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On the Root Function Expansion of a Nonlocal First Order Integro-Differential Operator [1]

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We consider the root function expansion of the spectral problem related to the nonlocal first order integro-differential operator: $y'(t)+\int_0^t V(t,u)y(u)du = \lambda y(t)$, $N(y)=0$, where N is a continuous linear functional in $C[-a,a]$. By certain assumptions imposed on the functional N, some theorems for the spectrum of this problem and for the uniqueness its of the root function expansion are proved. A theorem for a recovery of the functions of $L^1[-a,a]$ by means of their coefficients in this expansion is proved. The convolutional structure of of the root function expansion is studied. The question concerning the completeness of the root functions in some functional spaces is discussed as well.

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Consider the integro-differential expression

$$ly = y'(t) + \int_0^t V(t,u)y(u)du, \quad t \in [-a,a] \tag{1}$$

where $V(t,u) \in C(G), G = \{(t,u): -a \le t \le u \le 0\} \cup \{(t,u): 0 \le u \le t \le a\}$. Let $N(f) = \int_{-a}^{a} f\,dn$ with $n \in BV[-a,a]$ be arbitrary nonzero continuous linear functional in $C[-a,a]$ and let $C_N[-a,a] \stackrel{\text{def}}{=} \{f \in C[-a,a] : N(f) = 0\}$..

The *nonlocal Volterra integro-differential operator of first order* is said to be the operator D generated by the expression (1) in the space $X = L^1[-a,a]$ with domain $X_D = AC_N[-a,a] \stackrel{\text{def}}{=} \{f \in AC[-a,a] : N(f) = 0\}$. The present

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paper is devoted to the spectrum and root function expansion of the operator D. A necessary and sufficient condition for the validity of a uniqueness theorem for the root function expansion is found. A necessary and sufficient condition for a recovery of the functions of $L^1[-a,a]$ by means of their coefficients in the root function expansion is found as well. Also we give sufficient conditions for the completeness of the root functions in the space $L^p[-a,a]$, $1 \le p < \infty$. The convulutional structure of the operator D and its root function expansion is studied too.

Let $y(\lambda,t)$ be the solution of the problem $ly = \lambda y, y(\lambda,0) = 1$. Also, for each $f \in L^1[-a,a]$ and each $\lambda \in C$ let $\eta = R_\lambda^{(0)} f \in AC[-a,a]$ be the unique solution of the Cauchy problem $l\eta - \lambda\eta = f, \eta(0) = 0$. (The existence and uniqueness of the solutions of these problems is trivially proved reducing them to Volterra integral equations of second kind. It is clear that "the initial resolvent" $R_\lambda^{(0)}$ maps the space $L^1[-a,a]$ in $AC[-a,a]$ and that for each $f \in L^1[-a,a]$ the functions $R_\lambda^{(0)} f$ and $y(\lambda,t)$ are $C \to C[-a,a]$ entire functions of λ.)

It is easily proved that the operator D has a point spectrum $\sigma(D)$, which coincides with the set of zeroes $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$ of the entire function $E(\lambda) = N_t\{y(\lambda,t)\}$, if zeros exist, and let $\{m_1, m_2, m_3, \dots\}$ be their corresponding multiplicities. To each zero λ_k one may correspond the one-dimensional eigensubspace generated by the eigenfunction $y(\lambda_k,t)$.

For each λ of the resolvent set $\rho(D) = C \setminus \sigma(D) = \{\lambda \in C : E(\lambda) \ne 0\}$ the resolvent $R_\lambda = (D - \lambda I)^{-1}$ of the operator D is represented by the equality

$$(2) \qquad R_\lambda f = R_\lambda^{(0)} f - \frac{N\{R_\lambda^{(0)} f\}}{E(\lambda)} y(\lambda,t), \quad f \in L^1[-a,a].$$

So for each λ_k the corresponding root projection $P_{\lambda_k} = -\frac{1}{2\pi i} \int\limits_{\Gamma_k} R_\lambda \, d\lambda$ has the form

$$(3) \qquad P_{\lambda_k} f = \frac{1}{2\pi i} \int\limits_{\Gamma_k} \frac{y(\lambda,t) N\{R_\lambda^{(0)} f\}}{E(\lambda)} \, d\lambda, \quad f \in L^1[-a,a],$$

and it maps the space $L^1[-a,a]$ on the m_k-dimensional root subspace $\Pi_{\lambda_k} = \ker(D - \lambda_k I)^{m_k}$ of the operator D corresponding to λ_k. (Here Γ_k is a circle with center λ_k enclosing only the eigenvalue λ_k among the all eigenvalues of the operator D.) The root subspace Π_{λ_k} is generated by the basis of root functions (eigen and associated functions)

