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Applications of the Algebraic Derivatives to Solving Some Differential Equations of Fractional Order

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The differential equation of fractional order, $tD^{\delta+1}x(t)+(1-t)D^{\delta}x(t)-aD^{\delta-1}x(t)=0$ ($a \in C$, $\delta > 1$, $\delta \in R$), where D^{δ} is the Riemann-Liouville differential operator, and for $\delta = 1$ is the Laguerre differential equation, is solved using the Mikusinski's operational calculus. The obtained solutions are represented by

$$x_{a,\delta}(t) = \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k \frac{t^{k+\delta-1}}{\Gamma(k+\delta)} = \frac{t^{\delta-1}}{\Gamma(\delta)} \, {}_1F_1(a,\delta,t)$$

and satisfy

$$D^{\delta}[x_{a,\delta} * x_{b,\delta}](t) = x_{a+b,\delta}(t),$$

where $*$ is the convolution

$$(f * g)(t) = \int_0^t f(t-\xi)g(\xi)d\xi$$

and ${}_1F_1$ is the Kummer function, called also a confluent hypergeometric function.

Similar results are obtained also for the functions $x_{a,\delta,\alpha} = e^{\alpha t}x_{a,\delta}(t)$.

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1. Introduction

W. Kierat and K. Skornik [3], using the Mikusinski's operational calculus, have solved the differential equation

$$t\frac{d^2x}{dt^2} + (c-t)\frac{dx}{dt} - ax = 0 \qquad (c, a \in C)$$

which for $c = 1$ reduces to the Laguerre differential equation and has as one of its solutions,

$$x_a(t) = \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k \frac{t^k}{\Gamma(k+1)}.$$

We introduce the functions $x_{a,\alpha}(t) = e^{\alpha t}x_a(t)$, closelu related to $x_a(t)$ and prove the following propierties:

$$\frac{d}{dt}[x_a * x_b](t) = x_{a+b}(t)$$

$$(\frac{d}{dt} - \alpha)[x_{a,\alpha} * x_{b,\alpha}](t) = x_{a+b,\alpha}(t).$$

Obviously, the functions $x_a(t)$ and $x_{a,\alpha}(t)$ are generalizations of the Laguerre polynomials and functions respectively.

In this paper we generalize the preceding results, using a similar technique, to a differential equation containing the Riemann-Liouville fractional operators.

2. A generalization of the Laguerre differential equation and its solutions

Following J. Mikusinski [4], we consider the ring of functions

$$\mathcal{C}^n = \{f(t) : f(t) \in C^n([0,\infty)) \ , \ [D^k f(t)]_{t=0} = 0, \quad k = 0, 1, \ldots, n-2\}$$

and the differential equation of fractional order

$$tD^{\delta+1}x(t) + (1-t)D^{\delta}x(t) - aD^{\delta-1}x(t) = 0 \quad (a \in C \ , \ \delta > 1 \ , \ \delta \in R), \tag{2.1}$$

where

$$D^{\delta} = D^n I^{n-\delta} \qquad (n-1 < \delta \leq n) \tag{2.2}$$

and

$$\left.\begin{array}{c} I^{\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1} f(\xi)d\xi \ \ (\nu > 0) \\ I^0 f(t) = f(t) \end{array}\right\}, \tag{2.3}$$

I^{ν} being the Riemann-Liouville fractional integration operator of order ν.

Using (2.2) and (2.3), and substituting $y(t) = I^{n-\delta}x(t)$, the equation (2.1) is transformed into

$$tD^{n+1}y(t) + (1-t)D^n y(t) - aD^{n-1}y(t) = 0. \tag{2.4}$$

In order to solve (2.4) we take in account the following propositions.

