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Four Inequalities From PDE

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Presented by Bl. Sendov

In this note, we carry out some elementary computations concerning four inequalities associated with the function $|x|^p$ on $\mathbf{R}^n$ and the functional $\int_\Omega |\nabla u|^p \,dx$ on the Sobolev space $W_0^{1,p}(\Omega)$ and then apply those inequalities to establish the existence of multiple solutions of quasilinear elliptic boundary value problems.

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1. Introduction

In the study of nonlinear partial differential operators and equations, one meets quite often the functions $|x|^p$ on $\mathbf{R}^n$ and the functional $\int_\Omega |\nabla u|^p \,dx$ on $W_0^{1,p}(\Omega)$. The purpose of this note is to establish some inequalities about the two functions above, and its application in quasilinear elliptic boundary value problems.

2. Inequalities in $\mathbf{R}^n$

In this part, we study the convex function $|x|^p = (x_1^2 + x_2^2 + \ldots + x_n^2)^{p/2}$ in $\mathbf{R}^n$, where $p, n \geq 1$. The following inequalities have been known in the study of $p-$Laplace operator Δ_p, which is defined as $\Delta_p(u) = \div\{|\nabla u|^{p-2}\nabla u\}$ (see [2, 6, 8]).

Theorem 1. *Given constant $p > 1$, there exists a constant γ, which depends only on p, n such that*

a) *If $p \geq 2$, then*

$$D(x, y) \geq \gamma |x - y|^p, \quad \forall x, y \in \mathbf{R}^n.$$

b) *If* $2 \geq p > 1$, *then*

$$D(x,y) \leq \gamma |x-y|^p, \quad \forall x,y \in \mathbf{R}^n,$$

where $D(x,y) = (|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y)$ *and* '$\cdot$' *denotes the standard inner product in* $\mathbf{R}^n$.

In [8] Neta found the best constant $\gamma = 2^{2-p}$ in Theorem 1. Our first result is about the lower and upper bound estimates of $D(x,y)$ with the best constants.

Theorem 2. *The following inequalities hold with best constants.*

a) *If* $p \geq 2$, *then for all* $x, y \in \mathbf{R}^n$

$$2^{2-p}|x-y|^p \leq D(x,y) \leq (p-1)|x-y|^2(|y|+|x-y|)^{p-2}. \tag{2.1}$$

b) *If* $1 < p < 2$, *then for all* $x, y \in \mathbf{R}^n$

$$2^{2-p}|x-y|^p \geq D(x,y) \geq (p-1)|x-y|^2(|y|+|x-y|)^{p-2}. \tag{2.2}$$

Proof. We will show only (2.2) ((2.1) can be treated in similar way). By a change of variables, we see that the inequalities in (2.2) are equivalent to: For all $z, y \in \mathbf{R}^n$,

$$2^{2-p}|z|^p \geq (|y+z|^{p-2}(y+z) - |y|^{p-2}y)\cdot z \geq (p-1)|z|^2(|z|+|y|)^{p-2},$$

By rotation and the homogeneity, we see that they are also equivalent to

$$2^{2-p} \geq (|y+e|^{p-2}(y+e) - |y|^{p-2}y)\cdot e \geq (p-1)(1+|y|)^{p-2}, \quad \forall y \in \mathbf{R}^n, \tag{2.3}$$

where $e = (1, 0, \ldots, 0)$.

Consider the function

$$f_0(y) = (|y+e|^{p-2}(y+e) - |y|^{p-2}y)\cdot e$$

over $\mathbf{R}^n$. Note that if $y = (y_1, y_2, \ldots, y_n)$ then

$$\begin{aligned} f_0(y) &= (1+y_1)(1+2y_1+y_1^2+\ldots+y_n^2)^{\frac{p-2}{2}} - y_1(y_1^2+\ldots+y_n^2)^{\frac{p-2}{2}} \\ &= (1+t)(1+2t+r^2)^q - tr^{2q} = f(r,t) \end{aligned}$$

where $r = |y| = \sqrt{y_1^2+\ldots+y_n^2}$, $t = y_1$, $2q = p-2 \leq 0$. The inequalities in (2.3) are equivalent to

