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Oscillation of Solutions of Impulsive Nonlinear Hyperbolic Differential-Difference Equations

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Sufficient conditions for oscillation of the solutions of impulsive nonlinear hyperbolic differential-difference equations are obtained.

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1. Introduction

The theory of impulsive partial differential equations (PDE) marked rapid development in the last years [1]–[7]. The impulsive PDE can be successfully used for mathematical simulation in theoretical physics [9], population dynamics [6], optimal control [8] and in other processes and phenomena in science and technology.

The present paper is concerned with the oscillation of solutions of impulsive nonlinear hyperbolic differential-difference equations subject to certain boundary conditions. The oscillation properties of the solutions are investigated via averaging technique.

2. Preliminary notes

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. Suppose that $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ are given numbers and $t_{k+l} = t_k + \sigma$, $k = 0, 1, \ldots$, where $\sigma = \text{const} > 0$ and l is a fixed natural number.

Define $J_{imp} = \{t_k\}_{k=1}^{\infty}$, $\mathbb{R}_+ = [0, +\infty)$, $E^0 = [-\sigma, 0] \times \overline{\Omega}$, $E = (0, +\infty) \times \Omega$, $E^* = \mathbb{R}_+ \times \overline{\Omega}$, $E_{imp} = \{(t, x) \in E : t \in J_{imp}\}$, $E^*_{imp} = \{(t, x) \in E^* : t \in J_{imp}\}$.

Let $C_{imp}[E^0 \cup E^*, \mathbb{R}]$ be the class of all functions $u : E^0 \cup E^* \to \mathbb{R}$ such that:

(i) The restriction of u to the set $E^0 \cup E^* \setminus E^*_{imp}$ is a continuous function.

(ii) For each $(t, x) \in E^*_{imp}$ there exist the limits

$$\lim_{\substack{(q,s)\to(t,x)\\ q<t}} u(q, s) = u(t^-, x), \qquad \lim_{\substack{(q,s)\to(t,x)\\ q<t}} u(q, s) = u(t^+, x)$$

and $u(t, x) = u(t^+, x)$ for $(t, x) \in E^*_{imp}$.

The class of functions $C_{imp}[E^*, \mathbb{R}]$ is defined analogously as E^* is written instead of $E^0 \cup E^*$ in the above definition.

Let $C^t_{imp}[E^0 \cup E^*, \mathbb{R}]$ be the class of all functions $u \in C_{imp}[E^0 \cup E^*, \mathbb{R}]$ such that:

(i) u_t: $E^* \setminus E^*_{imp} \to \mathbb{R}$ and it is a continuous function.

(ii) For each $(t, x) \in E^*_{imp}$ there exist the limits

$$\lim_{\substack{(q,s)\to(t,x)\\ q<t}} u_t(q, s) = u_t(t^-, x), \qquad \lim_{\substack{(q,s)\to(t,x)\\ q<t}} u_t(q, s) = u_t(t^+, x)$$

and $u_t(t, x) = u_t(t^+, x)$ for $(t, x) \in E^*_{imp}$.

Consider the nonlinear hyperbolic differential-difference equation

$$(1) \qquad u_{tt}(t, x) - \Delta u(t, x) + p(t, x) f(u(t - \sigma, x)) = 0, \quad (t, x) \in E \setminus E_{imp},$$

subject to the impulsive conditions

$$(2) \qquad u(t, x) - u(t^-, x) = g(t, x, u(t^-, x)), \quad (t, x) \in E^*_{imp},$$

$$(3) \qquad u_t(t, x) - u_t(t^-, x) = h(t, x, u_t(t^-, x)), \quad (t, x) \in E^*_{imp}$$

and the boundary conditions

$$(4) \qquad \frac{\partial u}{\partial n}(t, x) + \gamma(t, x) u(t, x) = 0, \qquad (t, x) \in (\mathbb{R}_+ \setminus J_{imp}) \times \partial\Omega,$$

or

$$(5) \qquad u(t, x) = 0, \qquad (t, x) \in (\mathbb{R}_+ \setminus J_{imp}) \times \partial\Omega.$$

The functions $p: E^* \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $g: E^*_{imp} \times \mathbb{R} \to \mathbb{R}$, $h: E^*_{imp} \times \mathbb{R} \to \mathbb{R}$, $\gamma: \mathbb{R}_+ \times \partial\Omega \to \mathbb{R}$ are given.

Definition 1. The function $u: E^0 \cup E^* \to \mathbb{R}$ is called a solution of problem (1) – (4) ((1)–(3), (5)) if:

(i) $u \in C^t_{imp}[E^0 \cup E^*, \mathbb{R}]$, there exist the derivatives $u_{tt}(t, x)$, $u_{x_i x_i}(t, x)$, $i = 1, \ldots, n$ for $(t, x) \in E \setminus E_{imp}$ and u satisfies (1) on $E \setminus E_{imp}$.