$$\left\{ \frac{1}{s!} \frac{\partial^s}{\partial \lambda^s} y(\lambda_k, t) : \; 0 \le s \le m_k - 1 \right\}$$

of the operator D and the projection P_{λ_k} is represented with respect to this basis in the form

$$(4) \qquad P_{\lambda_k} f = \sum_{s=0}^{m_k-1} A^k_{m_k-1-s}(f)\frac{1}{s!}\frac{\partial^s}{\partial\lambda^s} y(\lambda_k, t), \qquad f \in L^1[-a,a],$$

where

$$(5) \qquad A^k_s(f) = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{(\lambda-\lambda_k)^{m_k-1-s} N\{R^{(0)}_\lambda f\}}{E(\lambda)}\,d\lambda, \qquad f \in L^1[-a,a],$$

$0 \le s \le m_k - 1$, $k = 0, 1, 2, \dots$ are the coefficient functionals of the projection P_{λ_k} with respect to this basis. Let $\{P_{\lambda_k}\}_{k=0}^{\infty}$ be the orthogonal projection system of the root projections of the operator D (i.e. $P_{\lambda_k}P_{\lambda_s} = 0$ for $k \ne s$).

Theorem 1. *The spectrum $\sigma(D)$ of the nonlocal Volterra integro-differential operator of first order operator D is either empty or infinite countable set. If $-a, a \in \operatorname{supp} N$, i.e. if $-a, a$ are points of variability of the representing function $n \in BV[-a,a]$ of the functional N, then the spectrum $\sigma(D)$ of the operator D is an infinite countable set.*

Proof. To prove the theorem as well as in the following considerations, we use a transmutation operator of the form

$$(6) \qquad Tf = f(t) + \int_0^t K(t,u) f(u)\,du$$

considered in [2] – [4], which transforms the operator D into the nonlocal operator $\tilde{D} = d/dt$ considered in $L^1[-a,a]$ with domain $X_{\tilde{D}} = \{f \in AC[-a,a] : \tilde{N}(f) = 0\}$, where the functional $\tilde{N} = N \circ T^{-1}$ and the inverse transmutation operator T^{-1} is a second kind Volterra integral operator as well.

The transmutation operator T is a continuous authomorphism of the space $L^1[-a,a]$ and its restriction is a continuous authomorphism of the corresponding space $X = L^p[-a,a]$, $1 \le p \le \infty$; $C[-a,a]$; $BV[-a,a]$; $AC[-a,a]$ and the equality

$$(7) \qquad lTf = T\frac{d}{dt}f \text{ holds for each } f \in AC[-a,a].$$

The operator T transforms the integro-differential operator D into the operator $\tilde{D} \overset{\text{def}}{=} d/dt$ considered in $L^1[-a,a]$ with domain

$$X_{\tilde{D}} = \{f \in AC[-a,a] : \tilde{N}(f) = 0\},$$

where $\tilde{N} \stackrel{\text{def}}{=} N \circ T^{-1} \in C^*[-a,a]$, i.e.

(8) $$T : L^1 \to L^1, \quad T(X_D) = X_{\tilde{D}}, \quad \text{and} \quad \tilde{D}T = TD \quad \text{in} \quad X_{\tilde{D}}.$$

The equality

(9) $$y(\lambda, t) = T_t^{-1}(e^{\lambda t}), \quad e^{\lambda t} = Ty(\lambda, t)$$

holds as well. The transmutation operator T allows us to reduce the nonlocal problem $Dy = \lambda y, \quad N(y) = 0$ to the corresponding nonlocal problem

(10) $$y' = \lambda y, \quad \tilde{N}(y) = 0$$

on the segment [-a,a] and from the equality (9) we get immediately that the entire function $E(\lambda) \stackrel{\text{def}}{=} N_t[y(\lambda,t)] = N[T^{-1}(e^{\lambda t})] = (N \circ T^{-1})[e^{\lambda t}] = \tilde{N}_t[e^{\lambda t}]$. These equalities show that both problems have one and the same spectrum, i.e. $\sigma(D) = \sigma(\tilde{D})$.

The nonlocal problem (10) is considered by many authors but in the following we essentially use the results of L.Schwartz [5], A.F.Leont'ev [6], A.M.Sedletzkii [7] and J.I.Ljubič [10].

