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Between $\mathcal{A}$- and $\mathcal{B}$-sets [1]

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The aim of this paper is to introduce the class of $\mathcal{AB}$-sets as the sets that are the intersection of an open and a semi-regular set. Several classes of well-known topological spaces are characterized via the new concept. A new decomposition of continuity is provided.

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1. Introduction

A subset A of a topological space (X, τ) is called *locally closed* if A is open in its closure or equivalently if $A = U \cap V$, where U is open and V is closed. Several classes of sets in general topological spaces have the above mentioned property. For example, all connected subsets of the real line as well as all locally compact subsets of Hausdorff spaces are locally closed. Moreover, a Tychonoff topological space X is locally closed in its Stone-Čech compactification βX if and only if X is locally compact. Spaces in which every subset is locally closed are known as submaximal. Recently, locally closed sets were studied in [9, 12].

In 1986 and in 1989, Tong [18, 19] introduced two new classes of set, namely $\mathcal{A}$-sets and $\mathcal{B}$-sets and using them obtained new decompositions of continuity. He defined a set A to be an $\mathcal{A}$-*set* [18] (resp. a $\mathcal{B}$-*set* [19]) if $A = U \cap V$, where U is open and V is regular closed (resp. semi-closed (= t-set)). Clearly every $\mathcal{A}$-set is locally closed and every locally closed set is a $\mathcal{B}$-sets. Several topological spaces can be characterized via the concepts of $\mathcal{A}$- and $\mathcal{B}$-sets [4].

The concepts of $\mathcal{A}$-sets, locally closed sets and $\mathcal{B}$-sets play important role when continuous functions are decomposed. If the reader is interested in different decompositions of continuity he (she) can refer to [1, 5, 8, 10, 11, 17,

18, 19]. Several new decomposition of continuous and related mappings were recently obtained in [6].

The aim of this paper is to introduce a class of sets very closely related to the classes of $\mathcal{A}$- and $\mathcal{B}$-sets, in fact properly placed between them. Under consideration are the sets that can be represented as the intersection of an open and a semi-regular set. A subset A of a (topological) space (X, τ) is called *semi-regular* [15] if it is both semi-open and semi-closed. In [15], Di Maio and Noiri pointed out that a set A is semi-regular if and only if there exists a regular open set U such that $U \subseteq A \subseteq \overline{U}$. Cameron [2] called semi-regular set *regular semi-open*. In this paper, the connection of $\mathcal{AB}$-sets to other classes of 'generalized open' sets is investigated as well as several characterizations of topological spaces via $\mathcal{AB}$-sets are given. The concept of $\mathcal{AB}$-continuity is also introduced. A new decomposition of continuity and a decomposition of $\mathcal{AB}$-continuity is produced at the end of the paper.

Recall that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called Ä-continuous [17] if for every open set V of (Y, σ), the set $f^{-1}(V)$ belongs to Ä, where Ä is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of Ä-continuity. However, for unknown concepts the reader may refer to [4, 10].

2. $\mathcal{AB}$-sets

Definition 1. *A subset A of a space (X, τ) is called an $\mathcal{AB}$-set if $A = U \cap V$, where U is open and V is semi-regular. The collection of all $\mathcal{AB}$-sets in X will be denoted by $\mathcal{AB}(X)$.*

Since regular closed sets are semi-regular and since semi-regular sets are semi-closed, then the following implications are obvious:

$$\mathcal{A}\text{-set} \Rightarrow \mathcal{AB}\text{-set} \Rightarrow \mathcal{B}\text{-set}$$

None of them of course is reversible as the following examples shows:

Example 2.1.   Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Set $A = \{b, c\}$. It is easily observed that A is an $\mathcal{AB}$-set but not an $\mathcal{A}$-set.

Example 2.2.   Let X be the space from Example 3.1 from [18], i.e. let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, X\}$. Set $A = \{c\}$. It is easily observed that A is a $\mathcal{B}$-set but not an $\mathcal{AB}$-set.

Clearly every open and every semi-regular set is an $\mathcal{AB}$-set. But the $\mathcal{AB}$-subset of the real line $\mathbf{R}$ (with the usual topology) $A = (\mathbf{R} \setminus \{0\}) \cap [-1, 1]$ is neither open nor semi-regular.

Moreover, since the intersection of an open set and a semi-regular set is always semi-open, then the following implication is clear:

$$\mathcal{AB}\text{-set} \Rightarrow \text{Semi-open set}$$

However, if one considers the space from Example 2.2 above, it becomes clear that not all semi-open sets are $\mathcal{AB}$-sets: The set $\{a, b\}$ is semi-open but not an $\mathcal{AB}$-set.

Next the relation between $\mathcal{B}$-sets and $\mathcal{AB}$-sets is shown but first consider the following, probably known lemma. Recall that a set $A \subseteq (X, \tau)$ is called β-open (= semi-preopen) if $A \subset \overline{\text{Int}\overline{A}}$. The semi-closure of a set $A \subseteq (X, \tau)$ is the intersection of all semi-closed supersets of A.