Proposition 1. *Let $y(t)$ be the function $y(t) = I^{n-\delta}x(t)$, then*

$$[D^k y(t)]_{t=0} = [\frac{d^k y(t)}{dt^k}]_{t=0} = 0, \quad k = 0, 1, ..., n-1.$$

Proof. The function $D^k y(t)$ can be expressed ([5], p.62, Th. 3) by

$$D^k y(t) = D^k I^{n-\delta} x(t) = I^{n-\delta} D^k x(t) + \sum_{r=0}^{k-1} \frac{t^{n-\delta-k+r}}{\Gamma(n-\delta-k+r+1)} [D^r x(t)]_{t=0},$$

and since $x(t) \in C^n$ and $k = 0, 1, ..., n-1$, it can be easily shown that

$$[D^k y(t)]_{t=0} = [I^{n-\delta} D^k x(t)]_{t=0} = 0.$$

■

From Proposition 1 and using the notion of algebraic derivative given in [4], $\mathcal{D}f(t) = -tf(t)$, we obtain

Proposition 2. *One of the solutions of the equation (2.4) can be written as*

$$y_{a,n}(t) = \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k \frac{t^{k+n-1}}{\Gamma(k+n)}$$

Proof. The equation (2.4) is transformed into

$$\mathcal{D}[s^{n+1} y(t) - y^{(n)}(0)] + s^n y(t) + \mathcal{D}[s^n y(t)] - as^{n-1} y(t) = 0,$$

by using the operational calculus' rule [4],

$$D^n f(t) = s^n f(t) - s^{n-1}[f'(t)]_{t=0} - s^{n-2}[f^{(2)}(t)]_{t=0} - \dots - [f^{(n-1)}(t)]_{t=0}$$

and Proposition 1. Then it can be rewritten as

$$(s^n - s^{n-1})\mathcal{D}y(t) + s^{n-1}(1 - a - ns)y(t) = 0,$$

so one can get the solution

$$y_{a,n}(t) = s^{a-n}(s-1)^{-a} = l^n(1-l)^{-a} = \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k \frac{t^{k+n-1}}{\Gamma(k+n)},$$

where $l = s^{-1}, l^\delta = t^{\delta-1}/\Gamma(\delta)$. ■

Remark. It is very easy to get $\frac{d^n}{dt^n}[y_{a,n} * y_{b,n}](t) = y_{a+b,n}(t)$.

Next we are going to prove similar propierties for the functions $x_{a,\delta}(t) = D^{n-\delta} y_{a,n}(t)$.

Proposition 3. *The solution $x_{a,\delta}(t)$ of equation (2.1) satisfies the following relations:*

a) $x_{a,\delta}(t) = \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k \frac{t^{k+\delta-1}}{\Gamma(k+\delta)} = l^{\delta}(1-l)^{-a}$,

b) $D^{\delta}[x_{a,\delta} * x_{b,\delta}](t) = x_{a+b,\delta}(t)$,

c) $x_{a,\delta}(t) = \frac{t^{\delta-1}}{\Gamma(\delta)}\, {}_1F_1(a, \delta, t)$.

Proof.

It is easy to prove a), using the next three equalities:

- $D^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}$ ([1]),
- $l^{\delta} = \frac{t^{\delta-1}}{\Gamma(\delta)}$ ([4]),
- $x_{a,\delta}(t) = D^{n-\delta} y_{a,n}(t)$.

b) is a straightforward corollary.

c) We can represent the function $x_{a,\delta}(t)$ by

$$x_{a,\delta}(t) = \frac{t^{\delta-1}}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{(a)_k}{(\delta)_k} \frac{t^k}{\Gamma(k+1)} = \frac{t^{\delta-1}}{\Gamma(\delta)}\, {}_1F_1(a, \delta, t),$$

where from [6],

- $\binom{-a}{k} = \frac{(-1)^k (a)_k}{\Gamma(k+1)}$
- $\Gamma(k+\delta) = (\delta)_k \Gamma(\delta)$
- $(a)_0 = 1$; $(a)_n = a(a+1)(a+2)...(a+n-1)$ $n = 1, 2, 3, \ldots$.

3. The functions $x_{a,\delta,\alpha}(t)$

Following a similar process to that in Section 2, we can obtain similar results for the functions $x_{a,\delta,\alpha}(t) = e^{\alpha t} x_{a,\delta}(t)$.

