},\ k+2k_0\le K.$$

and

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V(x,t)\right| \le M\tilde{\varepsilon}_1^{\,-k}\exp\left(-m\tilde{\varepsilon}_1^{\,-1}r(x,\Gamma)\right),\quad (x,t)\in\overline{G},\ k+2k_0\le K+2,$$

where m is an arbitrary number, $r(x,\Gamma)$ is the distance from x to the boundary Γ. Making this last estimate more precise, we obtain

$$\text{(4.15b)}\quad \left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V(x,t)\right| \le M\tilde{\varepsilon}_1^{\,-k_1}\left[1+\tilde{\varepsilon}_1^{\,-k_2+2}\right]\exp\left(m\tilde{\varepsilon}_1^{\,-1}r(x,\Gamma)\right),$$

$$(x,t)\in\overline{G},\quad k+2k_0\le K.$$

Theorem 4. *Let the data of problem (2.2), (2.1) and its solution $u(x,t)$ satisfy conditions (4.2), (4.5). Then, for the functions $U(x,t)$, $V(x,t)$, $W(x,t)$, i.e. for the components from representations (4.9), (4.13), estimates (4.11) and (4.15) are fulfilled under conditions (4.7), (4.8) and (4.7), (4.12), respectively.*

4.3. Now we estimate the solution $u(x,t)$ under the condition

$$\text{(4.16)}\qquad \varepsilon_1+\varepsilon_2\le M\varepsilon_3.$$

Let the parameters ε_1 and ε_2 be commensurable, i.e.,

$$\text{(4.17)}\qquad M^{-1}\varepsilon_2\le\varepsilon_1\le M\varepsilon_2.$$

We will also consider such a representation for the solution

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}. \tag{4.18a}$$

Here $U(x,t)$ is the restriction on $\overline{G}$ of the solution to problem (2.2) smoothly extended beyond the boundary S (beyond the sets S^L and S_0). The function $V(x,t)$ is the solution of the problem

$$L_{(2.2)}V(x,t) = 0, \ (x,t) \in G, \quad V(x,t) = \varphi(x,t) - U(x,t), \ (x,t) \in S.$$

The function $U(x,t)$ satisfies the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} U(x,t) \right| \le M \left[1 + \tilde{\varepsilon}_1^{\,2-l-k}\, \tilde{\varepsilon}_2^{\,l-2k_0} \right], \tag{4.19a}$$

$$(x,t) \in \overline{G}, \quad k + 2k_0 \le K, \quad l = 0, 2.$$

It is convenient to represent the function $V(x,t)$ in the form

$$V(x,t) = V_1(x,t) + V_0(x,t) + V_{10}(x,t), \ (x,t) \in \overline{G}, \tag{4.18b}$$

where $V_1(x,t)$, $V_0(x,t)$ and $V_{10}(x,t)$ are, respectively, the boundary, initial and corner layers. The functions $V_1(x,t)$ and $V_0(x,t)$ is the restrictions on $\overline{G}$ of the functions $V_j^{(j)}(x,t)$, $j = 0, 1$, which are defined by

$$L_{(2.2)}^{(j)} V_j^{(j)}(x,t) = 0, \quad (x,t) \in G^{(j)},$$
$$V_j^{(j)}(x,t) = \varphi^{(j)}(x,t) - U^*(x,t), \quad (x,t) \in S^{(j)}, \ j = 0, 1,$$

where $G^{(1)}$ and $G^{(0)}$ are extensions of the domain G beyond the sets S_0 and S^L respectively; $L_{(2.2)}^{(j)}$ and $\varphi^{(j)}(x,t)$ are defined by smooth continuation of the operator $L_{(2.2)}$ and the function $\varphi(x,t)$ respectively. Here the function $U^*(x,t)$, $(x,t) \in \overline{G}^*$ generates the function $U_{(4.18)}(x,t)$, and also

$$\varphi^{(1)}(x,t) - U^*(x,t) = 0, \ (x,t) \in S_0^{(1)}, \quad \varphi^{(1)}(x,t) = \varphi(x,t), \ (x,t) \in S^L,$$
$$\varphi^{(0)}(x,t) - U^*(x,t) = 0, \ (x,t) \in S^{(0)L}, \quad \varphi^{(0)}(x,t) = \varphi(x,t), \ (x,t) \in S_0.$$

The function $V_{10}(x,t)$ is the solution of the problem

$$L_{(2.2)} V_{10}(x,t) = 0, \quad (x,t) \in G,$$
$$V_{10}(x,t) = \varphi(x,t) - \left(U(x,t) + V_1(x,t) + V_0(x,t) \right), \quad (x,t) \in S.$$

For the functions $V_1(x,t)$, $V_0(x,t)$, $V_{10}(x,t)$ we obtain the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_1(x,t)\right| \le M\tilde{\varepsilon}_1^{-k_1}\left[1+\tilde{\varepsilon}_1^{2-l-k_2}\,\tilde{\varepsilon}_2^{l-2k_0}\right] \times\exp(-m_1\tilde{\varepsilon}_1^{-1}r(x,\Gamma)), \tag{4.19b}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_0(x,t)\right| \le M\tilde{\varepsilon}_2^{-2k_0}\left[1+\tilde{\varepsilon}_1^{2-k}\right]\exp(-m_2\tilde{\varepsilon}_2^{-2}t),$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_{10}(x,t)\right| \le M\tilde{\varepsilon}_1^{-k_1}\,\tilde{\varepsilon}_2^{-2k_0}\left[1+\tilde{\varepsilon}_1^{2-k_2}\right]\times$$

$$\times\min\left[\exp(-m_1\tilde{\varepsilon}_1^{-1}r(x,\Gamma)),\exp(-m_2\tilde{\varepsilon}_2^{-2}t)\right],$$

$$(x,t)\in\overline{G},\quad k+2k_0\le K,\quad l=0,2.$$

Here m_1, m_2 are arbitrary numbers such that $m_i = m_{i(4.19)} < m^0_{i(4.19)}$, $i=1,2$, where $m^0_{1(4.19)} = \left(1+M_{(4.16)}\right)^{-1} c_0^{1/2}(a^0)^{-1/2}$, $m^0_{2(4.19)} = \left(1+M_{(4.16)}\right)^{-2} c_0(p^0)^{-1}$.

Theorem 5. *Let the hypotheses of Theorem 4 be fulfilled. Then, under conditions (4.16), (4.17), the functions $U(x,t)$, $V_1(x,t)$, $V_0(x,t)$, $V_{10}(x,t)$ from representation (4.18) satisfy estimates (4.19).*

4.4. Let us estimate the solution of the problem provided that the parameters ε_i, $i=1,2,3$ satisfy (4.16); the fulfilment of (4.17) is not assumed.

