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Common Fixed Points by Ishikawa Iterates in Metric Linear Spaces

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In this paper we obtain a common fixed point theorem for the Ishikawa iterates of two self-mappings on a metric linear space under various contractive conditions.

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1. Introduction

In [3],[4] it has been shown that for a self-mapping T on a normed space or a Banach space X satisfying various contractive conditions, if the sequence of Ishikawa iterates associated with T converges, it converges to a fixed point of T. These results have been recently extended by L.A. Khan in [2] to the case of metric linear spaces.

In this paper, we consider two self-mappings S and T on a metric linear space X and show that if the sequence of Ishikawa iterates associated with S and T converges, it converges to a common fixed point of S and T.

In the sequel we assume that the topology on X is generated by an F-norm q which has the following properties (see [5], pp. 28-29).

(a) $q(x) \geq 0$ and $q(x) = 0$ if $x = 0$

(b) $q(x + y) \leq q(x) + q(y)$

(c) $q(ax) \leq q(x)$ for all (real or complex) scalars a with $|a| \leq 1$

(d) If $a_n \to a$ and $x_n \to x$, then $q(a_n x_n - ax) \to 0$.

Note that q is continuous on X, the equation $d(x,y) = q(x-y)$ defines a translation-invariant metric on X, and $q(ax) \leq q(bx)$ for all scalars a, b with $|a| \leq |b|$.

Let C be a nonempty convex subset of X and S, T self-maps on C. An Ishikawa scheme [1], for S, T is defined by

$$\begin{aligned} x_0 &\in C \\ y_n &= (1-\beta_n)x_n + \beta_n S x_n, \quad n \geq 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 0, \end{aligned} \tag{1}$$

where the real sequences $\{\alpha_n\}, \{\beta_n\}$ satisfy

(i) $0 \leq \alpha_n \leq 1, \quad 0 \leq \beta_n \leq 1$, for $n \geq 0$

(ii) $\lim_{n\to\infty} \alpha_n = \alpha > 0$

From (1) and condition (i) we obtain some inequalities which will be used later

$$\begin{aligned} q(y_n - x_n) &= q(\beta_n(Sx_n - x_n)) \\ &\leq q(Sx_n - Ty_n) + q(Ty_n - x_n), \end{aligned} \tag{2}$$

$$\begin{aligned} q(y_n - Ty_n) &= q(\beta_n(Sx_n - Ty_n)) + (1-\beta_n)(x_n - Ty_n) \\ &\leq q(Sx_n - Ty_n) + q(x_n - Ty_n), \end{aligned} \tag{3}$$

$$q(x_n - Sx_n) \leq q(x_n - Ty_n) + q(Ty_n - Sx_n) \tag{4}$$

$$\begin{aligned} q(y_n - Sx_n) &= q(1-\beta_n)(x_n - Sx_n) \\ &\leq q(x_n - Ty_n) + q(Ty_n - Sx_n) \end{aligned} \tag{5}$$

2. Main result

We present our result in the form of the following theorem:

Theorem 2.1 *Let C be a closed convex subset of X and let S, T be two self-mappings of C satisfying at least one of the following conditions*

(*I*) $q(Sx - Ty) \leq k\max\{q(x-y), q(x-Sx), q(y-Ty), q(x-Ty) + q(y-Sx)\}$

(*II*) $q(Sx - Ty) + q(x-Sx) + q(y-Ty) \leq c[q(x-Ty) + q(y-Sx)], 0 \leq c < 2$

(III) $\quad q(Sx-Ty) \leq k \max\{q(x-y), q(x-Sx), q(y-Ty), q(x-Ty), q(y-Sx)\}$,

for all $x, y \in C$, where $0 \leq k < 1$ and $0 \leq c < 2$. Suppose that for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ of Ishikawa iterates converges to a point u. Then u is a common fixed point of S and T. Moreover if (III) holds, then the common fixed point u is unique.

Proof. Suppose first that $Su = u$ for a point u in C. Then putting $x = y = u$ into any of the inequalities (I)-(III) we easily see that $Tu = u$. Similarly $Tu = u$ implies $Su = u$. Now let $\{x_n\}$ be a sequence of Ishikawa iterates with S and T such that

$$\lim_{n\to\infty} x_n = u.$$

From (1), we see that

$$Ty_n - x_n = \frac{1}{\alpha_n}\alpha_n(x_{n+1} - x_n).$$

Since

$$\lim_{n\to\infty} \alpha_n = \alpha > 0,$$

there exists an integer $N \geq 1$ such that

$$\frac{\alpha}{2} \leq \alpha_n$$

for all $n \geq N$. Hence for $n \geq N$

$$q(Ty_n - x_n) = q(\frac{\alpha}{2}(x_{n+1} - x_n)).$$

Since

$$\lim_{n\to\infty} x_n = u,$$

we have

$$q(Ty_n - x_n) \to 0 \quad \text{as} \quad n \to \infty$$

and hence

$$q(Ty_n - u) \to 0 \quad \text{as} \quad n \to \infty.$$

Now for $n \geq N$,

$$(6) \qquad (u - Tu) \leq q(u - Ty_n) + q(Sx_n - Ty_n) + q(Sx_n - Tu).$$

Now we need to show that

$$q(Sx_n - Ty_n) \to 0 \quad \text{and} \quad q(Sx_n - Tu) \to o \quad \text{as} \quad n \to \infty.$$

If x_n, y_n satisfy (I), then

$$\begin{aligned} q(Sx_n - Ty_n) \leq k \max \quad & \{q(Sx_n - Ty_n) + q(x_n - Ty_n), \\ & q(x_n - Ty_n) + q(Sx_n - Ty_n), \\ & q(Sx_n - Ty_n) + q(x_n - Ty_n), \\ & 2q(Sx_n - Ty_n) + q(x_n - Ty_n)\} = \\ & = k\,[2q(x_n - Ty_n) + q(Sx_n - Ty_n)]. \end{aligned}$$