Proposition 4. *The function $y_{a,n,\alpha}(t) = e^{\alpha t} y_{a,n}(t)$, where $y_{a,n}(t)$ is a solution of equation (2.4), satisfies:*

$$\left(\frac{d}{dt} - \alpha\right)^n [y_{a,n,\alpha} * y_{b,n,\alpha}](t) = y_{a+b,n,\alpha}(t).$$

Proof. We use that $e^{\alpha t}\frac{t^{n-1}}{\Gamma(n)} = \frac{1}{(s-\alpha)^n}$ (see [4]) and

$$\begin{aligned}(\tfrac{d}{dt} - \alpha)^n[l^n(1-\alpha l)^{-n}] &= \textstyle\sum_{k=0}^{n}\binom{n}{k}(-1)^k D^k \alpha^{n-k} l^n (1-\alpha l)^{-n} \\ &= (1-\alpha l)^{-n}\textstyle\sum_{k=0}^{n}\binom{n}{k}(-1)^k(\alpha l)^{n-k} = 1.\end{aligned}$$

■

Now we prove the same properties for the functions $x_{a,\delta,\alpha}(t) = e^{\alpha t}x_{a,\delta}(t)$, where $x_{a,\delta}(t)$ is a solution of equation (2.1). First we introduce the following definition.

Definition 1. Let $D = \frac{d}{dt}$ be the differentiation operator and $\delta > 1$ be a real number. Then we define the operator

$$(3.1)\qquad (D-\alpha)^\delta f(t) = \begin{cases} D^\delta f(t) & if\ \alpha = 0, \\ \displaystyle\sum_{k=0}^{\infty}\binom{\delta}{k}(-1)^k\alpha^{\delta-k}D^k f(t) & if\ \alpha \neq 0. \end{cases}$$

Obviously this definition can be only applied only to the set of functions f(t) for which the series $\sum_{k=0}^{\infty}\binom{\delta}{k}(-1)^k\alpha^{\delta-k}D^k f(t)$ is convergent. Let us observe that this set is not empty since, at least the polynomials belong to it.

In order to know if the operator $(D-\alpha)^\delta$ can be applied to the function $x_{a,\delta,\alpha}(t)$, it is necessary to work in the Mikusinski's quotient field [4], and so we can state the following proposition.

Proposition 5.

$$x_{a,\delta,\alpha}(t) = (s-\alpha)^{-\delta}(1 - \frac{1}{s-\alpha})^{-a} = l^\delta(1-\alpha l)^{-\delta}[\frac{1-(\alpha+1)l}{1-\alpha l}]^{-a}$$

Proof. It is a simple exercise taking into account that

$$x_{a,\delta}(t) = \sum_{k=0}^{\infty}\binom{-a}{k}(-1)^k\frac{t^{k+\delta-1}}{\Gamma(k+\delta)}$$

and

$$\frac{1}{(s-\alpha)^\delta} = l^\delta\sum_{k=0}^{\infty}\binom{-\delta}{k}(-1)^k(\alpha l)^k = \frac{t^{\delta-1}}{\Gamma(\delta)}\sum_{k=0}^{\infty}\frac{(\alpha t)^k}{\Gamma(k+1)} = \frac{t^{\delta-1}}{\Gamma(\delta)}e^{\alpha t}.$$

Definition 1 was given for any function, but it can be generalized for the elements of the Mikusinski's quotient field.

Now we can state

Proposition 6. *Let* $m_{a,\delta,\alpha} = (s-\alpha)^{-\delta}(1-\frac{1}{s-\alpha})^{-a}$ *be a Mikusinski operator, then*

$$(D-\alpha)^{\delta}\ [m_{a,\delta,\alpha}\ .\ m_{a,\delta,\alpha}] = m_{a+b,\delta,\alpha}.$$

P r o o f. It can be proved from

$$\begin{aligned}(D-\alpha)^{\delta}[l^{\delta}(1-\alpha l)^{-\delta}] &= \textstyle\sum_{k=0}^{\infty}\binom{\delta}{k}(-1)^{k}\alpha^{\delta-k}l^{\delta-k}(1-\alpha l)^{-\delta}\\ &= (1-\alpha l)^{-\delta}\textstyle\sum_{k=0}^{\infty}\binom{\delta}{k}(-1)^{k}(\alpha l)^{\delta-k)} = 1.\end{aligned}$$