(ii) u satisfies (2)–(4) ((2), (3), (5)).

Definition 2. The nonzero solution $u(t,x)$ of equation (1) is said to be nonoscillating if there exists a number $\mu \geq 0$ such that $u(t,x)$ has a constant sign for $(t,x) \in [\mu, +\infty) \times \Omega$.

Otherwise, the solution is said to oscillate.

For the function sign the following definition is adopted:

$$\text{sign}x = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

Introduce the following assumptions:

H1. $p \in C_{imp}[E^*, \mathbb{R}_+]$.

H2. $g(t_k, x, \xi) = L_k\xi$, $h(t_k, x, \xi) = L_k\xi$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}$, $k = 1, 2, \ldots$, $L_k \geq 0$ are constants.

H3. $\gamma \in C_{imp}[\mathbb{R}_+ \times \partial\Omega, \mathbb{R}_+]$.

H4. $f \in C(\mathbb{R}, \mathbb{R})$, $f(u) = -f(-u)$ for $u \geq 0$, f is a positive amd convex function in the interval $(0, +\infty)$.

In the sequel the following notations will be used:

$$P(t) = \min\{p(t,x) \colon x \in \overline{\Omega}\},$$

$$V(t) = \int_\Omega u(t,x)dx \left(\int_\Omega dx \right)^{-1}.$$

3. Main results

We give sufficient conditions for oscillation of the solutions of problem (1)–(4).

Lemma 1. *Let the following conditions hold:*

1. *Assumptions* H1–H4 *are fulfilled.*

2. $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *is a positive solution of the problem* (1) – (4) *in the domain* E.

Then the function $V(t)$ *satisfies for* $t \geq \sigma$ *the impulsive differential inequality*

(6) $$V''(t) + P(t)f(V(t-\sigma)) \leq 0, \quad t \neq t_k,$$

(7) $$V(t_k) = (1 + L_k)V(t_k^-),$$

(8) $$V'(t_k) = (1 + L_k)V'(t_k^-).$$

Proof. Let $t \geq \sigma$. Integrating equation (1) with respect to x over the domain Ω, we obtain

$$\frac{d^2}{dt^2}\int_\Omega u(t,x)dx - \int_\Omega \Delta u(t,x)dx \tag{9}$$

$$+\int_\Omega p(t,x)f(u(t-\sigma,x))dx = 0, \quad t \neq t_k.$$

From the Green formula and H3 it follows that

$$\int_\Omega \Delta u(t,x)dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS = -\int_{\partial\Omega} \gamma(t,x)u(t,x)dS \leq 0, \quad t \neq t_k. \tag{10}$$

Moreover, for $t \neq t_k$, the Jensen inequality enables us to get

$$\int_\Omega p(t,x)f(u(t-\sigma,x))dx \geq P(t)\int_\Omega f(u(t-\sigma,x))dx \tag{11}$$

$$\geq P(t)f\left(\int_\Omega u(t-\sigma,x)dx\left(\int_\Omega dx\right)^{-1}\right)\int_\Omega dx = P(t)f(V(t-\sigma))\int_\Omega dx.$$

In virtue of (10) and (11) we obtain from (9) that

$$V''(t) + P(t)f(V(t-\sigma)) \leq 0, \qquad t \neq t_k.$$

For $t = t_k$ we have that

$$V(t_k) - V(t_k^-) = L_k\left(\int_\Omega dx\right)^{-1}\int_\Omega u(t_k^-,x)dx = L_kV(t_k^-),$$

that is,

$$V(t_k) = (1+L_k)V(t_k^-),$$

and analogously

$$V'(t_k) = (1+L_k)V'(t_k^-).$$

■

Definition 3. The solution $V \in C^t_{imp}[[-\sigma,0]\cup\mathbb{R}_+,\mathbb{R}]\cap C^2(\cup_{k=0}^{\infty}(t_k,t_{k+1}),\mathbb{R})$ of the differential inequality (6)–(8) is called eventually positive (negative) if there exists a number $t^* \geq 0$ such that $V(t) > 0$ $(V(t) < 0)$ for $t \geq t^*$.

Theorem 1. *Let the following conditions hold:*

1. *Assumptions* H1–H4 *are fulfilled.*

2. *The differential inequality* (6)–(8) *has no eventually positive solutions.*

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of problem* (1) – (4) *oscillates in the domain* E.