Especially from a theorem of [6], pp. 106 – 110 we get that the spectrum of the problem (10) is either empty or infinite countable set. The spectrum of the problem (10) is empty if and only if the support of the functional $\tilde{N}$ consists of only one point. In the opposite case (if and only if the support of the functional $\tilde{N}$ contains at least two different points) the spectrum of the problem (10) is infinite countable set. It is clear that the last case is true, if and only if the representing function $\tilde{n} \in BV[-a,a]$ in the Riesz representation of the functional $\tilde{N}(f) = \int_{-a}^{a} f(u)\, d\tilde{n}(u)$, $f \in C[-a,a]$ has at least two different points of variability.

We shall prove that the endpoints $-a, a \in$ suppN, if and only if $-a, a \in$ supp$\tilde{\text{N}}$, i.e. $-a, a$ are points of variability of the representing function $n \in BV[-a,a]$ of the functional N, i.e. if and only if $-a, a$ are points of variability of the representing function $\tilde{n} \in BV[-a,a]$.

First we shall prove that $a \notin$ suppN, if and only if $a \notin$ supp$\tilde{\text{N}}$. Indeed, let $a \notin$ suppN. Then there is a neighbourhood of the form $(a-\delta, a]$ that $N(f) = 0$ for each $f \in C[-a,a]$ with suppf $\subset$ (a – δ, a] and from the Volterra form (6) of the transmutation operators T and T^{-1} it is clear that suppT^{-1}f $\subset$ (a – δ, a], hence $\tilde{N}(f) = N(T^{-1}f) = 0$. Therefore $a \notin$ supp$\tilde{\text{N}}$. The converse implication $a \notin$ supp$\tilde{\text{N}} \Longrightarrow a \notin$ suppN follows in the same way, since now the inclusion suppf $\subset$ (a – δ, a] implies the inclusion suppT^{-1}f $\subset$ (a – δ, a]. This shows that $a \in$ suppN, if and only if $a \in$ supp$\tilde{\text{N}}$. Analogously, it can be proved that $-a \in$ suppN, if and only if $-a \in$ supp$\tilde{\text{N}}$ and the last statement of the theorem will follow from previous remarks. ■

Other conditions, which imply the infinity of the spectrum can be formulated as well. However, the following theorems show that the case when $-a, a \in$ suppN is the most interesting for the investigation of the root function expansion, since this condition is necessary and sufficient a uniqueness theorem for this condition to be true, which gives a posiibility for a recovery of an arbitrary function of $L^1[-a,a]$ by means of its coefficients in the root function expansion of the operator D.

Now let $\sigma(D) = \{\lambda_k\}_{k=0}^{\infty}$ be an infinite countable set an suppose $\{P_{\lambda_k}\}_{k=0}^{\infty}$ be the *orthogonal* system of root projections of the operator D (i.e. $P_{\lambda_k} P_{\lambda_s} = 0$ for $k \neq s$). We say that this system is *total* in the space $L^1[-a,a]$, iff the equalities $P_{\lambda_k} f = 0,\ k = 0,1,2,\ldots$ for some $f \in L^1[-a,a]$ (i.e. $A_s^k(f) = 0,\ 0 \leq s \leq m_k - 1,\ k = 0,1,2,\ldots$) imply that $f = 0$ almost everywhere in $[-a,a]$. In other words, a uniqueness theorem for the root function expansion $f \sim \sum\limits_{k=0}^{\infty} P_{\lambda_k} f$ of the functions of $L^1[-a,a]$, i.e. for the formal expansion

$$(11) \qquad f \sim \sum_{k=0}^{\infty} \sum_{s=0}^{m_k-1} A_{m_k-1-s}^k(f) \frac{1}{s!} \frac{\partial^s}{\partial \lambda^s} y(\lambda_k, t),$$

on the root functions of the operator D is valid. Then the coefficient functionals A_s^k of this expansion generates the "traditional" integral transformation

$$(12) \qquad f \in L^1 \longrightarrow \tilde{f} = \{A_0^k(f), \ldots, A_{m_k-1}^k(f)\}_{k=0}^{\infty} \in \mathcal{X},$$

which is injective according to the totality and maps the space $L^1[-a,a]$ in the algebra $(\mathcal{X}, *_{\mathcal{X}})$ consisting of the cellular sequences of the form $\xi = \{\xi_0^k, \ldots, \xi_{m_k-1}^k\}_{k=0}^{\infty}$ provided with the inner Cauchy convolution $\xi *_{\mathcal{X}} \eta = \{\sum\limits_{i=0}^{s} \xi_{s-i}^k \eta_i^k : 0 \leq s \leq m_k - 1\}_{k=0}^{\infty}$ (see [1], pp.26 – 34).