■

Proposition 7. *For the functions* $x_{a,\delta,\alpha}(t)$ *and* $x_{b,\delta,\alpha}(t)$,

$$(D-\alpha)^{\delta}[x_{a,\delta,\alpha} * x_{b,\delta,\alpha}](t) = x_{a+b,\delta,\alpha}(t). \tag{3.2}$$

P r o o f. The functions $x_{a,\delta,\alpha}(t)$ and $x_{b,\delta,\alpha}(t)$ are continious and so, they belong to $\mathcal{C}$, the generating ring of the Mikusinski's quotient field. Thus they can be identified with the operators $m_{a,\delta,\alpha}$ and $m_{b,\delta,\alpha}$, respectively. Therefore, making use of Proposition 6, we can obtain (3.2).

4. More general differential equations

In [7], it is shown that the differential equation

$$4.1a_2ty^{(2)}(t) + (a_1t+b_1)y'(t) + (a_0t+b_0)y(t) = 0 \tag{4.1}$$

is convertible into

$$\frac{\mathcal{D}y}{y} = \frac{q(s)}{p(s)} = \frac{(-2a_2+b_1)s - a_1 + b_0}{a_2s^2 + a_1s + a_0}$$

and if the algebraic equation $p(z) = a_2z^2 + a_1z + a_0$, has two distinct roots z_1 and z_2, then
$y = C(s-z_1)^{\gamma_1}(s-z_2)^{\gamma_2}$ satisfies equation (4.1), where C is a non-zero constant and γ_1 , γ_2 are complex numbers satisfying

$$\frac{q(z)}{p(z)} = \frac{\gamma_1}{z-z_1} + \frac{\gamma_2}{z-z_2}.$$

Making use of Yosida's paper [7], W. Kierat [2] has determined particular solutions of the differential equations with linear coefficients

$$(4.2)\qquad a_n t x^{(n)}(t) + (a_{n-1}t + b_{n-1})x^{(n-1)}(t) + \ldots + (a_0 t + b_0)x(t) = 0,$$

transforming them into

$$\frac{\mathcal{D}x}{x} = \frac{Q(s)}{P(s)}.$$

He concludes that the element of Mikusinski's quotient field

$$(4.3)\qquad v = C\prod_{i=1}^{r}(s-\alpha_i)^{\gamma_{i1}}\prod_{j=2}^{k_i}\exp[\frac{\gamma_{ij}}{-j+1}(s-\alpha_i)^{-j+1}]$$

satisfies equation (4.2), where $C \in C$, the complex numbers α_i are the roots of the polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$ ($k_i \in N$, $k_1 + \ldots + k_r = \deg P$) and $\gamma_{ij} \in C$ satisfying

$$\frac{Q(s)}{P(s)} = \sum_{i=1}^{r}\sum_{j=1}^{k_i}\frac{\gamma_{ij}}{(s-\alpha_i)^j}.$$

The purpose of this section is to show that the method from the previous sections may be used to determine particular solutions of the differential equations of fractional order $\delta > 1$ (n is integer so that $n-1 < \delta \leq n$):

$$(4.4)\qquad a_n t D^{\delta}x(t) + (a_{n-1}t + b_{n-1})D^{\delta-1}x(t) + \ldots + (a_0 t + b_0)D^{\delta-n}x(t) = 0.$$

Let $y(t) = I^{n-\delta}x(t)$. The equation (4.4) can be written as

$$a_n t D^n y(t) + (a_{n-1}t + b_{n-1})D^{n-1}y(t) + \ldots + (a_0 t + b_0)y(t) = 0$$

and then, making use of (4.3) and $x(t) = D^{n-\delta}y(t)$, we can conclude

Proposition 8. *The element*

$$w = Cs^{n-\delta}\prod_{i=1}^{r}(s-\alpha_i)^{\gamma_{i1}}\prod_{j=2}^{k_i}\exp[\frac{\gamma_{ij}}{-j+1}(s-\alpha_i)^{-j+1}]$$

satisfies equation (4.4).

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