$$2^{-2q} \geq f(r,t) \geq (p-1)(1+r)^{2q}, \quad \forall (r,t) \in \mathcal{D} \tag{2.4}$$

where $\mathcal{D} = \{(r,t) \in \mathbf{R}^2; \quad |t| \le r\}$. Obviously, $f(r,t)$ is continuously differentiable in the interior of $\mathcal{D}$, and

$$f_r'(r,t) = 2q\{(1+2t+r^2)^{q-1}r(1+t) - r^{2q-1}t\}, \tag{2.5}$$

$$f_t'(r,t) = 2q(1+2t+r^2)^{q-1}(1+t) + (1+2t+r^2)^q - r^{2q}. \tag{2.6}$$

We see from (2.6) that if $1+2t \ge 0$, then $f_t'(r,t) < 0$ and consequently

$$\begin{aligned} f(r,t) &\ge f(r,r) = (1+r)^{p-1} - r^{p-1} \\ &= (p-1)\int_0^1 (r+\theta)^{p-2}\,d\theta \ge (p-1)(1+r)^{p-2}; \end{aligned}$$

If $1+t \le 0$, then $f(r,t)$ is increasing in t and therefore

$$f(r,t) \ge f(r,-r) = r^{p-1} - (r-1)^{p-1} \ge (1+r)^{p-1} - r^{p-1};$$

Finally, for $t \in (-1,-0.5)$, then $(1+2t+r^2)^q \ge r^{2q}$ and

$$f(r,t) \ge (1+t)r^{2q} - tr^{2q} = r^{p-2} \ge (p-1)(1+r)^{p-2},$$

which gives the inequality in the right-hand side of (2.4). Observe that

$$\frac{(1+r)^{p-1} - r^{p-1}}{(p-1)(1+r)^{p-2}} = \int_0^1 \left(\frac{r+\theta}{1+r}\right)^{p-2} d\theta \to 1,$$

as $r \to +\infty$, we see that the inequality is also sharp.

The left-hand inequality in (2.4) has been proved in [8], but we present here a different proof for the completeness. We calculate the maximum of $f(r,t)$ on $\mathcal{D}$. Dividing $\mathcal{D}$ into subdomains: $\mathcal{D}_1 = \{(r,t) \in \mathcal{D}; 1+2t \ge 0\}, \mathcal{D}_2 = \{(r,t) \in \mathcal{D}; -1 \le t \le 0.5\}, \mathcal{D}_3 = \{(r,t) \in \mathcal{D}; t \le -1\}$, we deduce from (2.6) that on $\mathcal{D}_1$, $f_t'(r,t) \le 0$ and thus

$$\max_{(r,t)\in\mathcal{D}_1} f(r,t) = \max\{\max_{0\le r\le 1/2} f(r,-r), \max_{r\ge 1/2} f(r,-1/2)\} = 2^{-2q} = 2^{2-p}.$$

On the other hand, $f(r,t)$ is increasing in t on $\mathcal{D}_3$, so we have

$$\max_{(r,t)\in\mathcal{D}_3} f(r,t) = \max_{r\ge 1} r^{2q} = 1.$$

Finally, we see from (2.5) that on $f_r'(r,t) \le 0$ on $\mathcal{D}_2$ and

$$\max_{(r,t)\in\mathcal{D}_2} f(r,t) \quad = \quad \max_{-1/2\ge t\ge -1} f(-t,t)$$

$$= \max\{(1-t)^{p-1} + t^{p-1}; 1/2 \le t \le 1\} = 2^{2-p}.$$

The proof is complete. ∎

Theorem 3. *For any $p \in (1,\infty)$, then there is a constant $c = c(p)$ such that the following inequalities hold with the sharp constants,*