Theorem 2. **(Uniqueness theorem)** *The condition $-a, a \in$ suppN is necessary and sufficient for the totality of the root projection system $\{P_{\lambda_k}\}_{k=0}^{\infty}$ in the space $L^1[-a,a]$, i.e. it is is necessary and sufficient a uniqueness theorem to be valid for the root function expansion* (11).

Proof. According to the proof of Theorem 1, the condition $-a, a \in$ suppN, is equivalent to the condition $-a, a \in$ supp$\tilde{N}$. Let $\tilde{R}_\lambda$ and $\tilde{R}_\lambda^{(0)}$ be respectively the resolvent and "the initial resolvent" of the operator $\tilde{D}$ in the space $L^1[-a,a]$ and it is clear that "the initial resolvent" is also an entire function of λ.

From the equality $\sigma(\tilde{D}) = \sigma(D)$ we proved above it follows that for each λ of the resolvent set $\rho(\tilde{D}) = \rho(D) = C \setminus \sigma(D) = \{\lambda \in C : E(\lambda) \neq 0\}$ the resolvent $\tilde{R}_\lambda = (\tilde{D} - \lambda I)^{-1}$ of the operator $\tilde{D}$ is represented by the equality

$$\tilde{R}_\lambda f = \tilde{R}_\lambda^{(0)} f - \frac{\tilde{N}\{\tilde{R}_\lambda^{(0)} f\}}{E(\lambda)} e^{\lambda t}, \quad f \in L^1[-a, a]. \tag{13}$$

From (7) and (8) it follows that

$$R_\lambda = T^{-1} \tilde{R}_\lambda T, \quad R_\lambda^{(0)} = T^{-1} \tilde{R}_\lambda^{(0)} T \tag{14}$$

in $L^1[-a, a]$. Then the last equalities imply the equality

$$P_{\lambda_k} = T^{-1} \tilde{P}_{\lambda_k} T \quad \text{in } L^1[-a, a], \tag{15}$$

where

$$\tilde{P}_{\lambda_k} = -\frac{1}{2\pi i} \int_{\Gamma_k} \tilde{R}_\lambda \, d\lambda, \quad f \in L^1[-a, a], \quad k = 0, 1, 2, \ldots \tag{16}$$

are the root projections of the operator $\tilde{D}$.

We shall prove that the root projecton system $\{P_{\lambda_k}\}_{k=0}^{\infty}$ is total in the space $L^1[-a, a]$, if and only if the system $\{\tilde{P}_{\lambda_k}\}_{k=0}^{\infty}$ is total in the space $L^1[-a, a]$. Indeed, suppose first that the system $\{\tilde{P}_{\lambda_k}\}_{k=0}^{\infty}$ is total in the space $L^1[-a, a]$ and let $P_{\lambda_k} f = 0$, $k = 0, 1, 2, \ldots$ for some $f \in L^1[-a, a]$. Then $P_{\lambda_k} f = T^{-1} \tilde{P}_{\lambda_k} T f = 0$ and since T^{-1} is an isomorphism of the space $L^1[-a, a]$ we get that $\tilde{P}_{\lambda_k} T f = 0$ and since the system $\{\tilde{P}_{\lambda_k}\}_{k=0}^{\infty}$ is total in the space $L^1[-a, a]$ we get that $Tf = 0$ almost everywhere in $[-a, a]$. But T is also an isomorphism of the space $L^1[-a, a]$. Therefore $f = 0$ almost everywhere in $[-a, a]$. This shows that the system $\{P_{\lambda_k}\}_{k=0}^{\infty}$ is total in the space $L^1[-a, a]$. The converse statement is proved in the same way.

Now let $-a, a \in \text{suppN}$. Then $-a, a \in \text{supp}\tilde{\text{N}}$ and according to a theorem of L.Schwartz–A.F.Leont'ev [5], [6] it follows the root projection system $\{\tilde{P}_{\lambda_k}\}_{k=0}^{\infty}$ is total in the space $L^1[-a, a]$. Hence the system $\{P_{\lambda_k}\}_{k=0}^{\infty}$ is total in the space $L^1[-a, a]$.