4.4.1. Let ε_1 and ε_2 take arbitrary values from the half-interval (0,1]:

$$\varepsilon_1,\ \varepsilon_2\in(0,1]. \tag{4.20}$$

The solution $u(x,t)$ can be written in the form of a sum

$$u(x,t) = U^0(x,t)+V_1^0(x,t)+V_0^0(x,t)+V_{10}^0(x,t)+v(x,t),\quad (x,t)\in\overline{G}, \tag{4.21}$$

where $U^0(x,t)$ and $V_1^0(x,t)$, $V_0^0(x,t)$, $V_{10}^0(x,t)$ are the principal terms (for small values of the parameters ε_1, ε_2) in the regular and singular components of the solution, $v(x,t)$ is the remainder term. These functions are the solutions of the problems

$$-\varepsilon_3^2\,c(x,t)U^0(x,t) = F(x,t,\varepsilon),\quad (x,t)\in\overline{G}; \tag{4.22}$$

$$L^1_{(4.22)}V_1^0(x,t) \equiv \left\{\varepsilon_1^2\sum_{s=1,2}a_s(x,t)\frac{\partial^2}{\partial x_s^2}-\varepsilon_3^2\,c(x,t)\right\}V_1^0(x,t)=0,$$

$$(x,t)\in\overline{G}\setminus\overline{S}^L,$$

$$\begin{aligned}
V_1^0(x,t) &= \varphi(x,t)-U^0(x,t), \quad (x,t)\in\overline{S}^L;\\
L^2_{(4.22)}V_0^0(x,t) &\equiv \left\{-\varepsilon_2^2\,p(x,t)\frac{\partial}{\partial t}-\varepsilon_3^2\,c(x,t)\right\}V_0^0(x,t)=0, \quad (x,t)\in\overline{G}\setminus
\end{aligned}$$

$$\begin{aligned}
V_0^0(x,t) &= \varphi(x,t)-U^0(x,t), \quad (x,t)\in S_0;\\
L_{(2.2)}V_{10}^0(x,t) &= 0, \quad (x,t)\in G,\\
V_{10}^0(x,t) &= \varphi(x,t)-\left(U^0(x,t)+V_1^0(x,t)+V_0^0(x,t)\right), \quad (x,t)\in S;\\
L_{(2.2)}v(x,t) &= F_{(4.22)}(x,t),\ (x,t)\in G, \quad v(x,t)=0,\ (x,t)\in S.
\end{aligned}$$

Here

$$\begin{aligned}
F_{(4.22)}(x,t) &= -\left\{\varepsilon_1^2\sum_{s=1,2}a_s(x,t)\frac{\partial^2}{\partial x_s^2}-\varepsilon_2^2\,p(x,t)\frac{\partial}{\partial t}\right\}U^0(x,t)+\\
&\quad+\varepsilon_2^2\,p(x,t)\frac{\partial}{\partial t}V_1^0(x,t)-\varepsilon_1^2\sum_{s=1,2}a_s(x,t)\frac{\partial^2}{\partial x_s^2}V_0^0(x,t).
\end{aligned}$$

Assume that, in the case of (4.16), (4.20), the problem data in a neighbourhood of the set γ_0 are such that the following condition is satisfied for every fixed set of the parameters $\varepsilon_i,\ i=1,2,3$:

(4.23a) $$V_{10}^0\in C^{l+\alpha,(l+\alpha)/2}(\overline{G}), \quad l\geq K, \quad \alpha>0.$$

The functions from representation (4.21) satisfy the estimates

(4.24) $$\begin{aligned}
\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}U^0(x,t)\right| &\leq M,\\
\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_1^0(x,t)\right| &\leq M\tilde\varepsilon_1^{-k_1}\left[1+\tilde\varepsilon_1^{2-k_2}\right]\exp\left(-m_1\tilde\varepsilon_1^{-1}r(x,\Gamma)\right),\\
\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_0^0(x,t)\right| &\leq M\tilde\varepsilon_2^{-2k_0}\exp\left(-m_2\tilde\varepsilon_2^{-2}t\right),\\
\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_{10}^0(x,t)\right| &\leq M\tilde\varepsilon_1^{-k_1}\,\tilde\varepsilon_2^{-2k_0}\left[1+\tilde\varepsilon_1^{2-k_2}\right]\times\\
&\quad\times\min\left[\exp\left(-m_1\tilde\varepsilon_1^{-1}r(x,\Gamma)\right),\exp(-m_2\tilde\varepsilon_2^{-2}t)\right],
\end{aligned}$$

$$|\,v(x,t)\,| \leq M(\widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2)^2,$$
$$(x,t) \in \overline{G},\ k + 2k_0 \leq K - 2,\ m_i = m_{i(4.19)},\ i = 1,2.$$

4.4.2. If condition (4.16) is fulfilled, then, provided that

$$\varepsilon_1 \leq M\varepsilon_2,\ \ \varepsilon_1,\ \varepsilon_2 \in (0,1], \tag{4.25}$$

we consider the solution $u(x,t)$ in the form

$$u(x,t) = U^0(x,t) + V_1^0(x,t) + v(x,t), \quad (x,t) \in \overline{G}. \tag{4.26}$$

Here $U^0(x,t)$ and $V_1^0(x,t)$ are the principal terms (for small values of the parameter ε_1, $\varepsilon_1 = o(\varepsilon_2)$) in the regular and singular components of the solution, $v(x,t)$ is the remainder term. The functions $U^0(x,t)$, $V_1^0(x,t)$, $v(x,t)$ are the solutions of the problems

$$\begin{aligned}
L^2_{(4.22)}U^0(x,t) &= F(x,t,\varepsilon),\ (x,t) \in \overline{G} \setminus S_0,\\
U^0(x,t) &= \varphi(x,t),\ (x,t) \in S_0;\\
L_{(2.2)}V_1^0(x,t) &= 0,\ (x,t) \in G,\\
V_1^0(x,t) &= \varphi(x,t) - U^0(x,t),\ (x,t) \in S;\\
L_{(2.2)}v(x,t) &= -\varepsilon_1^2 \sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2}U^0(x,t),\ (x,t) \in G,\\
v(x,t) &= 0,\ (x,t) \in S.
\end{aligned}$$

We suppose that, under conditions (4.16), (4.25), the problem data (in the neighbourhood of γ_0) are such that the following inclusion is fulfilled for every fixed set of the parameters ε_i, $i = 1,2,3$:

$$V_1^0 \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}),\quad l \geq K,\quad \alpha > 0. \tag{4.23b}$$

The components from representation (4.26) satisfy the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}U^0(x,t)\right| \leq M\,\widetilde{\varepsilon}_2^{\,-2k_0}, \tag{4.27}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_1^0(x,t)\right| \leq M\,\widetilde{\varepsilon}_1^{\,-k_1}\widetilde{\varepsilon}_2^{\,-2k_0}\left[1 + \widetilde{\varepsilon}_1^{\,2-k_2}\right]\exp\left(-m_1\widetilde{\varepsilon}_1^{\,-1}r(x,\Gamma)\right),$$

$$|\,v(x,t)\,| \leq M\,\widetilde{\varepsilon}_1^{\,2},\quad (x,t) \in \overline{G},\ k + 2k_0 \leq K,\ m_1 = m_{1(4.19)}.$$

4.4.3. Under condition (4.16), in the case if

$$\varepsilon_2 \le M\varepsilon_1, \quad \varepsilon_1,\ \varepsilon_2 \in (0,1] \tag{4.28}$$

the solution $u(x,t)$ can be written in such a form:

$$u(x,t) = U^0(x,t) + V_0^0(x,t) + v(x,t), \quad (x,t) \in \overline{G}. \tag{4.29}$$

Here $U^0(x,t)$ and $V_0^0(x,t)$ are the principal terms (for small values of the parameter ε_2, $\varepsilon_2 = o(\varepsilon_1)$) in the regular and singular components of the solution, $v(x,t)$ is the remainder term. The functions $U^0(x,t)$, $V_0^0(x,t)$, $v(x,t)$ are defined by

$$\begin{aligned}
L^1_{(4.22)} U^0(x,t) &= F(x,t,\varepsilon), \quad (x,t) \in \overline{G} \setminus \overline{S}^L, \\
U^0(x,t) &= \varphi(x,t), \quad (x,t) \in \overline{S}^L; \\
L_{(2.2)} V_0^0(x,t) &= 0, \quad (x,t) \in G, \\
V_0^0(x,t) &= \varphi(x,t) - U^0(x,t), \quad (x,t) \in S; \\
L_{(2.2)} v(x,t) &= \varepsilon_2^2 p(x,t) \frac{\partial}{\partial t} U^0(x,t), \quad (x,t) \in G, \\
v(x,t) &= 0, \quad (x,t) \in S.
\end{aligned}$$

In the case of conditions (4.16) and (4.28) we assume the fulfilment of the inclusion

$$V_0^0 \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}), \quad l \ge K, \quad \alpha > 0 \tag{4.23c}$$

for every fixed set of the parameters ε_i, $i = 1,2,3$.