a) *if $p \in [2,\infty)$, then for any $x, y \in \mathbf{R}^n$*

$$c_0(p)|y|^2(|x|+|y|)^{p-2} \ge C(x,y) \ge c(p)|y|^p. \tag{2.7}$$

b) *if $p \in (1,2]$, then for any $x, y \in \mathbf{R}^n$,*

$$c_0(p)|y|^2(|x|+|y|)^{p-2} \le C(x,y) \le c(p)|y|^p, \tag{2.8}$$

where $C(x,y) = |x+y|^p - |x|^p - p|x|^{p-2}x \cdot y$, $c_0(p) = p(p-1)/2$ and

$$c(p) = \begin{cases} \max\{(1-r)^p - r^p + pr^{p-1}; 0 \le r \le 1/2\}, & \text{if } 1 < p \le 2, \\ \min\{(1-r)^p - r^p + pr^{p-1}; 0 \le r \le 1/2\}, & \text{if } p \ge 2. \end{cases}$$

Remark. i) The right-hand side inequality in (2.7) with a constant $c = 1/(2^{p-1}-1)$ was obtained in [7] and a similar inequality as in the left hand side of (2.8) was also discussed in [7]. But our proof here is different from that in [7].

ii) It can be shown that

$$c(p) = \begin{cases} \sqrt{2+\sqrt{2}}/\sqrt{2}, & \text{if } p = 3/2, \\ 1, & \text{if } p = 2, \\ 2-\sqrt{2}, & \text{if } p = 3, \\ 1/3, & \text{if } p = 4, \end{cases}$$

Proof. We shall give only a proof of (2.7). Similarly as in the proof of Theorem 2, we see that inequalities in (2.7) are equivalent to

$$c_0(p)(1+r)^{p-2} \ge h(r,t) \ge c(p), \quad \forall (r,t) \in \mathcal{D}, \tag{2.9}$$

where $h(r,t) = (1+2t+r^2)^{p/2} - r^p - ptr^{p-2}$. An easy calculation shows that

$$h'_r(r,t) = p\{r(1+2t+r^2)^{(p-2)/2} - r^{p-1} - (p-2)r^{p-3}t\}, \tag{2.10}$$

$$h'_t(r,t) = p\{(1+2t+r^2)^{(p-2)/2} - r^{p-2}\}. \tag{2.11}$$

First we show the left-hand estimate in (2.9). It follows from (2.11) that if $1+2t \geq 0$, then $h'_t \geq 0$ and

$$\begin{aligned} h(r,t) &\leq h(r,r) = (1+r)^p - r^p - pr^{p-1} \\ &= p(p-1)\int_0^1 d\theta \int_0^1 (r+\theta\tau)^{p-2}\theta\, d\tau \\ &\leq \frac{1}{2}p(p-1)(1+r)^{p-2}. \end{aligned}$$

On the other hand, if $t < -1/2$, then $h'_t(r,t) \leq 0$ and thus

$$h(r,t) \leq h(r,-r) = |r-1|^p - r^p + pr^{p-1}$$

The estimate in (2.9) follows from the following

Claim : $|r-1|^p - r^p + pr^{p-1} = h(r,-r) \leq h(r,r) = (1+r)^p - r^p - pr^{p-1}.$

The sharpness of the inequality follows from the fact: as $r \to +\infty$.

$$(1+r)^{2-p}h(r,r) = p(p-1)\int_0^1 \theta\, d\theta \int_0^1 \left(\frac{r+\theta\tau}{r+1}\right)^{p-2} d\tau \to p(p-1)/2.$$

P r o o f of the claim. We study the two cases 1) $r \geq 1$, 2) $0 < r < 1$ separately. If $r \geq 1$,then we see that

$$\begin{aligned} h(r,r) - h(r,-r) &= (1+r)^p - (r-1)^p - 2pr^{p-1} \\ &= p\int_0^1 ((r+\theta)^{p-1} + (r-\theta)^{p-1})\, d\theta - 2pr^{p-1} \\ &= p(p-1)\int_0^1 \theta d\theta \int_0^1 ((r+\tau\theta)^{p-2} - (r-\tau\theta)^{p-2})\, d\tau \\ &= p(p-1)(p-2)\int_0^1 \theta^2 d\theta \int_0^1 \tau d\tau \int_{-1}^1 (r+\theta\tau s)^{p-3} ds \geq 0 \end{aligned}$$