Conversely, let the system $\{P_{\lambda_k}\}_{k=0}^{\infty}$ be total in the space $L^1[-a, a]$. Then the system $\{\tilde{P}_{\lambda_k}\}_{k=0}^{\infty}$ is total in $L^1[-a, a]$ and according to our theorem proved in [1], p.199 we get $-a, a \in \text{supp}\tilde{\text{N}}$. Therefore $-a, a \in \text{suppN}$, which finishes the proof of the theorem. ■

Next theorem shows that the necessary and sufficient condition for totality $-a, a \in \text{suppN}$ is necessary and sufficient for recovery of an arbitrary function $f \in L^1[-a, a]$ by means of its coefficients $A_s^k(f)$, $0 \le s \le m_k - 1$, $k = 0, 1, 2, \ldots$, i.e. that the functions of $L^1[-a, a]$ can be reconstructed by their coefficients in the root function expansion, if and only if the uniqueness theorem is true for the root function expansion (11). Also, this theorem shows that the recovery can be made always using the Abel summability method and the mentioned transmutation operator T. We note that in the case of an arbitrary continuous function $V(t,\tau)$ the function $y(\lambda,t)$ is in general not holomorphic with respect to the variable t. This shows that in the next theorem the using of transmutation operator T cannot be avoided, as in the case $V(t,\tau)\equiv 0$, $D=d/dt$ considered in [5] – [7].

Theorem 3. **(Recovery theorem)** *Let* $-a,a\in \text{suppN}$ *and suppose that* $\sigma(D)=\{\lambda_k\}_{k=0}^{\infty}$ *be the spectrum of the operator* D *and* T *be the transmutation operator mentioned before. Let* $f\in L^1[-a,a]$ *and* $A_s^k(f)$, $0\le s\le m_k-1$, $k=0,1,2,\ldots$ *be the coefficients (5) of this function in the root function expansion (11). Then by means of these coefficients* $A_s^k(f)$ *for every* $\delta>0$ *one may construct two functions* $F^+(z)$, $F^-(z)$, *whose are limits of two sequences of Dirichlet quasipolynomials of the form* $\sum\limits_{k=0}^{p}(\zeta_0^k+\zeta_1^k t+\ldots+\zeta_{m_k-1}^k t^{m_k-1})e^{\lambda_k t}$ *and which coefficients* ζ_s^k, $0\le s\le m_k-1$, $k=0,1,2,\ldots$ *are finite linear combinations of the numbers* $A_s^k(f)$, $0\le s\le m_k-1$, $k=0,1,2,\ldots$ *and their exponents are the eigenvalues* λ_k, $k=0,1,2,\ldots$ *of the operator* D. *The functions* $F^+(z)$, $F^-(z)$ *are holomorphic functions in the open rectangles*

$$\Omega_\delta^+=\{z\in C:\ -a+\delta<\operatorname{Re} z<a-\delta,\ 0<\operatorname{Im} z<\omega_\delta^+\},$$
$$\Omega_\delta^-=\{z\in C:\ -a+\delta<\operatorname{Re} z<a-\delta,\ -\omega_\delta^-<\operatorname{Im} z<0\}$$

with sufficiently small $\omega_\delta^+>0$, $\omega_\delta^->0$ *and the equality*

$$(Tf)(x) = \lim_{y\to 0+0}[F^+(x+iy) + F^-(x-iy)] \tag{17}$$

holds for almost all $x\in[-a+\delta,a-\delta]$. *Also the equality*

$$f(t) = T_x^{-1}\{\lim_{y\to 0+0}[F^+(x+iy) + F^-(x-iy)]\} \tag{18}$$

holds for almost all $t\in[-a+\delta,a-\delta]$.

If $f\in C[-a,a]$, *then the convergence in* (18) *is uniform with respect to* $x\in[-a+\delta,a-\delta]$ *and the equality* (18) *holds for every* $t\in[-a+\delta,a-\delta]$.

Proof. From the equality (14) we get $\tilde N\{\tilde R_\lambda^{(0)}f\} = N\{T^{-1}\tilde R_\lambda^{(0)}f\} = N\{R_\lambda^{(0)}Tf\}$. Hence from the equalities (13) and (16) and we obtain that

$$\tilde P_{\lambda_k}f=\frac{1}{2\pi i}\int\limits_{\Gamma_k}\frac{e^{\lambda t}\tilde N\{\tilde R_\lambda^{(0)}f\}}{E(\lambda)}\,d\lambda=\frac{1}{2\pi i}\int\limits_{\Gamma_k}\frac{e^{\lambda t}N\{R_\lambda^{(0)}Tf\}}{E(\lambda)}\,d\lambda, \tag{19}$$