If x_n, y_n satisfy (II), then

$$(3 - c)q(Sx_n - Ty_n) \leq 2(c - 1)q(x_n - Ty_n).$$

If x_n, y_n satisfy (III), then

$$\begin{aligned} q(Sx_n - Ty_n) \leq k \max \quad & \{q(x_n - Ty_n) + q(Sx_n - Ty_n), \\ & q(x_n - Ty_n) + q(Sx_n - Ty_n), \\ & q(Sx_n - Ty_n) + q(x_n - Ty_n), \\ & q(Sx_n - Ty_n) + q(x_n - Ty_n), \\ & q(Sx_n - Ty_n) + q(x_n - Ty_n)\} = \\ & = k\,[q(x_n - Ty_n) + q(Sx_n - Ty_n)]. \end{aligned}$$

Hence in any case,

$$q(Sx_n - Ty_n) \leq \max\left\{\frac{2k}{1-k}, \frac{2(1-c)}{3-c}, \frac{k}{1-k}\right\} q(x_n - Ty_n).$$

Letting $n \to \infty$, we obtain $q(Sx_n - Ty_n) \to o$. Then further $q(x_n - Sx_n)$ and $q(u - Sx_n)$ tend to zero as $n \to \infty$. Next if x_n, u satisfy (I), then

$$q(Sx_n - Tu) \leq k\,[q(x_n - u) + q(x_n - Sx_n) + q(Sx_n - Tu) + q(u - Sx_n)].$$

If x_n, u satisfy (II), then

$$(2 - c)q(Sx_n - Tu) \leq (c - 1)\,[q(x_n - Sx_n) + q(u - Sx_n)].$$

If x_n, u satisfy (III), then obviously satisfy (I) as well. Hence, we have

$$q(Sx_n - Tu) \leq \max\left\{\frac{k}{1-k}, \frac{(1-c)}{2-c}\right\} [q(x_n - u) + q(x_n - Sx_n) + q(u - Sx_n)].$$

Letting $n \to \infty$, we have $q(Sx_n - Tu) \to 0$. Thus it follows from (6) that $Tu = u$. In view our remark at the beginning of the proof u is a fixed point of S as well. In order to show the uniqueness of u in the case (III), let $v(v \neq u)$ be another common fixed point of S and T, then using (III) we have

$$q(u - v) = q(Su - Tv) \leq \max\{q(u - v), 0, 0, q(u - v)\},$$

whence $u = v$.

Finally we give some examples of a metric linear space and two mappings which satisfy the contractive conditions of T' orem 2.1.

Example 2.1. Let $X = R$, the set of real numbers and q be the F-norm defined by

$$q(x) = \frac{|x|}{1+|x|}$$

Let $C = [0,1]$ and $S, T : C \to C$ be defined by

$$Sx = \begin{cases} 0, & 0 \le x < 1 \\ \frac{1}{4}, & x = 1. \end{cases}, \qquad Tx = \begin{cases} 0, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1. \end{cases}$$

Then S, T satisfy condition (III) of Theorem 2.1 with $k = \frac{2}{3}$ as follows

(i) If $x = y = 1$, then

$$\begin{aligned} q(Sx - Ty) &= \frac{1}{5}, \\ \frac{2}{3}q(y - Sx) &= \frac{2}{3}q(\frac{3}{4}) = \frac{2}{7} > \frac{1}{5}, \\ \frac{2}{3}q(x - Ty) &= \frac{2}{3}q(\frac{1}{2}) = \frac{2}{9} > \frac{1}{5}, \end{aligned}$$

(ii) If $0 \le x, y \le 1$, then (III) is trivially satisfied.

(iii) If $x = 1$, and $0 \le y < 1$, then

$$\begin{aligned} q(Sx - Ty) &= q(\frac{1}{4}) = \frac{1}{5} \\ \frac{2}{3}q(x - Ty) &= \frac{2}{3}q(1) = \frac{1}{3} > \frac{1}{5}. \\ \frac{2}{3}q(x - Sx) &= \frac{2}{3}q(\frac{3}{4}) = \frac{2}{7} > \frac{1}{5}. \end{aligned}$$

(iv) If $0 \le x \le 1, y = 1$, then (III) (similarly as in (iii)).

Note that 0 is the unique common fixed point of S and T.

Example 2.2. Let X, q, C be as in example 2.1 and let $S, T : C \to C$ be defined by

$$Sx = \begin{cases} 1 - x, & 0 \le x < 1 \\ 0, & x = 1. \end{cases}, \qquad Tx = \begin{cases} \frac{1}{2}, & 0 \le x < 1 \\ 0, & x = 1. \end{cases}.$$

Then S, T satisfy condition (II) of Theorem 2.2 with $c = \frac{3}{2}$ as follows: If $x = 1$ and $0 \leq y < 1$, then

$$\begin{aligned} L &= q(Sx - Ty) + q(x - Sx) + q(y - Ty) \\ &= q(\frac{1}{2}) + q(1) + q(y - \frac{1}{2}) = q(1) + q(y) = \frac{1}{2} + q(y) \end{aligned}$$

$$R = q(x - Ty) + q(y - Sx) = q(\frac{1}{2}) + q(y) = \frac{1}{3} = q(y).$$

So that

$$\frac{L}{R} = \frac{\frac{1}{2} + q(y)}{\frac{1}{3} + q(y)} \leq \frac{3}{2}.$$

In this case $x = \frac{1}{2}$ is a common fixed point of S and T.

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