Lemma 2.3. *The semi-closure of every β-open set is semi-regular.*

Theorem 2.4. *For a subset A of a space X the following are equivalent:*
(1) A is an $\mathcal{AB}$-set.
(2) A is semi-open and a $\mathcal{B}$-set.
(3) A is β-open and a $\mathcal{B}$-set.

P r o o f. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Since A is a $\mathcal{B}$-set, then in the notion of Theorem 1 from [20], there exists an open sets U such that $A = U \cap \text{sCl}A$, where $\text{sCl}A$ denotes the semi-closure of A in X. By Lemma 2.3, $\text{sCl}A$ is semi-regular, since by (3) A is β-open. Thus A is an $\mathcal{AB}$-set. ∎

Recall that a space X is called *submaximal* if every dense subset of X is open. Let $\beta O(X)$ denote the collection of all β-open subset of X.

Corollary 2.5. *If (X, τ) is a submaximal space, then $\mathcal{AB}(X) = \beta O(X)$.*

P r o o f. Since X is submaximal, then by Theorem 3.1 from [4] every (β-open) subset of X is a $\mathcal{B}$-set. Thus by Theorem 2.4, every β-open subset of X is an $\mathcal{AB}$-set. On the other hand, every $\mathcal{AB}$-set is β-open. □

The class of locally closed sets is also properly placed between the classes of $\mathcal{A}$- and $\mathcal{B}$-sets but the concepts of $\mathcal{AB}$-sets and locally closed sets are independent from each other: If first, every locally closed set is an $\mathcal{AB}$-set, then it would be semi-open as well. But locally closed, semi-open sets are $\mathcal{A}$-sets [10, Theorem 1]; however not all locally closed sets are $\mathcal{A}$-sets. Second, if every $\mathcal{AB}$-set would be locally closed, then again it must be an $\mathcal{A}$-set but as shown above not all $\mathcal{AB}$-sets are $\mathcal{A}$-sets.

Theorem 2.6. *For a subset A of a space X the following are equivalent:*
(1) A is semi-regular.

(2) A *is semi-closed and an $\mathcal{AB}$-set.*

(3) A *is β-closed and an $\mathcal{AB}$-set.*

P r o o f. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Since A is β-closed and a $\mathcal{B}$-set, then A is semi-closed [4, Theorem 2.3]. On the other hand A is semi-open, since it is an $\mathcal{AB}$-set. Thus A is semi-regular, being both semi-open and semi-closed. ∎

Recall that a subset A of a space (X, τ) is called *interior-closed* (= ic-set) [11] if Int$A$ is closed in A. If $A \subseteq$ Int$\overline{A}$, then A is called *locally dense* [3] (= preopen).

Theorem 2.7. *For a subset A of a space X the following are equivalent:*

(1) A *is open.*

(2) A *is an $\mathcal{AB}$-set and A is either locally dense or an ic-set.*

P r o o f. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) If A is locally dense, then since A is also a $\mathcal{B}$-set, it follows from Proposition 9 in [19] that A is open. If A is an ic-set, then in the notion of Theorem 1 from [11], A is again open, since A is also semi-open. ∎

3. Some peculiar spaces

Recall that a space X is called *extremally disconnected* (= ED) if every open subset of X has open closure or equivalently if every regular closed set is open.

Theorem 3.1. *For a space (X, τ) the following are equivalent:*

(1) X *is ED.*

(2) $\tau = \mathcal{AB}(X)$.

(3) *Every $\mathcal{AB}$-set is open.*

P r o o f. (1) $\Rightarrow$ (2) Let $A \in \mathcal{AB}(X)$. Clearly A is semi-open. From Theorem 4.1 in [13] it follows that A is preopen, since X is ED. Moreover A is a $\mathcal{B}$-set and since it is preopen, it follows from Proposition 9 in [19] that $A \in \tau$. Hence $\mathcal{AB}(X) \subseteq \tau$. On the other hand it is obvious that $\tau \subseteq \mathcal{AB}(X)$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Let $A \subseteq X$ be regular closed. Thus A is an $\mathcal{AB}$-set. By (3) A is open. So, X is ED. ∎

Theorem 3.2. *For a space X the following are equivalent:*

(1) X *is submaximal.*

(2) *Every locally dense set is an $\mathcal{AB}$-set.*

(3) *Every dense set is an $\mathcal{AB}$-set.*

P r o o f. (1) $\Rightarrow$ (2) Let $A \subseteq X$ be locally dense (= preopen). By (1), A is open, since in submaximal spaces every locally dense set is open [14]. Hence A is an $\mathcal{AB}$-set.

(2) $\Rightarrow$ (3) every dense set is locally dense.

(3) $\Rightarrow$ (1) Let $A \subseteq X$ be dense. By (3), A is an $\mathcal{AB}$-set. Hence A is both preopen and a $\mathcal{B}$-set. From Proposition 9 in [19] it follows that A is open. Thus X is submaximal. ∎

Recall that a space X is called a *partition space* if every open subset of X is closed.

Theorem 3.3. *For a space X the following are equivalent:*

(1) *X is a partition space.*

(2) *Every $\mathcal{AB}$-set is clopen.*

(3) *Every $\mathcal{AB}$-set is (pre)closed.*

P r o o f. (1) $\Rightarrow$ (2) Let $A \subseteq X$ be an $\mathcal{AB}$-set. By (1) and Theorem 3.2 from [4], A is clopen, since it is a $\mathcal{B}$-set.