for $f \in L^1[-a,a]$ and $k = 0,1,2,\dots$ is the root projection of the operator $\tilde{D}$. The root projection maps the space $L^1[-a,a]$ on the m_k-dimensional root subspace $\tilde{H}_{\lambda_k} = \ker(\tilde{D} - \lambda_k I)^{m_k}$ of the operator $\tilde{D}$ corresponding to λ_k, which is generated by the basis of root functions $\left\{\frac{1}{s!}t^s e^{\lambda_k t} : 0 \le s \le m_k - 1\right\}$ of the operator $\tilde{D}$ and the projection $\tilde{P}_{\lambda_k}$ is represented with respect to this basis in the form

$$\tilde{P}_{\lambda_k} f = \sum_{s=0}^{m_k-1} \tilde{A}^k_{m_k-1-s}(f)\frac{1}{s!}t^s e^{\lambda_k t}, \qquad f \in L^1[-a,a], \tag{20}$$

where $\tilde{A}^k_s$, $0 \le s \le m_k - 1$, $k = 0,1,2,\dots$ are the coefficient functionals of the projection $\tilde{P}_{\lambda_k}$ with respect to this basis. Using (15), (20), (9) we obtain that the equalities

$$P_{\lambda_k} f = T^{-1}\tilde{P}_{\lambda_k} T f = \sum_{s=0}^{m_k-1} \tilde{A}^k_{m_k-1-s}(Tf)\frac{1}{s!}T^{-1}[t^s e^{\lambda_k t}]$$
$$= \sum_{s=0}^{m_k-1} \tilde{A}^k_{m_k-1-s}(Tf)\frac{1}{s!}\frac{\partial^s}{\partial\lambda^s}T^{-1}[e^{\lambda t}]\Big|_{\lambda=\lambda_k} = \sum_{s=0}^{m_k-1} \tilde{A}^k_{m_k-1-s}(Tf)\frac{1}{s!}\frac{\partial^s}{\partial\lambda^s}y(\lambda_k, t$$

hold for $f \in L^1[-a,a]$. Then from (4) we get that

$$\tilde{A}^k_s(Tf) = A^k_s(f) \quad \text{for} \quad f \in L^1[-a,a]. \tag{21}$$

Hence the statement of the theorem follows from a theorem due to L. Schwartz [5], A.F.Leont'ev [6], pp.435 - 447. A convenient formulation of this theorem can be found in [7] or in [1], p.198. ■

We note that the nonlocal operator D generated by the expression (1) by the same nonlocal boundary value condition can be considered in the spaces $Y = L^p[-a,a], 1 \le p \le \infty$ or $Y = C[-a,a]$; $BV[-a,a]$ with domain $X_{D,Y} = \{f \in AC[-a,a] : f' \in Y, N(f) = 0\}$. Now the resolvent R_λ, the root projections P_{λ_k} etc. are represented by the same formulae (2)–(12) but with $f \in Y$. Next theorem studies the convolutional structure of the operator D of the root function expansion (11) and of the integral transformation (12).

Let $*_{\tilde{N}}$ be the Berg–Dimovski convolution

$$f *_{\tilde{N}} g = \tilde{N}_x\left\{\int_t^x f(x+t-\tau)g(\tau)\,d\tau\right\}, \quad f,g \in L^1[-a,a]$$

generated by the functional $\tilde{N} = N \circ T^{-1}$ (see [8], [9], [1], Ch.2), where $\tilde{N}$ denotes the Lebesgue extension of the functional $\tilde{N} = N \circ T^{-1} \in C[-a,a]^*$, in the sense of Lebesgue–Stieltjes integral (see [1], Ch. 2). Consider the operation

(22) $$f * g \stackrel{\text{def}}{=} T^{-1}(Tf *_{\tilde N} Tg) \text{ for } f, g \in L^1[-a, a],$$

where T, T^{-1} is the transmutation operators considered before.

Theorem 4. a) *The operation $*$ is a continuous convolution for the operator D in every space $Y = L^p[-a,a], 1 \le p \le \infty; C[-a,a]; BV[-a,a]$, i.e. it is such a conitnuous, bilinear, commutative and associative operation in Y that $X_{D,Y}$ is an ideal of the algebra $(Y, *)$ and the equality $D(f * g) = (Df) * g$ holds for all $f \in Y$, $g \in X_{D,Y}$. Each space $Y = L^p[-a,a], 1 \le p \le \infty; C[-a,a]$; $C_N[-a,a] \stackrel{\text{def}}{=} \{C[-a,a] : N(f) = 0\}$; $AC_N[-a,a] \stackrel{\text{def}}{=} \{AC[-a,a] : N(f) = 0\}$ is an ideal of the convolutional algebra $(L^1[-a,a], *)$ and the operation $*$ is a $Y \times L^1[-a,a] \to Y$ continuous operation. The resolvent R_λ of the operator D in the space Y is represented by the equality*