For the components of the solution from representation (4.29) we have

$$\begin{aligned}
\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} U^0(x,t) \right| &\le M\,\tilde{\varepsilon}_1^{-k_1} \left[1 + \tilde{\varepsilon}_1^{2-k_2} \right], \\
\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} V_0^0(x,t) \right| &\le M\,\tilde{\varepsilon}_2^{-2k_0} \left[1 + \tilde{\varepsilon}_1^{2-k} \right] \exp\left(-m_2 \tilde{\varepsilon}_2^{-2} t \right), \\
|\, v(x,t) \,| \le M\,\tilde{\varepsilon}_2^2, \quad (x,t) \in \overline{G},\ & k + 2k_0 \le K,\ m_2 = m_{2(4.19)}.
\end{aligned} \tag{4.30}$$

Theorem 6. *Let the hypotheses of Theorem 4 be fulfilled. Then, under condition (4.16), and also under conditions (4.20), (4.23a) (conditions (4.25), (4.23b) and (4.28), (4.23c)) the components of the solution for problem (2.2), (2.1) from representation (4.21) (representations (4.26) and (4.29)) satisfy estimates (4.24) ((4.27) and (4.30) respectively).*

5. Grid approximations of boundary value problem (2.2), (2.1)

5.1. Let us construct a finite difference scheme based on classical approximations of the boundary value problem (2.2), (2.1) on uniform grids. On the set $\overline{G}$ we introduce the grid

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}_1 \times \omega_2 \times \overline{\omega}_0, \tag{5.1}$$

where $\overline{\omega}_1$, $\overline{\omega}_0$ are grids on the segments $[0, d_1]$ and $[0, T]$ respectively, ω_2 is a grid on the axis x_2; generally speaking, the grids $\overline{\omega}_1$ and $\overline{\omega}_0$ are nonuniform. We set $h_s^i = x_s^{i+1} - x_s^i$, $x_1^i, x_1^{i+1} \in \overline{\omega}_1$, $x_2^i, x_2^{i+1} \in \omega_2$, $h_s = \max_i h_s^i$, $h = \max_s h_s$, $s = 1, 2$, h_t is the step-size of the grid $\overline{\omega}_0$. By $N_1 + 1$ and $N_0 + 1$ we denote the number of nodes in the grids $\overline{\omega}_1$ and $\overline{\omega}_0$, by $N_2 + 1$ the minimal number of nodes in the grid ω_2 on any unit interlval of the $x - 2$-axis; let $N = \min_s N_s$, $s = 1, 2$, $h \le MN^{-1}$. On the grid $\overline{G}_h$, to problem (2.2), (2.1) we assign the difference scheme

$$\begin{aligned} \Lambda_{(5.2)} z(x,t) &= F(x,t,\varepsilon), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{5.2}$$

Here $G_h = G \bigcap \overline{G}_h$, $S_h = S \bigcap \overline{G}_h$,

$$\Lambda_{(5.2)} \equiv \varepsilon_1^2 \sum_{s=1,2} a_s(x,t) \delta_{\overline{xs}\,\widehat{xs}} - \varepsilon_2^2 p(x,t) \delta_{\overline{t}} - \varepsilon_3^2 c(x,t),$$

$\delta_{\overline{t}}\, z(x,t)$ and $\delta_{\overline{xs}\,\widehat{xs}}\, z(x,t)$ are the first (backward) and second difference derivatives, e.g.,

$$\begin{aligned} \delta_{\overline{x1}\,\widehat{x1}}\, z(x,t) &= 2\left(h_1^{i-1} + h_1^i\right)^{-1} (\delta_{x1} z(x,t) - \delta_{\overline{x1}} z(x,t)), \\ \delta_{x1} z(x,t) &= \left(h_1^i\right)^{-1} \left(z(x_1^{i+1}, x_2, t) - z(x,t)\right), \\ \delta_{\overline{x1}} z(x,t) &= \left(h_1^{i-1}\right)^{-1} \left(z(x,t) - z(x_1^{i-1}, x_2, t)\right), \quad x = (x_1^i, x_2). \end{aligned}$$

The difference operator $\Lambda_{(5.2)}$ is monotone [9] ε-uniformly.

Using the comparison theorems, we ascertain the ε-uniform boundedness of the solution for problem (5.2), (5.1):

$$|\, z(x,t)\,| \le M, \quad (x,t) \in \overline{G}_h. \tag{5.3a}$$

Taking into account the estimates of Theorem 3 (for $K = 2$) we find

(5.3b)

$$|u(x,t) - z(x,t)| \le M(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)[\varepsilon_1^{-1} N^{-1} + \varepsilon_2^{-2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) N_0^{-1}], \quad (x,t) \in \overline{G}_h,$$

that is, scheme (5.2), (5.1) converges for fixed values of the parameters ε_i. Note, from (5.3b) it follows that this scheme converges ε_3-uniformly for fixed ε_1 and ε_2.

5.2. Now we construct a grid condensing in the neighbourhood of the boundary and initial layers so that the discrete solution on this grid converges uniformly in the parameters ε_i, $i = 1,2,3$ satisfying (4.7) (or, in short, it converges ε-uniformly under condition (4.7)). On the set $\overline{G}$ we introduce the special grid

$$\overline{G}_h = \overline{D}_h^* \times \overline{\omega}_0^* = \overline{\omega}_1^* \times \omega_2 \times \overline{\omega}_0^*, \tag{5.4a}$$

where $\omega_2 = \omega_{2(5.1)}$, $\overline{\omega}_1^* = \overline{\omega}_1^*(\sigma_1)$ and $\overline{\omega}_0^* = \overline{\omega}_0^*(\sigma_2)$ are piecewise uniform grids on $[0, d_1]$ and $[0, T]$ respectively; σ_1, σ_2 are parameters depending on ε_i and N_1, N_0. The step-sizes of the grid $\overline{\omega}_1^*$ (cf. [8, 11, 15, 16]) are constant on the segments $[0, \sigma_1]$, $[d_1 - \sigma_1, d_1]$ and $[\sigma_1, d_1 - \sigma_1]$, and equal to $h_1^{(1)} = 4\sigma_1 N_1^{-1}$ and $h_1^{(2)} = 2(d_1 - 2\sigma_1)N_1^{-1}$ respectively. The step-sizes of the grid $\overline{\omega}_0^*$ are constant on the segments $[0, \sigma_2]$ and $[\sigma_2, T]$, and equal respectively to $h_0^{(1)} = 2\sigma_2 N_0^{-1}$ and $h_0^{(2)} = 2(T - \sigma_2)N_0^{-1}$. We choose the quantity σ_1 and σ_2 from the conditions

$$\begin{aligned} \sigma_1 &= \sigma_{1(5.4)}(\varepsilon, N_1) = \min\left[4^{-1}d_1,\ M_1\,\varepsilon_1\,(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^{-1} \ln N_1\right], \\ \sigma_2 &= \sigma_{2(5.4)}(\varepsilon, N_0) = \min\left[2^{-1}T,\ M_2\,\varepsilon_2^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^{-2} \ln N_0\right], \end{aligned} \tag{5.4b}$$

where M_1, M_2 are arbitrary numbers. The grid $\overline{G}_{h(5.4)}$ has been constructed.