If $r \in (0,1)$, then by the assumption $p \geq 2$ we get

$$\begin{aligned} h(r,r) - h(r,-r) &= (1+r)^p - (1-r)^p - 2pr^{p-1} \\ &\geq (1+r)^p - (1-r)^p - 2pr \\ &= p(p-1)r\int_0^1 \theta d\theta \int_0^1 ((1+r\tau\theta)^{p-1} - (1-r\tau\theta)^{p-2}))\, d\tau \\ &= p(p-1)(p-2)r^2\int_0^1 \theta^2 d\theta \int_0^1 \tau d\tau \int_{-1}^1 (1+\theta\tau s r)^{p-3} ds \geq 0. \end{aligned}$$

To show the right-hand estimate in (2.9), we need to find the minimum of $h(r,t)$ on $\mathcal{D}$. On the subdomain $\mathcal{D}^0 = \{(r,t) \in \mathcal{D}; 1+2t \geq 0\}$ of $\mathcal{D}$, $h(r,t)$ is increasing in t and

$$\min_{(r,t)\in\mathcal{D}^0} h(r,t) = \min \begin{cases} \min h(r,-\frac{1}{2}), & \frac{1}{2} \leq r \\ \min h(r,-r), & 0 \leq r \leq \frac{1}{2} \end{cases} = \min \begin{cases} p2^{1-p} \\ c(p) \end{cases} = c(p).$$

On the subset $\mathcal{D}_0 = \{(r,t) \in \mathcal{D}; 1+2t \leq 0\}$, h is decreasing in t and thus,

$$\min_{(r,t)\in\mathcal{D}_0} h(r,t) = \min_{r\geq 0.5} h(r,-\frac{1}{2}) = p2^{1-p}$$

Therefore, the inequality follows directly.

3. Inequalities on $W^{1,p}(\Omega)$

Given a subset $\Omega \subseteq \mathbf{R}^n$, let $\mathcal{L}^p(\Omega)$ be the space of p-integrable vector-valued functions from Ω to $\mathbf{R}^m$. In this part, we shall discuss some inequalities on $\mathcal{L}^p(\Omega)$ or on the Sobolev space $W^{1,p}(\Omega)$.

Theorem 4. *On $\mathcal{L}^p(\Omega)$, the following inequalities hold*

a) *If $1 < p < 2$, then for any $u, v \in \mathcal{L}^p(\Omega)$,*

$$(3.1) \qquad \begin{aligned} 2^{2-p}\|u-v\|_p^p &\geq \textstyle\int_\Omega (|u|^{p-2}u - |v|^{p-2}v)\cdot(u-v)\,dx \\ &\geq (p-1)\|u-v\|_p^2(\|u\|_p + \|u-v\|_p)^{p-2}, \end{aligned}$$

$$(3.2) \qquad \begin{aligned} c(p)\|v\|_p^p &\geq \|u+v\|_p^p - \|u\|_p^p - p\textstyle\int_\Omega |u|^{p-2}u\cdot v\,dx \\ &\geq c_0\|v\|_p^2(\|u\|_p + \|v\|_p)^{p-2}. \end{aligned}$$

b) *If $p \geq 2$, then for any pair u, v from $\mathcal{L}^p(\Omega)$,*

$$(3.3) \qquad \begin{aligned} 2^{2-p}\|u-v\|_p^p &\leq \textstyle\int_\Omega (|u|^{p-2}u - |v|^{p-2}v)\cdot(u-v)\,dx \\ &\leq (p-1)\|u-v\|_p^2(\|u\|_p + \|u-v\|_p)^{p-2}, \end{aligned}$$

$$(3.4) \qquad \begin{aligned} c(p)\|v\|_p^p &\leq \|u+v\|_p^p - \|u\|_p^p - p\textstyle\int_\Omega |u|^{p-2}u\cdot v\,dx \\ &\leq c_0\|v\|_p^2(\|u\|_p + \|v\|_p)^{p-2}. \end{aligned}$$

Here the constants $c_0(p)$ and $c(p)$ are given in Theorem 3 and $\|u\|_p^p = \int_\Omega |u(x)|^p\,dx$.