(23) $$R_\lambda f = \left\{ -\frac{y(\lambda, t)}{E(\lambda)} \right\} * f \quad \text{for } f \in Y,\ \lambda \in C \text{ with } E(\lambda) \ne 0.$$

b) *For each eigenvalue λ_k the corresponding root projection P_{λ_k} is convolutionally represented in the form*

(24) $$P_{\lambda_k} f = f * u^k_{m_k - 1}, \quad f \in L^1[-a,a], \quad \text{where } u^k_{m_k-1} = \frac{1}{2\pi i} \int_{\Gamma_k} \frac{y(\lambda, t)}{E(\lambda)}\, d\lambda$$

is an associated function of the operator D of hihgest order $m_k - 1$ corresponding to λ_k and the equality $u^k_{m_k-1} * u^k_{m_k-1} = u^k_{m_k-1}$ holds. The operation $*$ is a continuous convolution for the root function expansion (11) *and for the integral transform* (12) *in the sense of* [1], 1.2.6.

Proof. a) Since the convolution $*$ is of the form (22) where $*_{\tilde N}$ is the Berg–Dimovski convolution (see [8], [9]) with respect to the functional $\tilde N = T \circ N$, then from our results of [1], Ch.2 it follows that the convolution $*$ is is a continuous bilinear operation in the space $L^1[-a,a]$, which is a Banach algebra with respect to this operation. Also, its commutativity and associativity follow from the corresponding commutativity and associativity of the Berg–Dimovski convolution. Since the transmutation operators T, T^{-1} are continuous authomotphisms of each space Y of mentioned form and the , the Berg–Dimovski convolution is $Y \times Y \to Y$ continuous, commutative and associative operation as well as $Y \times L^1[-a,a] \to L^1[-a,a]$ continuous operation, then it follows immediately that the convolution $*$ is also operation of the same kind. Using the fact that $E(\lambda) \stackrel{\text{def}}{=} N_t[y(\lambda,t)] = N[T^{-1}(e^{\lambda t})] = (N \circ T^{-1})[e^{\lambda t}] = \tilde N_t[e^{\lambda t}]$ we obtain that formula (23) follows from (14), (9) and analogous formula $\tilde{R}_\lambda f = \{-e^{\lambda t}/\tilde{N}(e^{\lambda t})\} *_{\tilde{N}} f \in L^1[-a,a]$ for the Berg–Dimovski convolution established in [1], Ch.2.

b) Since $P_{\lambda_k} = -\frac{1}{2\pi i}\int_{\Gamma_k} R_\lambda d\lambda$ and since the convolution $*$ is $Y \times L^1[-a,a] \to Y$ continuous operation it is clear that the statement follows from (23) by means of a contour integration under the convolution sign. From the general results in [1], Ch.1, 1.5 it follows that $*$ is a continuous convolution for the root function expansion (11) and for the integral transform (12). ∎

The system of root functions

$$\left\{\frac{1}{s!}\frac{\partial^s}{\partial\lambda^s}y(\lambda_k,t):\ 0 \le s \le m_k - 1\right\}_{k=0}^{\infty} \tag{25}$$

of the operator D (or equivalently the root subspace system $\{\Pi_{\lambda_k}\}_{k=0}^{\infty}$ of the operator D) is said to be *complete* in a linear topological space $Y \subset L^1[-a,a]$, if the set of the finite linear combinations of the root functions is dense in Y.

In the next theorem we obtain a necessary condition for the completeness of the root functions in every linear topological space $Y \subset L^1[-a,a]$, which are dense linear subspace of $L^1[-a,a]$ and is *topological imbedded* in $L^1[-a,a]$, i.e. if the topology of Y is stronger than the topology induced in Y by $L^1[-a,a]$.