The estimates of Theorem 4 (for $K = 4$) implies the ε-uniform (under condition (4.7)) convergence of scheme (5.2), (5.4). We estimate the convergence rate using the techniques from [11, 15, 16]. In that case when M_2 satisfies the condition

$$M_2 > (m^0_{(4.11)})^{-1}, \tag{5.5}$$

for the solution of the difference scheme we obtain the estimate

$$|u(x,t) - z(x,t)| \le M[N^{-1} \ln N + N_0^{-1} \ln N_0], \quad (x,t) \in \overline{G}_h. \tag{5.6}$$

Theorem 7. *Let the data of boundary value problem (2.2), (2.1) satisfy conditions (2.3). Then the solution of difference scheme (5.2), (5.1) is bounded ε-uniformly. Let, in addition, for the solution of problem (2.2), (2.1) the estimates of Theorem 3 be fulfilled for $K = 2$. Then the solution of scheme (5.2), (5.1) converges to the solution of the boundary value problem for fixed values of the parameters ε_i, $i = 1,2,3$. If for the solution $u(x,t)$ the estimates of*

Theorem 4 are fulfilled for $K = 4$, the solution of difference scheme (5.2), (5.4) converges ε-uniformly under condition (4.7). For the solutions of schemes (5.2), (5.1) and (5.2), (5.4), (5.5) the estimates (5.3) are (5.6) are valid, respectively.

R e m a r k. The singularly perturbed differential equations

$$L^1_{(5.7)}u(x,t) \equiv \{\varepsilon^2 \sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2} - p(x,t)\frac{\partial}{\partial t} - c(x,t)\}u(x,t) = f(x,t), \tag{5.7a}$$

$$L^2_{(5.7)}u(x,t) \equiv \{\sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2} - \varepsilon^2 p(x,t)\frac{\partial}{\partial t} - c(x,t)\}u(x,t) = f(x,t), \tag{5.7b}$$

with $\varepsilon \in (0,1]$ are equations (2.2a) in which the parameters satisfy conditions (4.7), (4.12) (in the case of (5.7a)) and (4.7), (4.8) (in the case of (5.7b)). Thus, in the case of a Dirichlet problem for (5.7a) and (5.7b), the difference scheme (5.2), (5.4) converges ε-uniformly.

5.3. The a-priori estimates of Theorem 5 (for $K = 4$) implies the ε-uniform convergence of scheme (5.2), (5.4) under conditions (4.16), (4.17). In the case of the grid $\overline{G}_{h(5.4)}$, if

$$M_i > (m^0_{i(4.19)})^{-1}, \ i = 1,2, \tag{5.8}$$

for the solution of the difference scheme we have the estimate

$$|u(x,t) - z(x,t)| \leq M\left[N^{-1}\ln N + N_0^{-1}\ln N_0\right], \ (x,t) \in \overline{G}_h. \tag{5.9}$$

The following assertion is valid.

Theorem 8. *Let the solution of boundary value problem (2.2), (2.1) satisfy the estimates of Theorem 5 (for $K = 4$). Then the solution of difference scheme (5.2), (5.4) converges to the solution of the boundary value problem ε-uniformly under conditions (4.16), (4.17). For the solution of scheme (5.2), (5.4), (5.8) estimate (5.9) is valid.*

R e m a r k. The singularly perturbed differential equation

$$L_{(5.10)}u(x,t) \equiv \{\varepsilon^2[\sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2} - p(x,t)\frac{\partial}{\partial t}] - c(x,t)\}u(x,t) = f(x,t),$$
$$(x,t) \in G, \quad \varepsilon \in (0,1] \tag{5.10}$$

is equation (2.2a) in which the parameters satisfy (4.16) and (4.17) (with $M_{(4.16)} = M_{(4.17)} = 1$). Thus, in the case of a Dirichlet problem for (5.10), scheme (5.2), (5.4) converges ε-uniformly.

5.4. Let us study the convergence of difference scheme (5.2), (5.4) if the parameters ε_i satisfy only (4.16); the fulfilment of (4.17) is not assumed. Suppose that the solution of the boundary value problem satisfies the estimates of Theorem 6 (for $K = 6$).

It follows from (4.24) that scheme (5.2), (5.4) converges provided that $N,\ N_0 \to \infty,\ (\varepsilon_1+\varepsilon_2)(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-1} \to 0$. Under condition (5.8) we have

(5.11)

$$|u(x,t) - z(x,t)| \le M(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-2}\left[(\varepsilon_1+\varepsilon_2)^2 + N^{-1}\ln N + N_0^{-1}\ln N_0\right],$$

$$(x,t) \in \overline{G}_h,\quad \overline{G}_h = \overline{G}_{h(5.4)}.$$

By estimates (4.27) (estimates (4.30)) scheme (5.2), (5.4) converges as $N,\ N_0 \to \infty,\ \varepsilon_1(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-1} \to 0$ (as $N,\ N_0 \to \infty,\ \varepsilon_2(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-1} \to 0$), if the parameter ε_2 (parameter ε_1) is kept fixed. On the grid $\overline{G}_{h(5.4)}$ under condition (5.8), the solution of the difference scheme satisfies the estimates

$$(5.12a)\qquad |u(x,t) - z(x,t)| \le M(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-2}\left[\varepsilon_1^2 + (\varepsilon_1+\varepsilon_2+\varepsilon_3)^2 N^{-1}\ln N + \varepsilon_2^{-2}(\varepsilon_1+\varepsilon_2+\varepsilon_3)^4 N_0^{-1}\right],$$

$$(5.12b)\qquad |u(x,t) - z(x,t)| \le M(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-2}\left[\varepsilon_2^2 + \varepsilon_1^{-1}(\varepsilon_1+\varepsilon_2+\varepsilon_3)^3 N^{-1} + (\varepsilon_1+\varepsilon_2+\varepsilon_3)^2 N_0^{-1}\ln N_0\right],\quad (x,t) \in \overline{G}_h.$$

The difference scheme (5.2), (5.4) converges for fixed values of the parameters ε_1, ε_2 with an error bound given by (5.3). Thus, it follows from (4.27) (from (4.30)) that, in the case of the parameters satisfying (4.16), scheme (5.2), (5.4) converges $(\varepsilon_1,\varepsilon_3)$–uniformly ($(\varepsilon_2,\varepsilon_3)$–uniformly) for fixed ε_2 (fixed ε_1). In the case of scheme (5.2), (5.4), (5.8) the inequalities (5.3), (5.12a) and (5.3), (5.12b) imply the estimates

$$(5.13)\ |u(x,t) - z(x,t)| \le M\left[(N^{-1}\ln N)^{2/3} + \varepsilon_2^{-2}(\varepsilon_1+\varepsilon_2+\varepsilon_3)^2 N_0^{-1}\ln N_0\right],$$

$$|u(x,t) - z(x,t)| \le M\left[\varepsilon_1^{-1}(\varepsilon_1+\varepsilon_2+\varepsilon_3)N^{-1}\ln N + (N_0^{-1}\ln N_0)^{1/2}\right],\quad (x,t) \in \overline{G}_h.$$

It follows from (5.11) and (5.13) that

$$(5.14)\ |u(x,t) - z(x,t)| \le M\left[(N^{-1}\ln N)^{2/3} + (N_0^{-1}\ln N_0)^{1/2}\right],\quad (x,t) \in \overline{G}_h.$$

Thus, difference scheme (5.2), (5.4), (5.8) converges ε-uniformly under condition (4.16).

