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Simple States of Non-Linear Electric Field Model in $\mathbf{R}^2$

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A three-dimensional non-homogeneous system of quasi-linear PDEs on R^2 is considered. By means of a geometrical approach we study the existence and uniqueness of solution constructed by Riemann invariants.

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1. Introduction. Notation

In this paper we study three-dimensional systems of quasi-linear PDEs in two independent variables t (time) and x (space) of the form

$$(1)\qquad \left|\begin{array}{l} \partial_t u^1 + \partial_x u^2 = 0 \\ \partial_t u^2 - a^2 \partial_x u^1 + 2a\partial_x u^2 + h = 0 \\ \partial_x u^3 - g = 0, \end{array}\right.$$

where $\partial_s \equiv \partial/\partial s$, and $a = a(u^2/u^1)$, $g = g(u^1)$, $h = h(u)$ are sufficiently smooth functions of their arguments; $u \equiv (u^1, u^2, u^3)$.

Such systems are physical models of non-linear electric field's oscillations in continuous media and were investigated by U. P. Emec, I. I. Reppa in [4], who reduce (1) by a suitable transformation into a second order ODE; they analyse only the behaviour of the solution in a neighbourhood of singular points, with very special assumptions about the functions a, g, h. For certain conservation law systems A. M. Grundland, R. Zelazzny (refer to [2] and [3]) proposed an algebraic approach for finding of solutions constructed by means of Riemann invariants (for non-homogeneous systems they are called simple states; see A.

Jeffrey [1]), but it is unfit for (1). In the present work we study a more general system, applying a purely geometric method (see J. Tabov [6-9]). For that purpose for the functions a, g, h we suppose:

H1. $a,\ g \in C^m(\mathbf{R}^1)$ $(m \geq 2)$, $h \in C^n(\mathbf{R}^3)$ $(n \geq 2)$;

H2. *For* $|\, s \,| > 0$, *and* $p,\ q,\ r \in \mathbf{R}^1$ *are fulfilled* $|\, g(s) \,| > 0,\ \ |\, h(s,p,q) \,| > 0,\ \ |\, ps^{-1}g(s) + r \,| > 0$.

The main result of our paper consists in the statement that there exists a pair of vector-fields

$$\eta_1 = (-y, g, 0, 0, 0, 0)\,,\ \ \eta_2 = (1, 0, gK, gKy^{-1}, -K, g),$$

such that the system

$$(2)\qquad \left|\begin{array}{l} \eta_1\Phi \equiv -y\partial_x\Phi + g\partial_t\Phi = 0 \\ \eta_2\Phi \equiv \partial_x\Phi + gK\partial_y\Phi + gKy^{-1}\partial_{u^1}\Phi - K\partial_{u^2}\Phi + g\partial_{u^3}\Phi = 0, \end{array}\right.$$

is involutive; here $\Phi(t, x, y, u^1, u^2, u^3)$ is unknown function, $K = ghy(ga + y)^{-2}$. The involutivity implies that the commutator $[\eta_1, \eta_2]$ belongs to the linear hull of η_1, η_2 (it is easy verifiable); besides, the system (2) has four functionally independent solutions Φ_i $(i = 1,2,3,4)$ forming the system $\Phi_i(t, x, y, u^1, u^2, u^3) = c_i$ $(i = 1,2,3,4)$ (the constants c_i $(i = 1,2,3,4)$ depend on the initial conditions), hence by classical implicit function theorem we obtain the implicit functions $y = y(t,x)$, $u^j = u^j(t,x)$ $(j = 1,2,3)$; thus $u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ is a Riemann wave type solution for (1) with suitable initial data.

2. Tree-dimensional distribution in $\mathbf{R}^6$

Lemma 1 (A. M. Grundland [2], our interpretation). *The system (1) possesses Riemann wave type solution if and only if at least two among following three equalities hold,*

$$(3)\qquad \langle \nabla u^1,\ J\nabla u^2 \rangle = 0,\ \ \langle \nabla u^1,\ J\nabla u^3 \rangle = 0,\ \ \langle \nabla u^2,\ J\nabla u^3 \rangle = 0,$$

where $\langle .,. \rangle$ *denotes scalar product, and the matrix* J *is defined as*

$$J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Thus u is a solution of (1), constructed by means of Riemann invariants, if and only if it satisfies the system

$$(4)\qquad \left|\begin{array}{l} \partial_t u^1 = -\partial_x u^2 \\ \partial_t u^2 = a^2\partial_x u^1 - 2a\partial_x u^2 - h \\ \partial_x u^3 = g, \\ \langle \nabla u^1, J\nabla u^2\rangle = 0 \\ \langle \nabla u^2, J\nabla u^3\rangle = 0. \end{array}\right.$$