Theorem 5. *Let the system of root functions of the operator D is complete in a linear topological space $Y \subset L^1[-a,a]$, densely topologically imbedded in $L^1[-a,a]$. Then the root projection system $\{P_{\lambda_k}\}_{k=0}^{\infty}$ is total in the space $L^1[-a,a]$, i.e. the condition $-a, a \in \mathrm{suppN}$ is necessary for the completeness of the root functions in the space Y. In particular the condition $-a, a \in \mathrm{suppN}$ is necessary for the completeness of the root functions in the space $Y = L^p[-a,a], 1 \le p < \infty$. It is also necessary for the completeness of the root functions in the space $C_N[-a,a]$, if the functional $N \in C[-a,a]^*$ is not continuous in the space $L^1[-a,a]$, e.g. if its representing function $n \in BV[-a,a]$ has at least one jump in $[-a,a]$.*

P r o o f. The theorem is a consequence of the convolutional structure established in Theorem 4 and our general theorem proved in [1], p.53 (Theorem 1.3.4). ∎

Next theorem gives some sufficient conditions for completeness of root functions in some subspaces $Y \subset L^1[-a,a]$

Theorem 6. *Let the representing function $n \in BV[-a,a]$ normalized by the condition $n(t-0) = n(t)$, $t \in (-a,a]$ have at least one point of discontinuity in $[-a,a]$. Then:*

a) There exists L^1-bounded approximate identity in each Banach algebra $(Y,*)$, where $Y = L^p[-a,a], 1 \le p < \infty$; $C[-a,a]$; $C_N[-a,a]$; $AC_N[-a,a]$, i.e.

there is a sequence $e_0 \in Y$ which is bounded with respect to L^1-norm such that $e_0 * f \to f$ with respect to the Y-norm for each $f \in Y$.

b) *If the root functions of the operator D are complete in the space $L^1[-a,a]$, then they are complete in each space $Y = L^p[-a,a], 1 \le p < \infty$; $C[-a,a]$; $C_N[-a,a]$; $AC_N[-a,a]$. In particular, they are complete in these spaces, if one of the endpoints $-a$, a of the segment $[-a,a]$ is a jump point and the other is a point of variability of the representing function n.*

Proof. Indeed, using the Volterra form (6) of the transmutation operator T it is easily to prove that for each $f \in C[-a,a]$ the functional $\tilde{N} = N \circ T$ is represented in the form $\tilde{N}(f) = N(Tf) = \int_{-a}^{a} f(\tau)\, d\tilde{n}(\tau)$, where its representing function $\tilde{n} \in BV[-a,a]$ is represented in the form

$$
(26) \qquad \tilde{n}(t) = \begin{cases} n(t) - \int_0^t d\tau \int_{-a}^{\tau} K(v,\tau)\, dn(v), & -a \le t < 0 \\ n(t) - \int_0^t d\tau \int_{\tau}^{a} K(v,\tau)\, dn(v), & 0 \le t \le a. \end{cases}
$$

The last equality shows that the functions of bounded variation $\tilde{n}$ and n differ with an absolutely continuous function, hence we obtain that the function $\tilde{n}$ has the same jumps as the function n. Then according to a theorem ([1], p.144) for the Berg–Dimovski convolution $*_{\tilde{N}}$, there exists L^1-bounded family of functions $\{\tilde{e}_0\}_{k=0}^{\infty}$ in each space Y. Now since the transmutation operator T maps continuously Y onto itself, it is easily seen that the family $e_0 \stackrel{\text{def}}{=} T^{-1}(\tilde{e}_0)$ is a L^1-bounded approximate identity for the convolution $*$ in Y.

b) follows from a) in the same way as it is done for an analogous result for Berg–Dimovski convolution ([1], p.151, Theorem 2.1.16). ■

We note that other sufficient conditions for completenes of the root functions in the space $L^p[-a,a], 1 \le p < \infty$ can be formulated as well. They are similar to conditions of J.I. Ljubič [10] who considered the special case $D = d/dt$, $V(t,\tau) \equiv 0$. A.M. Sedletzkii [7] proved that the conditions

$$
(27) \qquad n(\pm a \mp 0) \ne n(\pm a), \quad \inf_{k \ne s} |\lambda_k - \lambda_s| > 0,
$$

are sufficient for basis property of the exponent system (root functions of D in the special case $D = d/dt$, $V(t,\tau) \equiv 0$) in the space $L^p[-a,a], 1 < p < \infty$. In the general case of the nonlocal integro-differential operator of first order D it turns out that the conditions (27) are also sufficient for existence of L^p-basis of root functions of the operator D in the space $L^p[-a,a], 1 < p < \infty$. Moreover, these conditions are sufficient for L^1-summability of the root function expansion

(11) by means of Riesz means for each function $f \in L[-a, a]$. They are sufficient for C-summability of the root function expansion (11) by means of Riesz means for each function $f \in C_N[-a, a]$. The conditions (27) are also sufficient for the uniform convergence of the root function expansion (11) for instance for each function f with bounded variation of $C_N[-a, a]$.

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