Theorem 9. *Let the solution of boundary value problem (2.2), (2.1) satisfy the estimate of Theorems 3 and 6 for $K = 6$. Then the solution of difference scheme (5.2), (5.4) converges to the solution of the boundary value problem ε-uniformly under condition (4.16); for the solution of difference scheme (5.2), (5.4), (5.8) the estimates (5.13), (5.14) are valid.*

R e m a r k. The singularly perturbed differential equation

$$\Lambda_{(5.15)}u(x,t) \equiv \{\bar{\varepsilon}_1^2 \sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2} - \bar{\varepsilon}_2^2\, p(x,t)\frac{\partial}{\partial t}u(x,t) - \tag{5.15}$$

$$-c(x,t)\}u(x,t) = f(x,t), \quad (x,t) \in G, \quad \bar{\varepsilon}_i \in (0,1], \; i = 1,2$$

is equation (2.2a) in which the parameters satisfy (4.16). Thus, in the case of a Dirichlet problem for (5.15), scheme (5.2), (5.4) converges ($\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$) uniformly.

5.5. It follows from the results of Subsections 5.2, 5.4 that scheme (5.2), (5.4) converges ε-uniformly under the estimates of Theorems 4, 6 (for $K = 6$). On the grid $\overline{G}_{h(5.4)}$, where

$$M_1 > (m^0_{1(4.19)})^{-1}, \quad M_2 > \max\left[(m^0_{(4.11)})^{-1}, (m^0_{2(4.19)})^{-1}\right], \tag{5.16}$$

the solution of the difference scheme satisfies (5.13), (5.14).

The following theorem is valid.

Theorem 10. *Let the estimates of Theorems 3 – 6, for $K = 6$, be fulfilled for the solution of boundary value problem (2.2), (2.1) and for its components. Then the solution of difference scheme (5.2), (5.4) converges ε-uniformly to the solution of the boundary value problem; for the solution of difference scheme (5.2), (5.4), (5.16) the estimates (5.13), (5.14), and also under conditions (4.7) or (4.16), (4.17), the estimate (5.9) are valid.*

6. Grid approximations of boundary value problem (2.4), (2.5)

6.1. The solution of problem (2.4), (2.5) satisfies (4.1). One can derive this estimate according to the estimation plan for problem (2.2), (2.1). Here we suppose that the data of the boundary value problem satisfy (4.5), (4.2a) and provide the fulfilment of (4.2b).

For the solution of problem (2.4), (2.5) the estimate (4.4) holds. Let us give the estimates if the parameters ε_i satisfy (4.7). In the case of (4.7), (4.8) we represent the solution in the same form as in (4.9). The additive components from (4.9) satisfy (4.11), where

$$m^0_{(4.11)} = m^0_{(6.1)} \equiv \left(1 + M_{(4.7)}\right)^{-2} \left(1 + M_{(4.8)}\right)^{-2} \pi^2 a_0 (p^0)^{-1} \max_s d_s^{-2}, \; s = 1,2. \tag{6.1}$$

Under conditions (4.7), (4.12) the function $V_{(4.13)}(x,t)$ can be written in the form

$$(6.2)\qquad V(x,t) = V_1(x,t) + V_2(x,t) + V_{12}(x,t), \quad (x,t) \in \overline{G}.$$

The functions $V_j(x,t)$ are the restrictions on $\overline{G}$ of the functions $V_j^{(j)}(x,t)$, $(x,t) \in \overline{G}^{(j)}$, where $V_j^{(j)}(x,t)$ is the solution of the problem

$$\begin{aligned} L_{(2.4)}^{(j)} V_j^{(j)}(x,t) &= 0, \quad (x,t) \in G^{(j)}, \\ V_j^{(j)}(x,t) &= \varphi^{(j)}(x,t) - U^{**}(x,t), \quad (x,t) \in S^{(j)}, \ j = 1,2. \end{aligned}$$

Here the domain $G^{(j)}$ is an extension of G beyond the set S^L in the direction of the axis x_j, the operator $L_{(2.4)}^{(j)}$ and the function $\varphi^{(j)}(x,t)$ are continuations of the operator $L_{(2.4)} = L_{(2.2)}$ and the function $\varphi(x,t)$, $\varphi^{(j)}(x,t) = \varphi(x,t)$, $(x,t) \in S^{(j)} \cap S$; $U^{**}(x,t) = U_{(4.14)}^{**}(x,t)$. The function $V_{12}(x,t)$, i.e., the corner boundary layer, is defined by

$$\begin{aligned} L_{(2.4)} V_{12}(x,t) &= 0, \quad (x,t) \in G, \\ V_{12}(x,t) &= \varphi(x,t) - (U(x,t) + V_1(x,t) + V_2(x,t)), \quad (x,t) \in S. \end{aligned}$$

The components of the solution $u(x,t)$ from representations (4.13), (6.2) satisfy (4.15a) and also the estimates

$$(6.3)\qquad \begin{aligned} \left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} V_j(x,t) \right| &\le M \tilde{\varepsilon}_1^{-k_j} \left[1 + \tilde{\varepsilon}_1^{2-k_{3-j}} \right] \exp\left(-m \tilde{\varepsilon}_1^{-1} r(x, \Gamma_j) \right), \\ \left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} V_{12}(x,t) \right| &\le M \tilde{\varepsilon}_1^{-k_1-k_2} \min_{s=1,2} \left[\exp\left(-m \tilde{\varepsilon}_1^{-1} r(x, \Gamma_s) \right) \right], \\ & (x,t) \in \overline{G}, \quad k + 2k_0 \le K, \ j = 1,2, \end{aligned}$$

where $m = m_{(4.15)}$, Γ_j is that part of the boundary Γ which is orthogonal to the axis x_j.

Thus, if the parameters ε_i satisfy (4.7), for the components of the solution from representation (4.9) (representation (4.13), (6.2)) the estimates (4.11), (6.1) (estimates (4.15a), (6.3)) are valid under condition (4.8) (condition (4.12)) respectively.

6.2. Now we estimate the solution if the parameters ε_i satisfy (4.16), (4.17). For this we use representation (4.18a). The function $V(x,t)$ from (4.18a) can be written in the form

$$(6.4)\qquad V(x,t)=\sum_{j=0}^{2}V_j(x,t)+\sum_{\substack{i,j=0,1,2,\\ i<j}}V_{ij}(x,t)+V_{012}(x,t),\quad (x,t)\in\overline{G}.$$