Letting $y \equiv \partial_t u^3$, $K \equiv \partial_t u^1$, $L \equiv \partial_t u^2$ and replacing them in (4) we obtain the following algebraic system in two unknowns K, L,

$$(5)\qquad \left|\begin{array}{l} K^2 + gy^{-1}K(a^2gy^{-1}K + 2aK - h) = 0 \\ yK + gL = 0, \end{array}\right.$$

which possesses two real solutions: (i) $K_1 = L_1 = 0$ (ii) $K_2 = ghy(ga + y)^{-2}$, $L_2 = -hy^2(ga + y)^{-2}$.

For $K = L = 0$ and $g \neq 0$ the system (1) has no solution, and a trivial solution exists provided $g \equiv h \equiv 0$.

In the non-trivial case $K = K_2(y, u)$, $L = L_2(y, u)$ determined by (ii), we apply the idea proposed by J. Tabov in [7] (see [8], [9] as well) .

3. Involutive subdistributions

The Pfaff' system corresponding to (4) is

$$(6)\qquad \left|\begin{array}{l} \omega^1(dz) \equiv ydu^1 - Kdu^3 = 0 \\ \omega^2(dz) \equiv gdu^2 + Kdu^3 = 0 \\ \omega^3(dz) \equiv du^3 - ydt - gdx = 0, \end{array}\right.$$

where $z \equiv (x, t, y, u^1, u^2, u^3) \in \mathbf{R}^6$; the differential forms $\omega^j(dz)$ $(j = 1, 2, 3)$ define a three-dimensional codistribution $\tilde{\theta}$. The vector-fields

$$(7)\qquad \begin{array}{c} \xi_1 = (0, 0, 1, 0, 0, 0), \quad \xi_2 = (0, 1, 0, K, L, y) \\ \xi_3 = (1, 0, 0, gy^{-1}K, -K, g), \end{array}$$

annuling the forms $\omega^j(\xi_k) \equiv 0 \quad (j,k = 1,2,3)$ determine a three-dimensional distribution $\theta(z)$ which is their linear hull. Our next purpose is to investigate whether $\theta(z)$ has two-dimensional involutive subdistributions.

Lemma 2 (Compare with Th. 3 in J. Tabov [7]) *For $k = 1,2,3$ there exists exactly one (up to a scalar multiplier) vector field $\eta = \eta_k$, satisfying the system*

(8) $$\omega^i(\eta) = 0, \quad \partial\omega^k(\xi_j, \eta) = 0 \quad (i = 1,2,3;\ j = 1,2,3).$$

If the system (6) has a two-dimensional resolving distribution $\theta_1(z)$, then $\eta_k \in \theta_1(z)$.

The vector-fields $\eta_1, \eta_2, \eta_3 \in \theta(z)$ satisfying the systems (8), respectively are linearly dependent, and if $g \neq 0$ their rank equals 2.

Proof. Omitting the technical details, we write directly the possible solutions of that systems, respectively

$$\eta_1 = -gy^{-1}\xi_2 + \xi_3$$

$$\eta_2 = -g\xi_2 + y\xi_3$$

$$\eta_3 = K\xi_1 + \xi_3,$$

or written in six-dimensional coordinate basis $(\partial_x, \partial_t, \partial_y, \partial_{u^1}, \partial_{u^2}, \partial_{u^3})$,

(9) $$\begin{aligned} \eta_1 &= (K - y\partial_y K,\ -gy^{-1}(K - y\partial_y K),\ 0,\ 0,\ 0,\ 0) \\ \eta_2 &= (y\partial_y K,\ -g\partial_y K,\ 0,\ 0,\ 0,\ 0) \\ \eta_3 &= (1,\ 0,\ K,\ gy^{-1}K,\ -K,\ g). \end{aligned}$$

Obviously, the vector-fields η_1 and η_2 are colinear.

Thus we obtain two linearly independent vector-fields (up to a multiplier) as follws:

(10) $$\eta_1 = (-y, g, 0, 0, 0, 0), \quad \eta_2 = (1, 0, gK, gKy^{-1}, -K, g).$$

Theorem 1. *If the system (6) possesses two-dimensional involutive subdistribution $\theta_1(z)$ of $\theta(z)$, then the vector-fields $\eta_j \ (j = 1,2)$ belong to $\theta_1(z)$.*

P r o o f: reduces to check that $\eta_j \ (j = 1,2)$ satisfy the systems (8), respectively.