Proof. Suppose the conclusion of the theorem is not true, i.e., $u(t,x)$ is a nonzero solution of problem (1)–(4) which is of the class $C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ and it has a constant sign in the domain $E_\mu = [\mu, +\infty) \times \Omega$, $\mu \geq 0$. Without loss of generality we may assume that $u(t,x) > 0$ in E_μ. Then from Lemma 1 it follows that the function $V(t)$ is a positive solution of the differential inequality (6)–(8) for $t \geq \mu + \sigma$ which contradicts condition 2 of the theorem. ■

Theorem 2. *Let the following conditions hold:*

1. $P \in C_{imp}[\mathbb{R}_+, \mathbb{R}_+]$, $\int\limits_{t^*}^{\infty} P(\tau)d\tau = +\infty$ *for each* $t^* \geq 0$.

2. $\sum\limits_{k=1}^{\infty} L_k < +\infty$, $L_k \geq 0$, $k = 1, 2, \ldots$, *are constants.*

3. $f(u) \geq Mu$, $u \geq 0$, $M > 0$ *is a constant.*

Then the differential inequality (6)–(8) *has no eventually positive solutions.*

Proof. Suppose that the conclusion of the theorem is not true and let $V(t)$ be a positive solution of differential inequality (6)–(8) in the interval $[t^*, +\infty)$, $t^* \geq 0$. Then we have for $t \geq t^* + \sigma$

$$V''(t) + MP(t)V(t - \sigma) \leq 0, \quad t \neq t_k,$$
$$V(t_k) = (1 + L_k)V(t_k^-),$$
$$V'(t_k) = (1 + L_k)V'(t_k^-),$$

and since $V''(t) \leq 0$, $t \neq t_k$, $t \geq t^* + \sigma$, we obtain for each $\tilde{t}_1 \geq t^* + \sigma$ that

$$V'(t) \leq \prod_{\tilde{t}_1 < t_k \leq t} (1 + L_k)V'(\tilde{t}_1). \tag{12}$$

We will prove the inequality

$$V'(t) \geq 0 \quad \text{for} \quad t \geq t^* + \sigma. \tag{13}$$

Suppose that there exists a number $\tilde{t}_2 \geq t^* + \sigma$ such that $V'(\tilde{t}_2) = -c < 0$. Then for any $t \geq \tilde{t}_2$ the following inequality holds

$$V'(t) \leq -\prod_{\tilde{t}_2 < t_k \leq t} (1 + L_k)c.$$

Integrating the last inequality over the interval $[\tilde{t}_2, t]$, we get

$$V(t) \leq \prod_{\tilde{t}_2 < t_k \leq t} (1 + L_k)[V(\tilde{t}_2) - c(t - \tilde{t}_2)],$$

which leads to a contradiction with the fact that $V(t)$ is eventually positive. Therefore (13) holds and it implies that

$$\prod_{t^* + \sigma < t_k \leq t} (1 + L_k) V(t^* + \sigma) \leq V(t). \tag{14}$$

Direct calculation gives us

$$V'(t) \leq \prod_{t^* + \sigma < t_k \leq t} (1 + L_k) V'(t^* + \sigma) - M \int_{t^* + \sigma}^{t} \prod_{\tau < t_k \leq t} (1 + L_k) P(\tau) V(\tau - \sigma) d\tau.$$

From (14), conditions 1 and 2 of the theorem we conclude that for $t \geq t^* + 2\sigma$

$$V'(t) - \prod_{t^* + \sigma < t_k \leq t} (1 + L_k) V'(t^* + \sigma) + \tag{15}$$

$$+MV(t^* + \sigma) \left(\int_{t^* + 2\sigma}^{t} \frac{P(\tau) d\tau}{\prod_{\tau - \sigma < t_k \leq \tau} (1 + L_k)} \right) \prod_{t^* + \sigma < t_k \leq t} (1 + L_k) \leq 0.$$

Inequality (15) implies in virtue of (12) that

$$\int_{t^* + 2\sigma}^{\infty} \frac{P(\tau) d\tau}{\prod_{\tau - \sigma < t_k \leq \tau} (1 + L_k)} < +\infty$$

and consequently

$$\int_{t^* + 2\sigma}^{\infty} P(\tau) d\tau < +\infty,$$

which is a contradiction. ■

Corollary 1. *Let the following conditions hold:*

1. *Assumptions* H1–H4 *are fulfilled.*
2. $\sum_{k=1}^{\infty} L_k < +\infty$.
3. $\int_{t^*}^{\infty} P(\tau)d\tau = +\infty$ *for each* $t^* \geq 0$.
4. $f(u) \geq Mu,\ u \geq 0,\ M > 0$ *is a constant.*

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of problem* (1) – (4) *oscillates in the domain* E.