The functions $V_j(x,t)$ are the restrictions on $\overline{G}$ of the functions $V_j^{(j)}(x,t)$ defined by

$$\begin{aligned} L^{(j)}_{(2.4)}V_j^{(j)}(x,t) &= 0,\quad (x,t)\in G^{(j)},\\ V_j^{(j)}(x,t) &= \varphi^{(j)}(x,t)-U^*(x,t),\quad (x,t)\in S^{(j)},\ j=0,1,2, \end{aligned}$$

where $G^{(j)}$ is an extension of the domain G beyond the set $\bigcup_k S_k$, $k=0,1,2$, $k\neq j$ ($S_j=\Gamma_j\times(0,T]$, $j=1,2$); $\varphi^{(j)}(x,t)-U^*(x,t)=0$ for $(x,t)\in S^{(j)}\setminus S_j^{(j)}$, $\varphi^{(j)}(x,t)=\varphi(x,t)$ for $(x,t)\in S^{(j)}\cap S$, $j=0,1,2$. The functions $V_{ij}(x,t)$ are the restrictions on $\overline{G}$ of the functions $V_{ij}^{(ij)}(x,t)$, which are the solutions for the problems

$$\begin{aligned} L^{(ij)}_{(2.4)}V_{ij}^{(ij)}(x,t) &= 0,\quad (x,t)\in G^{(ij)},\\ V_{ij}^{(ij)}(x,t) &= \varphi^{(ij)}(x,t)-\left(U^*(x,t)+V_i^{(i)}(x,t)+V_j^{(j)}(x,t)\right),\\ &\quad (x,t)\in S^{(ij)},\quad i,j=0,1,2,\quad i<j, \end{aligned}$$

where $G^{(ij)}$ is an extension of the domain G beyond the set S_k, $k=0,1,2$, $k\neq i,j$, and also

$$\varphi^{(ij)}(x,t)-U^*(x,t)=0,\ (x,t)\in S_k^{(ij)},\quad \varphi^{(ij)}(x,t)=\varphi(x,t),\ (x,t)\in S_k^{(ij)}\cap S,$$
$$i,\ j,\ k=0,1,2,\quad i<j,\quad k\neq i,j.$$

The function $V_{012}(x,t)$ is the solution of the problem

$$L_{(2.4)}V_{012}(x,t)=0,\quad (x,t)\in G,$$

$$V_{012}(x,t)=\varphi(x,t)-(U(x,t)+\sum_{j=0}^{2}V_j(x,t)+\sum_{\substack{i,j=0,1,2,\\ i<j}}V_{ij}(x,t)),\ (x,t)\in S.$$

The components of the solution from representations (4.18a), (6.4) satisfy (4.19a) and also the estimates

$$(6.5)\qquad \left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_j(x,t)\right| \leq M\,\tilde{\varepsilon}_1^{-k_j}\left[1+\tilde{\varepsilon}_1^{2-l-k_{3-j}}\,\tilde{\varepsilon}_2^{l-2k_0}\right]\times$$

$$\times \exp\left(-m_1\tilde{\varepsilon}_1^{-1} r(x,\Gamma_j)\right), \quad j=1,2,$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}} V_0(x,t)\right| \le M\,\tilde{\varepsilon}_2^{-2k_0}\left[1+\tilde{\varepsilon}_1^{2-k}\right]\exp\left(-m_2\tilde{\varepsilon}_2^{-2}t\right),$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}} V_{12}(x,t)\right| \le M\,\tilde{\varepsilon}_1^{-k}\left[1+\tilde{\varepsilon}_2^{2-2k_0}\right]\min_{s=1,2}\left[\exp\left(-m_1\tilde{\varepsilon}_1^{-1}r(x,\Gamma_s)\right)\right],$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}} V_{0j}(x,t)\right| \le M\,\tilde{\varepsilon}_1^{-k_j}\,\tilde{\varepsilon}_2^{-2k_0}\left[1+\tilde{\varepsilon}_1^{2-k_{3-j}}\right]\times$$

$$\times\min\left[\exp\left(-m_1\tilde{\varepsilon}_1^{-1}r(x,\Gamma_j)\right),\exp\left(-m_2\tilde{\varepsilon}_2^{-2}t\right)\right], \quad j=1,2,$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}} V_{012}(x,t)\right| \le M\,\tilde{\varepsilon}_1^{-k}\,\tilde{\varepsilon}_2^{-2k_0}\times$$

$$\times\min\left[\min_{s=1,2}\left[\exp\left(-m_1\tilde{\varepsilon}_1^{-1}r(x,\Gamma_s)\right)\right],\exp\left(-m_2\tilde{\varepsilon}_2^{-2}t\right)\right],$$

$$(x,t)\in\overline{G},\quad k+2k_0\le K,\ l=0,2,\ m_i=m_{i(4.19)},\ i=1,2.$$

6.3. In a similar way we construct proper representations of the solution and find estimates for their components under assumption (4.16) and in the case of one of conditions (4.20), (4.25) or (4.28). In the case of (4.20) the solution can be written in the form

$$u(x,t)=U^0(x,t)+\sum_{j=0}^{2}V_j^0(x,t)+\sum_{\substack{i,j=0,1,2,\\ i<j}}V_{ij}^0(x,t)+V_{012}^0(x,t)+v(x,t),\ (x,t)\in\overline{G}.$$

(6.6)

For the components of the solution we have the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}} U^0(x,t)\right| \le M, \tag{6.7}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}} V_j^0(x,t)\right| \le M\tilde{\varepsilon}_1^{-k_j}\left[1+\tilde{\varepsilon}_1^{2-k_{3-j}}\right]\exp\left(-m_1\tilde{\varepsilon}_1^{-1}r(x,\Gamma_j)\right),\ j=1,2,$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}} V_0^0(x,t)\right| \le M\tilde{\varepsilon}_2^{-2k_0}\exp\left(-m_2\tilde{\varepsilon}_2^{-2}t\right),$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_{12}^0(x,t)\right| \le M\widetilde{\varepsilon}_1^{-k}\min_{s=1,2}\left[\exp\left(-m_1\widetilde{\varepsilon}_1^{-1}r(x,\Gamma_s)\right)\right],$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_{0j}^0(x,t)\right| \le M\widetilde{\varepsilon}_1^{-k_j}\widetilde{\varepsilon}_2^{-2k_0}\times$$

$$\times\min\left[\exp\left(-m_1\widetilde{\varepsilon}_1^{-1}r(x,\Gamma_j)\right),\exp\left(-m_2\widetilde{\varepsilon}_2^{-2}t\right)\right],\quad j=1,2,$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_{012}^0(x,t)\right| \le M\widetilde{\varepsilon}_1^{-k}\widetilde{\varepsilon}_2^{-2k_0}\times$$

$$\times\min\left[\min_{s=1,2}\left[\exp\left(-m_1\widetilde{\varepsilon}_1^{-1}r(x,\Gamma_s)\right)\right],\exp\left(-m_2\widetilde{\varepsilon}_2^{-2}t\right)\right],$$

$$|v(x,t)| \le M\left(\widetilde{\varepsilon}_1^2+\widetilde{\varepsilon}_2^2\right),$$

$$(x,t)\in\overline{G},\quad k+2k_0\le K-2,\quad m_i=m_{i(4.19)},\quad i=1,2.$$

If (4.25) is fulfilled, then we consider the solution as a sum of the functions:

$$u(x,t)=U^0(x,t)+\sum_{j=1,2}V_j^0(x,t)+V_{12}^0(x,t)+v(x,t),\ (x,t)\in\overline{G}. \tag{6.8}$$

For these components we obtain the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}U^0(x,t)\right| \le M\widetilde{\varepsilon}_2^{-2k_0}, \tag{6.9}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_j^0(x,t)\right| \le M\widetilde{\varepsilon}_1^{-k_j}\widetilde{\varepsilon}_2^{-2k_0}\left[1+\widetilde{\varepsilon}_1^{2-k_{3-j}}\right]\times$$

$$\times\exp\left(-m_1\widetilde{\varepsilon}_1^{-1}r(x,\Gamma_j)\right),\ j=1,2,$$