Theorem 2. *The subdistribution $\theta_1(z)$ of $\theta(z)$ defined as the linear hull of $\eta_j \ (j = 1,2)$ is involutive.*

Proof. The commutator of η_j $(j = 1, 2)$ equals

$$[\eta_1, \eta_2] = (\partial_{u^1} g) K y^{-1} \eta_1.$$

Hence $[\eta_1, \eta_2] \in \theta_1(z)$, which proves Theorem 2.

Theorem 3. *Let for the system (1) $H1$ and $H2$ hold. Since the system (2) is involutive and the conditions for the existence of implicit function are fulfilled, it follows that the system (2) has three functionally independent solutions* $\Phi_i(t, x, y, u^1, u^2, u^3)$ $(i = 1, 2, 3, 4)$, *which form the system* $\Phi_i(t, x, y, u^1, u^2, u^3) = \Phi_i(t_0, x_0, y_0, u_0^1, u_0^2, u_0^3)$ $(i = 1, 2, 3, 4)$, *specifying the implicit functions* $y = y(t, x)$, $u^j = u^j(t, x)$ $(j = 1, 2, 3)$; *thus* $u(t, x) = (u^1(t, x), u^2(t, x), u^3(t, x))$ *is an unique Riemann wave type solution for (1) with initial data* $u^j(t_0, x_0) = u_0^j$ $(j = 1, 2, 3)$, $y_0 = \partial_t u^3(t_0, x_0)$.

Proof. From Th. 1 and Th. 2 it follows that the system (4) has four functionally independent solutions $\Phi_i(t, x, y, u^1, u^2, u^3)$ $(i = 1, 2, 3, 4)$. By the classical Pfaff theory, the implicit functions y, u^1, u^2, u^3 determined by the equations $\Phi_i(t, x, y, u^1, u^2, u^3) = \Phi_i(t_0, x_0, y_0, u_0^1, u_0^2, u_0^3)$ $(i = 1, 2, 3, 4)$ give all the solutions of (4), and Theorem 3 is proved.

Since (2) is equivalent to the system

$$\left|\begin{aligned} & dx/dt = -yg^{-1} \\ & dy/dt = -yK \\ & du^1/dt = -K \\ & du^2/dt = yKg^{-1} \\ & du^3/dt = -y, \end{aligned}\right.$$

which possesses integrals

$$y = y_0 \exp(u^1 - u_0^1), \quad u^2 = u_0^2 - y_0 \int_{u_0^1}^{u^1} \exp(s - u_0^1) g^{-1}(s) ds,$$

and since K already depends solely on u^1, u^3, i.e. $K = K(u^1, u^3)$, we can reduce the problem of solving (11) to the equation

$$du^1/du^3 = K(u^1, u^3) y_0^{-1} \exp(u_0^1 - u^1).$$

The applied method for solving of the considered problem is applicable for more general systems of quasi-linear PDEs, with coefficients dependig on both dependent and independent variables, and the Grundland's condition (3) can be replaced with another one, like this $F(x, t, u, u_x, u_t) = 0$. Then it makes possible to be obtained solutions not only of Riemann wave type, but with complete different properties.

References

[1] A. J e f f r e y. Equations of evolution and waves, In: *Wave Phenomena: Modern Theory and Applications*, North-Holland Math. Studies **97**, Elsevier, N.Y., 1984, 1-17.
[2] A. M. G r u n d l a n d. Riemann invariants for non-homogeneous systems of quasilinear partial differential equations, *Bull. Acad. Polon. Sci., Ser. Sci. Techn.* **22**, No 4, 1974, 273-282.
[3] A. M. G r u n d l a n d, R. Z e l a z z n y. Simple waves in quasilinear hyperbolic systems (Part 1 & Part 2), *J. Math. Phys.* **24**, No 9, 1983.
[4] U. P. E m e c, I. I. R e p p a. Nonlinear variations of electric field in continuous medium, *Appl. Math. and Techn. Phys.* (In Russian) **3**, 1969, 55-57.
[5] J. T a b o v. Existence of resolving distributions for Pfaff' systems, *Russ. Math. Surv.* **29**, 1974, 183-185.
[6] J. T a b o v. Existence of resolving distributions of dimension $n - r - 1$ and $n - r - 2$ for Pfaff' systems of rank r in $\mathbf{R}^n$, *Russ. Math. Surv.* **30**, 1975, 187-188.
[7] J. T a b o v. Riemann invariants for a class of PDE' systems, *Diff. Equations* (In Russian). To appear.
[8] J. T a b o v. Simple waves for certain systems of PDEs in $\mathbf{R}^3$, In: *Fifth Intern. Colloq. on Diff. Eqs in Plovdiv,* SCT Publishing, Singapore, 1995, 217-224.

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