Proof. We consider only the inequalities in (3.1). The left hand side inequality follows directly from Theorem 2 by integration. To show the second one, it suffices by Theorem 2 to prove that

$$(3.5) \qquad \int_\Omega |v(x)|^2(|v(x)| + |u(x)|)^{p-2}\,dx \geq \|v\|_p^2(\|v\|_p + \|u\|_p)^{p-2}$$

Notice $2/p$ and $2/(2-p)$ are conjugate to each other, so it follows from the Hölder and triangle inequalities that

$$\int_\Omega |v(x)|^p dx \leq \left(\int_\Omega |v(x)|^2(|v(x)|+|u(x)|)^{p-2}dx\right)^{\frac{p}{2}} \left(\int_\Omega (|v(x)|+|u(x)|)^p dx\right)^{\frac{2-p}{2}}$$

$$\leq \left(\int_\Omega |v(x)|^2(|v(x)|+|u(x)|)^{p-2}\,dx\right)^{\frac{p}{2}} (||v||_p+||u||_p)^{\frac{p(2-p)}{2}}$$

which is equivalent to (3.5). This completes the proof. ■

In the following, we work on the Sobolev space $W_0^{1,p}(\Omega)$ and use the norm $||u||^p = \int_\Omega |\nabla u|^p\,dx$. As an immediate consequence of Theorem 4 and the Sobolev imbedding theorem, we have the following strong monotonicity result of $p-$Laplace operator.

Corollary 1. i) *If $p \geq 2$, then for any $u, v \in W_0^{1,p}(\Omega)$*

$$\int_\Omega D(\nabla u(x), \nabla v(x))\,dx \geq c(p)||u-v||^p. \tag{3.6}$$

ii) *If $1 < p < 2$, then for any pair $u, v \in W_0^{1,p}(\Omega)$*

$$\int_\Omega D(\nabla u(x), \nabla v(x))\,dx \geq c(p)(||u||+||v||)^{p-2}||u-v||^2. \tag{3.7}$$

Corollary 2. *For any $u_0, v \in W_0^{1,p}(\Omega)$, we have the following estimate*
a) *If $1 < p < 2$, then*

$$c_0(p)||v||^2(||v||+||u_0||)^{p-2} \leq E_0(v) \leq c(p)||v||^p. \tag{3.8}$$

b) *If $p \geq 2$, then*

$$c_0(p)||v||^2(||v||+||u_0||)^{p-2} \geq E_0(v) \geq c(p)||v||^p, \tag{3.9}$$

where

$$E_0(v) = \int_\Omega (|\nabla (v+u_0)|^p - |\nabla u_0|^p - p|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla v)\,dx.$$

4. Applications

In this section we study the quasilinear elliptic boundary value problem on a bounded domain,

$$\begin{cases} -\Delta_p u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

To establish the existence of solutions of (4.1) by the variational method, we study the functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,u)\,dx, \quad F(x,u) = \int_0^u f(x,t)\,dt,$$

on the Sobolev space $W_0^{1,p}(\Omega)$ [3, 9, 10]. The critical points of $E(u)$ are the solutions of (4.1). We assume that $f(x,u)$ is measurable in x and is continuous in u and satisfies

$$|f(x,u)| \le a(x) + b|u|^q, a(x) \in L^r, b > 0, p < q+1 < p^*, r \ge p^*/(p^*-1), \tag{4.2}$$

where p^* is the critical Sobolev exponent.

Theorem 5. *Let K be any closed convex subset in $W_0^{1,p}(\Omega)$, $u_0 \in K$ and $\delta_0 > 0$ be a constant such that $E(u) - E(u_0) > 0, \forall u \in K, 0 < ||u-u_0|| \le \delta_0$. If the assumption (4.2) holds, then for any $\delta \in (0,\delta_0]$, there exists a $\rho = \rho(\delta) > 0$ so that $E(u) - E(u_0) \ge \rho$, if $u \in K$ and $\delta \le ||u-u_0|| \le \delta_0$.*