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_{12}^0(x,t)\right| \le M\widetilde{\varepsilon}_1^{-k}\widetilde{\varepsilon}_2^{-2k_0}\min_{s=1,2}\left[\exp\left(-m_1\widetilde{\varepsilon}_1^{-1}r(x,\Gamma_s)\right)\right],$$

$$|v(x,t)| \le M\widetilde{\varepsilon}_1^2,$$

$$(x,t)\in\overline{G},\quad k+2k_0\le K,\quad m_1=m_{1(4.19)}.$$

Under condition (4.28), for the components of such a representation for the solution:

$$u(x,t)=U^0(x,t)+V_0^0(x,t)+v(x,t),\ (x,t)\in\overline{G} \tag{6.10}$$

we obtain the estimate

$$(6.11)\qquad \begin{aligned} \left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}U^0(x,t)\right| &\le M\tilde{\varepsilon}_1^{-k}, \\ \left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}V_0^0(x,t)\right| &\le M\tilde{\varepsilon}_1^{-k}\tilde{\varepsilon}_2^{-2k_0}\exp\left(-m_2\tilde{\varepsilon}_2^{-2}t\right), \\ |v(x,t)| &\le M\tilde{\varepsilon}_2^2, \\ (x,t)\in\overline{G},\quad k+2k_0\le K,&\quad m_2=m_{2(4.19)}. \end{aligned}$$

To derive (6.7), (6.9), (6.11), we suppose that the problem data in the neighbourhood of corner points and edges of the sets γ_0 and $\gamma_1 = \left\{\overline{\Gamma}_1 \bigcap \overline{\Gamma}_2\right\} \times [0,T]$ are such that the following inclusions are fulfilled for every fixed set of the parameters ε_i, $i = 1,2,3$:

(6.12a)

$$V^0_{ij(6.6)},\ V^0_{012(6.6)} \in C^{l+\alpha,(l+\alpha)/2}\left(\overline{G}\right),\ i,j=0,1,2,\ i<j \quad \text{for (4.16), (4.20)},$$

$$(6.12b)\qquad V^0_{12(6.8)} \in C^{l+\alpha,(l+\alpha)/2}\left(\overline{G}\right) \quad \text{for (4.16), (4.25)},$$

$$(6.12c)\qquad U^0_{(6.10)},\ V^0_{0(6.10)} \in C^{l+\alpha,(l+\alpha)/2}\left(\overline{G}\right) \quad \text{for (4.16), (4.28)},$$

where $l \ge K$, $\alpha > 0$. The following theorem is valid.

Theorem 11. *For the solution of the boundary value problem (2.4), (2.5) estimate (4.1) is valid. Let the data of the problem and its solution satisfy (4.2), (4.5). Then estimate (4.4) holds for the solution. If, besides this, the parameters ε_i, $i = 1,2,3$ obey conditions (4.7), (4.8) (conditions (4.7), (4.12); (4.16), (4.17); (4.16), (4.20); (4.16), (4.25) or (4.16), (4.28)), then the components of the solution from representation (4.9) (representations (4.13), (6.2); (4.18a), (6.4); (6.6); (6.8) or (6.10)) satisfy estimates (4.11), (6.1) (estimates (4.15a), (6.3); (4.19a), (6.5); (6.7); (6.9) or (6.11) respectively). Conditions (6.12a), (6.12b), (6.12c) are assumed to be fulfilled if the parameters satisfy (4.16), (4.20); (4.16), (4.25); (4.16), (4.28) respectively.*

6.4. To solve problem (2.4), (2.5), we first study a classical finite difference scheme on (possibly) nonuniform meshes. On the set $\overline{G}$ we introduce the grid

$$(6.13)\qquad \overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}_1 \times \overline{\omega}_2 \times \overline{\omega}_0,$$

where $\overline{\omega}_0 = \overline{\omega}_{0(5.1)}$; $\overline{\omega}_s$ are arbitrary grids on the intervals $[0, d_s]$, $s = 1,2$. By $N_s + 1$ we denote the number of nodes in the grid $\overline{\omega}_s$, $h = \max_s h_s$, $s = 1,2$;

let $h \le MN^{-1}$. On the grid $\overline{G}_h$ we use the finite difference scheme

(6.14)
$$\begin{aligned} \Lambda_{(5.2)} z(x,t) &= F(x,t,\varepsilon), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned}$$

Scheme (6.14), (6.13) converges for fixed values of the parameters ε_i :

(6.15) $$|u(x,t) - z(x,t)| \le M(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)[\varepsilon_1^{-1} N^{-1} + \varepsilon_2^{-2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) N_0^{-1}], \quad (x,t) \in \overline{G}_h.$$

6.5. On the set $\overline{G}_h$ we introduce the special grid

(6.16a)
$$\overline{G}_h = \overline{D}_h^* \times \overline{\omega}_0^* = \overline{\omega}_1^* \times \overline{\omega}_2^* \times \overline{\omega}_0^*$$

which is condensed in the neighbourhood of the boundary and initial layers. Here $\overline{\omega}^*_{0(6.16)} = \overline{\omega}^*_{0(5.4)}$, the grid $\overline{\omega}^*_s$ is the grid $\overline{\omega}^*_{1(5.4)}$ with d_1 and σ_1 being d_s and σ_1^s respectively, the parameter σ_1^s is defined by the relation
(6.16b)
$$\sigma_1^s = \sigma^s_{1(6.16)}(\varepsilon_1, N_s) = \min\left[4^{-1} d_s,\ M_1^s \varepsilon_1 (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^{-1} \ln N_s\right], \quad s = 1, 2;$$

M_1^1, M_1^2 and M_2 from $\sigma_2(\varepsilon_2, N_0) = \sigma_2(\varepsilon_2, N_0; M_2)$ that defines $\overline{\omega}^*_{0(6.16)}$ are arbitrary numbers. The grid $\overline{G}_{h(6.16)}$ has been constructed.

Estimates (4.11), (6.1), (4.15a) and (6.3) for the components of the solution imply the ε-uniform convergence of scheme (6.14), (6.16) under condition (4.7). Provided that

(6.17)
$$M_2 > \left(m^0_{(6.1)}\right)^{-1}$$

we obtain the estimate

(6.18)
$$|u(x,t) - z(x,t)| \le M\left[N^{-1} \ln N + N_0^{-1} \ln N\right], \quad (x,t) \in \overline{G}_h.$$

Estimates (4.19a), (6.5) for the components of the solution imply the ε-uniform convergence of scheme (6.14), (6.16) under conditions (4.16), (4.17). The solution of the difference scheme on the grid $\overline{G}_{h(6.16)}$ satisfies estimate (6.18) provided that

(6.19)
$$M_1^s > \left(m^0_{1(4.19)}\right)^{-1}, \quad M_2 > \left(m^0_{2(4.19)}\right)^{-1}, \quad s = 1, 2.$$

Theorem 12. *Let the solution of the boundary value problem (2.4), (2.5) satisfy estimate (4.4) for $K = 2$. Then the solution of the difference* *scheme (6.14), (6.13) converges to the solution of the boundary value problem for fixed values of the parameters ε_i, $i = 1,2,3$. Assume that the components of the solution for problem (2.4), (2.5) satisfy estimates (4.11), (6.1) for $K = 4$ under conditions (4.7), (4.8) and estimates (4.15a), (6.3) for $K = 4$ under conditions (4.7), (4.12). Then the difference scheme (6.14), (6.16) converges ε-uniformly under condition (4.7). If the components of the solution satisfy estimates (4.19a), (6.5) under conditions (4.16), (4.17), then scheme (6.14), (6.16) converges ε-uniformly under conditions (4.16), (4.17). For the solutions of scheme (6.14), (6.13), and also schemes (6.14), (6.16), (6.17) and (6.14), (6.16), (6.19) the estimates (6.15) and (6.18) are valid respectively.*

Remark. It follows from Theorem 12 that the difference scheme (6.14), (6.16) converges ε-uniformly in the case of a Dirichlet problem for Eqs. (5.7a), (5.7b) and (5.10).