(2) $\Rightarrow$ (3) every clopen set is preclosed.

(3) $\Rightarrow$ (1) Let $A \subseteq X$ be open. Then A is an $\mathcal{AB}$-set and by (3) it is preclosed. Since every preclosed (semi-)open set is (regular) closed, then X is a partition space. ∎

Theorem 3.4. *For a space X the following are equivalent:*

(1) *X is indiscrete.*

(2) *$\mathcal{AB}(X) = \{\emptyset, X\}$.*

P r o o f. The theorem follows from Theorem 3.3 from [4], since the class of $\mathcal{AB}$-sets is (properly) placed between the classes of $\mathcal{A}$- and $\mathcal{B}$-sets. ∎

Theorem 3.5. *For a space X the following are equivalent:*

(1) *X is discrete.*

(2) *Every subset of X is an $\mathcal{AB}$-set.*

(3) *Every singleton is an $\mathcal{AB}$-set.*

P r o o f. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Let $x \in X$. By (3), $\{x\}$ is an $\mathcal{AB}$-set and hence semi-open. Then $\{x\}$ must contain a non-void open subset. Since the only possibility is $\{x\}$ itself, then each singleton is open or equivalently X is discrete. ∎

Recall that a space X is called *hyperconnected* if every open subset of X is dense in X.

Theorem 3.6. *For a space X the following are equivalent:*

(1) *X is hyperconnected.*

(2) *Every $\mathcal{AB}$-set is dense.*

P r o o f. (1) $\Rightarrow$ (2) Let $A \subseteq X$ be an $\mathcal{AB}$-set. Then A is semi-open and hence there exist an open subset U such that $U \subseteq A \subseteq \overline{U}$. By (1), U is dense. Hence its superset A is also dense.

(2) $\Rightarrow$ (1) Every open subset of X is an $\mathcal{AB}$-set and hence by (2) dense.

∎

Recall that a space X is called *semi-connected* [16] if X cannot be expressed as the disjoint union of two non-void semi-open sets.

Theorem 3.7. *For a space X the following are equivalent:*
(1) X *is semi-connected.*
(2) X *is not the union of two disjoint non-void $\mathcal{AB}$-sets.*

P r o o f. (1) $\Rightarrow$ (2) If X is the union of two disjoint non-void $\mathcal{AB}$-sets, then X is not semi-connected, since $\mathcal{AB}$-sets are semi-open.

(2) $\Rightarrow$ (1) If X is not semi-connected, then X has a non-trivial semi-open subset A with semi-open complement. Since both A and $B = X \setminus A$ are semi-regular, then A and B are $\mathcal{AB}$-sets. So X is the union of two disjoint non-void $\mathcal{AB}$-sets, contradictory to (2).   ∎

4. $\mathcal{AB}$-continuous functions

Definition 2. *A function $f : (X, \tau) \to (Y, \sigma)$ is called $\mathcal{AB}$-continuous , if the preimage of every open subset of Y is an $\mathcal{AB}$-set in X.*

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is called *strongly irresolute* [7] if $f(sClA) \subseteq f(A)$ for every subset A of X. It is easily observed that a function $f : (X, \tau) \to (Y, \sigma)$ is strongly irresolute if and only if the inverse image of every subset of Y is semi-regular in X.

The last four theorems, below, are consequences of results from the beginning of this paper, therefore their proofs are omitted. Theorem 4.1 gives the relations between $\mathcal{AB}$-continuous functions and other forms of 'generalized continuity'. Note that none of the implications in Theorem 4.1 is reversible. Theorem 4.2 gives a decomposition of $\mathcal{AB}$-continuity, while Theorem 4.3 is an improvement of Theorem 4 (i) from [10] and it follows from Theorem 2.4 in [4]. Theorem 4.4 gives a decomposition of continuity dual to $\mathcal{AB}$-continuity.

Theorem 4.1. (i) *Every $\mathcal{A}$-continuous function is $\mathcal{AB}$-continuous,*
(ii) *Every strongly irresolute function is $\mathcal{AB}$-continuous,*
(iii) *Every $\mathcal{AB}$-continuous function is $\mathcal{B}$-continuous,*
(iv) *Every $\mathcal{AB}$-continuous function is semi-continuous.*

Theorem 4.2. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:*

(1) f is $\mathcal{AB}$-continuous.

(2) f is semi-continuous and $\mathcal{B}$-continuous.

(3) f is β-continuous and $\mathcal{B}$-continuous.

Theorem 4.3. *For a function $f:(X,\tau) \to (Y,\sigma)$, the following conditions are equivalent:*

(1) f is $\mathcal{A}$-continuous.

(2) f is β-continuous and LC-continuous.

Theorem 4.4. *For a function $f:(X,\tau) \to (Y,\sigma)$, the following conditions are equivalent:*

(1) f is continuous.

(2) f is $\mathcal{AB}$-continuous and either precontinuous or ic-continuous.

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