Corollary 1 follows from Theorem 1 and Theorem 2.

Now we give sufficient conditions for oscillation of the solutions of problem (1)–(3), (5). Consider the following Dirichlet problem

$$\begin{aligned} &\Delta\varphi + \alpha\varphi = 0 \quad \text{in} \quad \Omega, \\ &\varphi|_{\partial\Omega} = 0, \end{aligned} \tag{16}$$

where $\alpha = \text{const}$. It is known that the smallest eigenvalue α_0 of the problem (16) is positive and the corresponding eigenfunction $\varphi_0(x) > 0$ for $x \in \Omega$. Without loss of generality we may assume that φ_0 is normalized, i.e., $\int_{\Omega} \varphi_0(x)dx = 1$.

Introduce the notation:

$$W(t) = \int_{\Omega} u(t,x)\varphi_0(x)dx.$$

Lemma 2. *Let the following conditions hold:*

1. *Assumptions* H1, H2, H4 *are fulfilled.*
2. $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *is a positive solution of the problem* (1)–(3), (5) *in the domain* E.

Then the function $W(t)$ *satisfies for* $t \geq \sigma$ *the impulsive differential inequality*

$$W''(t) + \alpha_0 W(t) + P(t)f(W(t-\sigma)) \leq 0, \quad t \neq t_k, \tag{17}$$

$$W(t_k) = (1 + L_k)W(t_k^-), \tag{18}$$

$$W'(t_k) = (1 + L_k)W'(t_k^-). \tag{19}$$

P r o o f. Let $t \geq \sigma$. We multiply both sides of equation (1) by the eigenfunction $\varphi_0(x)$ and integrating with respect to x over Ω, we obtain

$$\frac{d^2}{dt^2}\int_\Omega u(t,x)\varphi_0(x)dx - \int_\Omega \Delta u(t,x)\varphi_0(x)dx + \int_\Omega p(t,x)f(u(t-\sigma,x))\varphi_0(x)dx = 0, \quad t \neq t_k. \tag{20}$$

From the Green formula it follows that

$$\int_\Omega \Delta u(t,x)\varphi_0(x)dx = \int_\Omega u(t,x)\Delta\varphi_0(x)dx = -\alpha_0\int_\Omega u(t,x)\varphi_0(x)dx = -\alpha_0 W(t), \quad t \neq t_k, \tag{21}$$

where $\alpha_0 > 0$ is the smallest eigenvalue of the problem (16).

Moreover, from the Jensen inequality

$$\int_\Omega p(t,x)f(u(t-\sigma,x))\varphi_0(x)dx \geq P(t)\int_\Omega f(u(t-\sigma,x))\varphi_0(x)dx \geq P(t)f\left(\int_\Omega u(t-\sigma,x)\varphi_0(x)dx\right) = P(t)f(W(t-\sigma)), \quad t \neq t_k. \tag{22}$$

Making use of (21) and (22), we obtain from (20) that

$$W''(t) + \alpha_0 W(t) + P(t)f(W(t-\sigma)) \leq 0, \qquad t \neq t_k.$$

For $t = t_k$ we have that

$$W(t_k) - W(t_k^-) = L_k\int_\Omega u(t_k^-,x)\varphi_0(x)dx = L_k W(t_k^-),$$

that is,

$$W(t_k) = (1 + L_k)W(t_k^-),$$

and analogously,

$$W'(t_k) = (1 + L_k)W'(t_k^-).$$

Analogously to Theorem 1 we can prove the following theorem:

Theorem 3. *Let the following conditions hold:*

1. *Assumptions* H1, H2, H4 *are fulfilled.*
2. *The differential inequality* (17)–(19) *has no eventually positive solutions.*

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of problem* (1)–(3), (5) *oscillates in the domain* E.

Theorem 4. *Let the following conditions hold:*

1. $P \in C_{imp}[\mathbb{R}_+, \mathbb{R}_+]$.
2. $\sum_{k=1}^{\infty} L_k < +\infty$, $L_k \geq 0$, $k = 1, 2, \ldots$, *are constants.*
3. $f(u) \geq 0$ *for* $u \geq 0$.

Then the differential inequality (17)–(19) *has no eventually positive solutions.*

The proof of Theorem 4 is analogous to the proof of Theorem 2. It is omitted here.

Corollary 2. *Let the following conditions hold:*

1. *Assumptions* H1, H2, H4 *are fulfilled.*
2. $\sum_{k=1}^{\infty} L_k < +\infty$.

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of problem* (1)–(3), (5) *oscillates in the domain* E.

Corollary 2 follows from Theorem 3 and Theorem 4.

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