6.6. Let us study the ε-uniform convergence of scheme (6.14), (6.16) in that case if (4.16) is fulfilled; the fulfilment of (4.17) is not assumed.

Taking into account (6.7), we verify that scheme (6.14), (6.16) converges for $N, N_0 \to \infty$, $(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^{-1} \to 0$. From (6.9) and (6.11) we find, respectively, that this scheme converges under condition $N, N_0 \to \infty$, $\varepsilon_1(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-1} \to 0$, if the parameter ε_2 is fixed, and under condition $N, N_0 \to \infty$, $\varepsilon_2(\varepsilon_1+\varepsilon_2+\varepsilon_3)^{-1} \to 0$, if the parameter ε_1 is fixed. In addition, the scheme converges for fixed values of the parameters ε_1 and ε_2 (see (6.15)). From here it follows that scheme (6.14), (6.16) converges ε-uniformly under condition (4.16).

The following estimate is valid for the solution of scheme (6.14), (6.16), (6.19), namely:

$$\begin{aligned} |u(x,t) - z(x,t)| &\le M\left[\left(N^{-1}\ln N\right)^{2/3} + \varepsilon_2^{-2}(\varepsilon_1+\varepsilon_2+\varepsilon_3)^2 N_0^{-1}\ln N_0\right], \\ |u(x,t) - z(x,t)| &\le M\left[\varepsilon_1^{-1}(\varepsilon_1+\varepsilon_2+\varepsilon_3)N^{-1}\ln N + \left(N_0^{-1}\ln N_0\right)^{1/2}\right], \\ |u(x,t) - z(x,t)| &\le M\left[\left(N^{-1}\ln N\right)^{2/3} + \left(N_0^{-1}\ln N_0\right)^{1/2}\right], \end{aligned}$$

$$(x,t) \in \overline{G}_h. \qquad (6.20)$$

Theorem 13. *Let the components of the solution for the boundary value problem (2.4), (2.5) satisfy estimates (6.7), (6.9) and (6.11) with $K = 6$ under conditions (4.16), (4.20); (4.16), (4.25) and (4.16), (4.28) respectively. Then the solution of the difference scheme (6.14), (6.16) converges to the solution of the boundary value problem ε-uniformly under condition (4.16). For the solution of scheme (6.14), (6.16), (6.19) estimates (6.20) are valid.*

Remark. It follows from Theorem 13 that scheme (6.14), (6.16) converges $(\overline{\varepsilon}_1, \overline{\varepsilon}_2)$ – uniformly in the case of a Dirichlet problem for Eq. (5.15).

6.7. The fact that scheme (6.14), (6.16) converges ε-uniformly under condition (4.7) (see Subsection 6.4) and under condition (4.16) (see Subsection 6.6) implies ε-uniform convergence of this scheme. Under the condition (6.21)

$$M_1^s > \left(m^0_{1(4.19)}\right)^{-1}, \quad s = 1, 2, \quad M_2 > \max\left[\left(m^0_{(6.1)}\right)^{-1}, \left(m^0_{2(4.19)}\right)^{-1}\right],$$

estimates (6.20) hold for the solution of scheme (6.14), (6.16).

Theorem 14. *Let the solution of the boundary value problem (2.4), (2.5) and its components satisfy the estimates of Theorem 11 for $K = 6$. Then the solution of the difference scheme (6.14), (6.16) converges to the solution of the boundary value problem ε-uniformly. For the solution of the difference scheme (6.14), (6.16), (6.21) the estimates (6.20) and also, under conditions (4.7) or (4.16) and (4.17), estimate (6.18) are valid.*

7. Remarks and generalizations

7.1. The conditions (4.2b), (4.23) for problem (2.2), (2.1) are sufficiently restrictive. These conditions are fulfilled, for example, if the functions $f(x,t)$, $\varphi(x,t)$ vanish in the neighbourhood of the set γ_0. In that case when the compatibility conditions on the set γ_0, except (4.2a), are not fulfilled (under these assumptions, generally speaking, $u \notin C^{2,1}(\overline{G})$; (4.2b) and (4.23) are not satisfied), difference schemes (5.2), (5.1) and (5.2), (5.4) do converge, respectively, for fixed values of the parameters ε_i, $i = 1, 2, 3$ and ε-uniformly. However, the accuracy of the discrete solutions deteriorates (the analysis of their convergence is to be done with the use of the technique given in [11, Chapter II].)

In the case of problem (2.4), (2.5) the convergence of scheme (6.14), (6.13) for fixed values of the parameters and the ε-uniform convergence of scheme (6.14), (6.16) hold also if (4.2b), (6.12) are violated.

7.2. The issues arising in the numerical solution of singularly perturbed equations with (small) parameters by classical finite difference methods remain urgent in that case if finite element or finite volume methods [6] is used for the construction of the schemes.

7.3. The results obtained for linear equations allow us to construct ε-uniformly convergent finite difference scheme in the case of quasilinear equations. On the strip $\overline{D}_{(2.1)}$, we consider the quasilinear singularly perturbed parabolic equation

$$
(7.1) \qquad \begin{aligned} L_{(2.2)}u(x,t) &= (\varepsilon_1+\varepsilon_2+\varepsilon_3)^2 g(x,t,u(x,t)), \quad (x,t)\in G,\\ u(x,t) &= \varphi(x,t), \quad (x,t)\in S. \end{aligned}
$$

The function $g(x,t,u)$ is sufficiently smooth on the set $\overline{G}\times R$: $g\in C^{l+\alpha}(\overline{G}\times R)$, $l\geq K\geq 6$, $\alpha>0$, moreover, $-M\tilde{\varepsilon}_2^2 \leq (\partial/\partial u)g(x,t,u)<\infty$, $(x,t,u)\in\overline{G}\times R$. The coefficients of $L_{(2.2)}$ and the boundary condition satisfy the conditions mentioned for problem (2.2), (2.1).

The estimates for the solution of problem (7.1), (2.1) are similar to those in the case of problem (2.2), (2.1). To problem (7.1), (2.1) we put in correspondence the finite difference scheme

$$
(7.2) \qquad \begin{aligned} \Lambda_{(5.2)}z(x,t) &= (\varepsilon_1+\varepsilon_2+\varepsilon_3)^2 g(x,t,z(x,t)), \quad (x,t)\in G_h,\\ z(x,t) &= \varphi(x,t), \quad (x,t)\in S_h. \end{aligned}
$$

Here $\overline{G}_h$ is one of the grids $\overline{G}_{h(5.1)}$ or $\overline{G}_{h(5.4)}$.

The solutions of schemes (7.2), (5.1) and (7.2), (5.4) converge to the solution of problem (7.1), (2.1), respectively, for fixed values of the parameters ε_i, $i=1,2,3$ and ε-uniformly. The assertions similar to Theorems 7–10 are valid for the solutions of these problems. The special schemes for problem (7.1), (2.5) are being constructed in a similar way. Schemes (7.2), (5.1) and (7.2), (5.4) are nonlinear; for their solution one can use iterative algorithms developed in [9, 10].

7.4. The techniques examined in this work allow one to construct special finite difference schemes for singular perturbation problems with mixed boundary conditions and also with concentrated factors (e.g., sources; see [9, Chapter VII].)

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