Proof. Suppose that there were a sequence $\{u_n\}$ such that $\delta \le ||u_n - u_0|| \le \delta_0$, but $E(u_n) \to E(u_0)$. Since a closed ball in $W_0^{1,p}(\Omega)$ is weakly compact [11] and $K_0 = K \cap \{||u-u_0|| \le \delta_0\}$ is weakly closed, therefore there exists a $\bar{u} \in K_0$ such that $u_n \rightharpoonup \bar{u}$ weakly. By the Rellich-Kondrachov compact embedding theorem, we see that $E(u)$ is weakly lower semi-continuous under the assumption (4.2) [5] and consequently, $\bar{u} = u_0$. On the other hand, we have

$$E_1(u_n) = \int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla(u_n - u_0)\,dx \to 0, n \to \infty.$$

$$E_2(u_n) = \int_\Omega (F(x,u_n) - F(x,u_0))\,dx \to 0, n \to \infty.$$

Whence,

$$E_0(u_n - u_0)/p = E(u_n) - E(u_0) - E_1(u_n) - E_2(u_n) \to 0, n \to \infty.$$

However, we obtain from Corollary 2 that $E_0(u_n - u_0) \ge \tau(\delta) > 0$ if $||u_n - u_0|| \ge \delta$. This is a contradiction and the claim is proved. ∎

Next we consider the nonhomogeneous problem,

$$\begin{cases} -\Delta_p u = g(u) + \lambda h(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where $h(x) \neq 0$ and $\lambda > 0$ is a parameter. We are mainly interested in the existence of multiple positive solutions. We assume that $g(u) = 0, u \leq 0$, and

$$|g(u)| \leq \alpha_1 |u|^{q_1 - 1} + \alpha_2 |u|^{q_2 - 1}, u \geq 0, \quad h(x) \in L^{p_*}(\Omega), \quad p_* = \frac{p^*}{p^* - 1} \tag{4.4}$$

for some $\alpha_i > 0, q_i \in (p, p^*), i = 1, 2$, and let $u_0 = (-\Delta_p)^{-1}(h)$, i.e., u_0 solves the following boundary value problem

$$\begin{cases} -\Delta_p u = h(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Then we have the following existence theorem.

Theorem 6. *Under the hypothesis (4.4), there is a $\lambda_0 > 0$ such that if $\lambda \in (0, \lambda_0]$, then (4.3) has at least one solution u_1, which is a local minimizer of $E(u)$ and is also non-negative under an extra condition either $h(x) \geq 0$ or $u_0(x) \geq 0$ and $g(u) \geq 0$ for $u \geq 0$. If additionally there is a $q_0 \in (p, p^*)$ such that $ug(u) \geq Cu^{q_0}$ for large u, then (4.3) has at least two ordered positive solutions $u_1(x), u_2(x)$ and $u_1 \leq u_2$.*

Proof. Now we consider the associated functional

$$E(u) = \int_\Omega (\frac{1}{p} | \nabla u|^p \, dx - G(u) - \lambda h(x) u) \, dx, \quad G(u) = \int_0^u g(t) \, dt.$$

Let $k = k(s)$ be the best constant in the embedding $||u||_s^s \leq k|| \nabla u||_p^s, u \in W_0^{1,p}(\Omega)$. If $||u|| = t > 0$ then by the assumption (4.4)

$$E(u) \geq \frac{t^p}{p} - \beta_1 t^{q_1} - \beta_2 t^{q_2} - \lambda \beta_3 t = t[e(t) - \lambda \beta_3],$$

where $\beta_1 = \alpha_1 k(q_1)/q_1, \beta_2 = \alpha_2 k(q_2)/q_2, \beta_3 = k(p^*)||h||_{p_*}$. It can be easily shown that $e(t)$ has a unique maximum at point $t = t_0 > 0$, which is the only root of equation

$$(p-1)/p = (q_1 - 1)\beta_1 t^{q_1 - p} + (q_2 - 1)\beta_2 t^{q_2 - p}$$

and $e(t_0) > 0$, since $e(t) > 0$ for small $t > 0$. So if we choose $\lambda_0 = e(t_0)/\beta_3$, then for any $\lambda \in (0, \lambda_0]$ we have $E(u) \geq 0, ||u|| = t_0$. Furthermore, E is bounded from below on $\{u; ||u|| \le t_0\}$ and if $\varphi(x) \in W_0^{1,p}(\Omega)$ such that $\int_\Omega h(x)\varphi(x)\,dx > 0$, then $E(t\varphi) < 0$ for sufficient small $t > 0$. Therefore, $E(u)$ has a local minimizer u_1 inside the ball $\{u; ||u|| \le t_0\}$, which is a solution of (4.3) and $E(u_1) < 0$.

To show that $u_1 \ge 0$, we note that when the assumption $u_0(x) \ge 0$ and $g(u) \ge 0, u \ge 0$ is satisfied, we deduce by the comparison principle that $u_1(x) \ge u_0(x) \ge 0$. If the condition $h(x) \ge 0$ holds, then we use $u_1^- = \min\{u_1, 0\}$ as a test function of (4.3), we see that

$$\begin{aligned} 0 \le \int_\Omega |\nabla u_1^-|^p &= \int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla u_1^- \,dx \\ &= \int_{\{u_1 \le 0\}} [g(u_1) + \lambda h(x)] u_1^- \,dx = \int_{\{u_1 \le 0\}} \lambda h(x) u_1^- \,dx \le 0, \end{aligned}$$

which implies $\nabla u_1^- = 0$ on Ω and consequently, $u_1^- = 0$ since $u_1^- = 0$ on $\partial\Omega$. If $g(u) \ge 0, u \ge 0$, then $u_1 > 0$ on Ω by the strong maximal principle.

To get the second solutions $u_2 \ge u_1$, we use the truncation technique [1, 3] and consider the functional

$$E_+(v) = \int_\Omega (\frac{1}{p}|\nabla (v + u_0)|^p - G_+(x, v))\,dx - E(u_0),$$

where

$$G_+(x, v) = \int_0^v g_+(x, t)\,dt, \quad g_+(x, t) = \begin{cases} g(t + u_0(x)) + \lambda h(x), & \text{if } t \ge 0, \\ g(u_0(x)) + \lambda h(x), & \text{if } t \le 0. \end{cases}$$

Then we see that $E_+(0) = 0$, and $E_+(v) \ge -E(u_1) > 0$, if $||v + u_1|| = t_0$. The mountain pass theorem [9, 10] imlpies that E_+ admitts a critical point $v_0 \ne 0$ (see [3] for details). Finally, we use Corollary 1 to derive $v_0 \ge 0$. Since u_1 solves (4.3) and $u_2 = v_0 + u_1$ solves the equation $-\Delta_p u = g_+(x, u)$, using $v_0^+ = \min\{0, v_0\}$ as a test function for both equations we obtain

$$\begin{aligned} \int_\Omega D(\nabla(u_1 + v_0^-), \nabla u_1)dx &= \int_{\{v_0 \le 0\}} (|\nabla u_2|^{p-2} \nabla u_2 - |\nabla u_1|^{p-2} \nabla u_1) \cdot \nabla v_0 dx \\ &= \int_{\{v_0 \ge 0\}} (g_-(x, u_1 + v_0) - g(u_1) - \lambda h(x))dx = 0. \end{aligned}$$

But, according to Corollary 1,

$$\int_\Omega D(\nabla(u_1 + v_0^-), \nabla u_1)\,dx \ge \begin{cases} c(p)(||u_1 + v_0^-|| + ||u_1||)^{p-2}||v_0^-||^2, & \text{if } p \in (1, 2), \\ c(p)||v_0^-||^p, & \text{if } p \ge 2. \end{cases}$$

Therefore $v_0^- = 0$. Thus $u_2 = v_0 + u_1 \ge u_1$ is a solution of the original problem (4.3). The proof is complete. ∎

R e m a r k. 1) The result in Theorem 6 remains valid if h is just a bounded linear functional on $W_0^{1,p}(\Omega)$ (with the constant β_3 replaced by the norm of h in the dual space of $W_0^{1,p}(\Omega)$).

2) Further applications of Corollaries 1 and 2 will be given in the forthcoming paper [5], where the Hölder countinuity of the inverse of $p-$Laplace operator is discussed.

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